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ON HALF CYCLICALLY ORDERED GROUPS

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Abstract. In this paper we introduce and investigate the notion of half cyclically ordered group generalizing the notion of half partially ordered group whose study was begun by Giraudet and Lucas.

Keywords: half partially ordered group, half cyclically ordered group, half lc-group, lexicographic product

MSC 2000: 06F15

Introduction

In this paper we introduce the notion of a half cyclically ordered group generalizing the notion of half partially ordered group which has been studied by Giraudet and Lucas [5] (cf. also Giraudet and Rachůnek [6], Černák [2a], [2b], Ton [19], Černák and the author [12], and the author [9], [11]). In particular, we deal with half ℓc-groups (which generalize the half linearly ordered groups from [5]).

For the terminology, cf. Section 1. Let M be an ℓ-cyclically ordered set with $\operatorname{card} M \geqslant 3$; we denote by $P(M)$ the system of all monotone permutations on M. We recall that the idea of dealing with $P(M)$ goes back to Droste, Giraudet and Macpherson [3].

We introduce in a natural way the group operation and the relation of cyclic order on the set $P(M)$. In Section 2 we show that if M is finite and $\operatorname{card} M \geqslant 3$, then the just mentioned structure on $P(M)$ is a half cyclically ordered group. If M is infinite, then the analogous result need not be valid in general.

In Section 3 we prove the following result:

(A) Let G be a half ℓc-group such that

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(i) the decreasing part $G\downarrow$ of G is nonempty,

(ii) for each $y \in G\downarrow$, $y^2 = e$.

Then the increasing part $G\uparrow$ of G is abelian.

If G is a half linearly ordered group, then the condition (ii) above is satisfied. Hence (A) generalizes a result of Giraudet and Lucas [5] concerning half linearly ordered groups.

In Section 4 we define the notion of lexicographic product decomposition of a half cyclically ordered group G which fails to be cyclically ordered (i.e., such that $G\downarrow \neq \emptyset$).

The increasing part $G\uparrow$ of a half cyclically ordered group G is a cyclically ordered group. To each lexicographic product decomposition α of G there corresponds a lexicographic product decomposition β of the cyclically ordered group $G\uparrow$; we say that β is generated by α.

Let β_1 be a lexicographic product decomposition of $G\uparrow$. In Section 5 we find a necessary and sufficient condition for the existence of a lexicographic product decomposition α_1 of G such that β_1 is generated by α_1.

Fundamental results on lexicographic products of linearly ordered groups have been proved by Maľcev [13]. Further, lexicographic product decompositions of some types of ordered algebraic structures were dealt with in the papers [1], [7], [10], [11].

1. Preliminaries

For the sake of completeness, we start by recalling some notions which will be systematically used below.

For the following definition cf. Novák and Novotný [14], [15] and Quilot [16]; cf. also the author's paper [6].

1.1. Definition. A nonempty set M endowed with a ternary relation C is said to be cyclically ordered if the following conditions are satisfied:

(I) If $(x, y, z) \in C$, then $(y, x, z) \notin C$.

(II) If $(x, y, z) \in C$, then $(z, x, y) \in C$.

(III) If $(x, y, z) \in C$ and $(x, z, u) \in C$, then $(x, y, u) \in C$.

The relation C is called a cyclic order on M.

1.2. Definition. Suppose that C is a cyclic order on M satisfying the condition

(IV) whenever x, y and z are mutually distinct elements of M, then either $(x, y, z) \in C$ or $(z, y, x) \in C$.

Then M is said to be ℓ-cyclically ordered and C is called an ℓ-cyclic order on M.

We remark that in some papers (cf., e.g., [3] and [17]) a different terminology was applied; namely "cyclic order" was understood as the above term ℓ-cyclic order. Further, ℓ-cyclic order is called "complete cyclic order" in [16].

If $(M;C)$ is a cyclically ordered set and if M_1 is a nonempty subset of M, then we consider M_1 as cyclically ordered under the induced cyclic order (i.e., under the relation $C \cap M_1^3$).

1.3. Definition. Let G be a group. Further, suppose that G is at the same time a cyclically ordered set satisfying the condition

(V) if $(x_1, x_2, x_3) \in C$, $a \in G$, $y_i = ax_i$, $z_i = x_i a$ $(i = 1, 2, 3)$, then $(y_1, y_2, y_3) \in C$ and $(z_1, z_2, z_3) \in C$.

Then G is called a cyclically ordered group. In particular, if G is an ℓ-cyclically ordered set, then G is called an ℓc-group.

1.4. Definition. A cyclically ordered group G is said to be a dc-group if, whenever x and y are distinct elements of G, then there exists $z \in G$ such that either $(x, y, z) \in C$ or $(y, x, z) \in C$.

It is clear that each ℓc-group is a dc-group. Lexicographic products of dc-groups have been investigated in [1].

1.5. Example. Let $(G; \leqslant)$ be a partially ordered group with a non-trivial partial order. We denote by C the set of all triples (x, y, z) of elements of G such that one of the conditions

$$x < y < z, \quad y < z < x, \quad z < x < y$$

is valid. Then $(G; C)$ is a cyclically ordered group. If, moreover, $(G; \leqslant)$ is a linearly ordered group, then $(G; C)$ is an ℓc-group.

Now suppose that $(G, \cdot)$ is a group and that, at the same time, $(G; C)$ is a cyclically ordered set.

We denote by $G{\uparrow}$ (and $G{\downarrow}$) the set of all $x \in G$ such that, whenever $(y_1, y_2, y_3) \in C$, then $(xy_1, xy_2, xy_3) \in C$ (or $(xy_3, xy_2, xy_1) \in C$, respectively).

1.6. Definition. Let $(G; \cdot, C)$ be as above. G is said to be a half cyclically ordered group, if the following conditions are satisfied:

1) the system C is nonempty;
2) if $x \in G$ and $(y_1, y_2, y_3) \in C$, then $(y_1 x, y_2 x, y_3 x) \in C$;
3) $G = G{\uparrow} \cup G{\downarrow}$;
4) if $(x, y, z) \in C$, then either $\{x, y, z\} \subseteq G{\uparrow}$ or $\{x, y, z\} \subseteq G{\downarrow}$.

If, moreover, $G\uparrow$ is a dc-group (or an ℓc-group), then G is called a half dc-group (or a half ℓc-group, respectively).

$G\uparrow$ (and $G\downarrow$) is called the increasing part (or the decreasing part, respectively) of G.

If $(G,\cdot,C)$ is a half cyclically ordered group and if G_1 is a subgroup of G such that the induced cyclic order $C_1 = C \cap G_1^3$ is nonempty, then we call $(G_1,\cdot,C_1)$ a hc-subgroup of $(G;\cdot,C)$. We often write G_1 instead of $(G_1,\cdot,C_1)$.

Each cyclically ordered group G is a half cyclically ordered group (we have $G\uparrow = G$ and $G\downarrow = \emptyset$). Hence in view of 1.5 and according to the definitions contained in [5] we have

(i) the class of all half partially ordered groups is a subclass of the class of all half cyclically ordered groups;

(ii) the class of all half linearly ordered groups is a subclass of all half ℓc-groups.

The notion of isomorphism of half cyclically ordered groups is defined in the usual way.

From 1.6 we immediately obtain

1.7. Lemma. *Let G be a half cyclically ordered group and $x_1, x_2 \in G\uparrow$, $y_1, y_2 \in G\downarrow$. Then $x_1x_2 \in G\uparrow$, $y_1y_2 \in G\uparrow$, $x_1y_1 \in G\downarrow$, $y_1x_1 \in G\downarrow$.*

2. Monotone permutations on an ℓ-cyclically ordered set

Let $(M;C)$ be an ℓ-cyclically ordered set with card $M \geqslant 3$. We denote by $P(M)(+)$ the system of all permutations p on M such that

$$(x,y,z) \in C \Rightarrow \big(p(x),p(y),p(z)\big) \in C;$$

further, let $P(M)(-)$ be the set of all permutations q on M with

$$(x,y,z) \in C \Rightarrow \big(q(z),q(y),q(x)\big) \in C.$$

We put $P(M) = P(M)(+) \cup P(M)(-)$. The elements of $P(M)$ will be called monotone permutations on $(M;C)$.

For $\varphi_1,\varphi_2 \in P(M)$ let φ be the permutation on M with $\varphi(x) = \varphi_1(\varphi_2(x))$ for each $x \in M$. Then $\varphi \in P(M)$. Also, $\varphi_1^{-1} \in P(M)$. Denote $\varphi = \varphi_1\varphi_2$. Then $P(M)$ turns out to be a group.

We define $\overline{C}$ to be the set of all triples $(\varphi_1,\varphi_2,\varphi_3)$ of elements of $P(M)$ such that for each $x \in M$ the relation

$$\big(\varphi_1(x),\varphi_2(x),\varphi_3(x)\big) \in C$$

is valid.

The following assertion is easy to verify; the proof will be omitted.

2.1. Lemma. *$(P(M);C)$ is a cyclically ordered set.*

Under the notation as in Section 1 we have

$$G{\uparrow} = P(M)(+), \quad G{\downarrow} = P(M)(-), \quad G = G{\uparrow} \cup G{\downarrow},$$

where $G = P(M)$.

Let $(\varphi_1, \varphi_2, \varphi_3) \in \overline{C}$, $\varphi \in G$ and let $t \in M$. Then

$$\big(\varphi_1(\varphi(t)), \varphi_2(\varphi(t)), \varphi_3(\varphi(t))\big) \in C,$$

whence $(\varphi_1\varphi, \varphi_2\varphi, \varphi_3\varphi) \in \overline{C}$.

Thus in view of 1.6 and 2.1 we have

2.2. Lemma. *Let M be an ℓ-cyclically ordered set. Suppose that*

(i) *the corresponding system $\overline{C}$ is nonempty;*

(ii) *if $(\varphi_1, \varphi_2, \varphi_3) \in \overline{C}$, then either $\{\varphi_1, \varphi_2, \varphi_3\} \subseteq P(M)(+)$ or $\{\varphi_1, \varphi_2, \varphi_3\} \subseteq P(M)(-)$.*

Under these assumptions $(P(M), \cdot, \overline{C})$ is a half cyclically ordered group.

2.3. Proposition. *Let M be a finite ℓ-cyclically ordered set,* $\operatorname{card} M \geqslant 3$. *Put $G = (P(M), \cdot, \overline{C})$. Then G is a half cyclically ordered group.*

P r o o f. Without loss of generality we can assume that $M = \{0, 1, 2, \ldots, n-1\}$, $n \geqslant 2$ and that for $x, y, z \in M$ the relation $(x, y, z) \in C$ holds if and only if one of the conditions

$$x < y < z, \quad y < z < x, \quad z < x < y$$

is valid, where the symbol $<$ for elements of M has the usual meaning.

Let the operations $+$ and $-$ on M be taken $\operatorname{mod} n$.

Let p be a permutation on M. Then p belongs to $P(M)(+)$ if and only if there is $k_1 \in M$ such that

$$p(x) = k_1 + x \quad \text{for each } x \in M.$$

Similarly, p is an element of $P(M)(-)$ if and only if there is $k_2 \in M$ such that

$$p(x) = k_2 - x \quad \text{for each } x \in M.$$

There are $k_1, k_2, k_3 \in M$ with $k_1 < k_2 < k_3$. Put $p_i(x) = k_i + x$ for each $i \in \{1, 2, 3\}$ and each $x \in M$. Then $(p_1, p_2, p_3) \in \overline{C}$, whence $\overline{C} \neq \emptyset$. Thus the condition (i) from 2.2 holds.

We have to verify that the condition (ii) from 2.2 is valid. By way of contradiction, assume that this condition fails to hold.

Hence there are $\varphi_i \in P(M)$ $(i = 1, 2, 3)$ such that

$$(\varphi_1, \varphi_2, \varphi_3) \in \overline{C},\ \{\varphi_1, \varphi_2, \varphi_3\} \cap P(M)(+) \neq \emptyset \neq \{\varphi_1, \varphi_2, \varphi_3\} \cap P(M)(-).$$

It is easy to verify that without loss of generality we can assume that

$$\varphi_1, \varphi_2 \in P(M)(+), \quad \varphi_3 \in P(M)(-).$$

Thus there are $k_1, k_2, k_3 \in M$ such that

$$\varphi_1(x) = k_1 + x, \quad \varphi_2(x) = k_2 + x, \quad \varphi_3(x) = k_3 - x$$

for each $x \in M$.

Then we have $(\varphi_1(0), \varphi_2(0), \varphi_3(0)) \in C$, hence one of the conditions

$$k_1 < k_2 < k_3, \quad k_2 < k_3 < k_1, \quad k_3 < k_1 < k_2$$

is satisfied.

a) First suppose that $k_1 < k_2 < k_3$. If $k_3 - k_1$ is even, then there exists $t \in M$ such that $k_3 - t = k_1 + t$, yielding that $\varphi_3(t) = \varphi_1(t)$. Then $\big(\varphi_1(t), \varphi_2(t), \varphi_3(t)\big)$ does not belong to C, which is impossible. Thus $k_3 - k_1$ is odd. In the same way we obtain that $k_3 - k_2$ is odd.

Therefore $k_1 + 1 < k_2$ and $k_3 - (k_1 + 1)$ is even. Hence there is $0 \neq t \in M$ with $2t = k_3 - (k_1 + 1)$. Thus

$$\varphi_3(t) = k_3 - t = k_1 + 1 + t < k_2 + t = \varphi_2(t),$$
$$\varphi_1(t) < \varphi_3(t).$$

Then we cannot have $(\varphi_1, \varphi_2, \varphi_3) \in \overline{C}$, which is a contradiction.

b) Further, suppose that $k_3 < k_1 < k_2$. Assume that $k_1 - k_3$ is even. Hence there is $0 \neq z \in M$ with $k_1 - k_3 = 2z$. Put $z_1 = -z$. (Recall that the operation $-$ is taken $\bmod n$, whence $z_1 = n - z$.) We have $k_1 - k_3 = -2z_1$, yielding that $k_1 + z_1 = k_3 - z_1$, thus $\varphi_1(z_1) = \varphi_3(z_1)$, which is impossible. Therefore $k_1 - k_2$ is odd.

Hence $k_1 - k_3 + 1$ is even and there exists $t_1 \in M$ such that $k_1 - k_3 + 1 = 2t_1$. Thus

$$k_1 - t_1 + 1 = k_3 + t_1.$$

Put $t = -t_1$. Hence $k_1 + t + 1 = k_3 - t$, therefore

$$\varphi_1(t) + 1 = \varphi_3(t). \tag{1}$$

From $(\varphi_1, \varphi_2, \varphi_3) \in \overline{C}$ we conclude that $\big(\varphi_1(t), \varphi_2(t), \varphi_3(t)\big) \in C$ is valid, hence one of the relations

$$\varphi_1(t) < \varphi_2(t) < \varphi_3(t), \quad \varphi_2(t) < \varphi_3(t) < \varphi_1(t), \quad \varphi_3(t) < \varphi_1(t) < \varphi_2(t)$$

must hold. In view of (1), all these relations fail to be satisfied. In this way we have arrived at a contradiction.

c) Finally, suppose that $k_2 < k_3 < k_1$. Similarly as in the previous cases we verify that $k_3 - k_2$ must be odd. Hence there is $t \in M$, $t \neq 0$ such that

$$k_2 \leqslant k_3 - t < k_2 + t \leqslant k_3,$$
$$(*) \qquad (k_2 + t) - (k_3 - t) = 1.$$

Hence $\varphi_3(t) < \varphi_2(t)$.

Further, from the above relations we conclude that either

$$(\alpha) \qquad k_1 + t > k_1$$

or

$$(\beta) \qquad k_1 + t < k_2 + t.$$

If (α) is valid, then $\varphi_2(t) < \varphi_1(t)$, whence $(\varphi_1, \varphi_2, \varphi_3) \notin \overline{C}$, which is impossible. Suppose that (β) holds. We have $\varphi_1(t) \neq \varphi_3(t)$, hence in view of $(*)$ we obtain

$$k_1 + t < k_3 - t,$$

thus $\big(\varphi_1(t), \varphi_3(t), \varphi_2(t)\big) \in C$, yielding that the relation $(\varphi_1, \varphi_2, \varphi_3) \in \overline{C}$ cannot hold. Therefore the relation $k_2 < k_3 < k_1$ cannot be valid. □

If M is an infinite ℓ-cyclically ordered set, then we can ask whether the assertion analogous to 2.3 is valid for M.

2.4. Example. Let $\mathbb{Z}$ be the set of all integers with the natural linear order. We define the ℓ-cyclic order on $\mathbb{Z}$ as in 1.5.

We can apply for $\mathbb{Z}$ the same steps as in the proof of 2.3 with the distinctions that

α_1) the operations $+$ and $-$ are now not taken $\bmod n$, but they have the usual meaning;

α_2) part c) of the proof of 2.3 is now to be modified in the sense that only the condition (α) is taken into account; the condition (β) cannot be valid in the present case.

Therefore we have:

$(**)$ If $G = (P(\mathbb{Z}), \cdot, \overline{C})$, then G is a half cyclically ordered group.

2.5. Example. Let $\mathbb{N}$ be the set of all positive integers. The ℓ-cyclic order on $\mathbb{N}$ is defined similarly as in 2.4. Then the only element of $P(\mathbb{N})$ is the identical permutation. Hence $\overline{C}$ is the empty set. Thus we obtain:

If $G = (P(\mathbb{N}), \cdot, \overline{C})$, then G fails to be a half cyclically ordered group.

The following question remains open: Let M be an infinite ℓ-cyclically ordered set such that (under the notation as above) the set $\overline{C}$ is nonempty. Must $(P(M), \cdot, \overline{C})$ be a half cyclically ordered group? In other words: must the condition (ii) from 2.2 be satisfied?

3. On half ℓc-groups

In this section we assume that G is an ℓc-group such that $G{\downarrow} \neq \emptyset$.

3.1. Lemma. *Let $y \in G{\downarrow}$. Then $y^4 \neq e \Rightarrow y^2 = e$.*

P r o o f. Let $y^4 \neq e$. By way of contradiction, assume that $y^2 \neq e$. Hence $y^{-1} \neq y$ and clearly $y^{-1} \in G{\downarrow}$. Moreover, the relation $y^4 \neq e$ yields $y^2 \neq y^{-2}$. We also have $y^{-2} \neq e$. Since $G{\uparrow}$ is an ℓ-cyclically ordered set we get that either (i) $(y^{-2}, e, y^2) \in C$, or (ii) $(y^2, e, y^{-2}) \in C$.

Assume that (i) is valid. Then

$$(y^{-2} \cdot y, y, y^2 \cdot y) \in C \quad \text{and} \quad (y \cdot y^2, y, y \cdot y^{-2}) \in C,$$

i.e., $(y^{-1}, y, y^3) \in C$ and $(y^3, y, y^{-1}) \in C$, which is a contradiction.

If (ii) is valid, then we proceed analogously. □

In particular, if $G{\uparrow}$ is linearly ordered, then for each $x \in G{\uparrow}$ with $x \neq e$ we have $x^2 \neq e$. Since $y \in G{\downarrow}$ implies that $y^2 \in G{\uparrow}$, in view of 3.1 we obtain

3.2. Corollary (cf. [5]). *If G is a half linearly ordered group and $y \in G{\downarrow}$, then $y^2 = e$.*

3.3. Example. Let K be the set of all reals x with $0 \leqslant x < 1$ with the natural linear order. We define the ℓ-cyclic order on K as in 1.5. We consider the group operation on K which is defined to be addition mod 1. Then K turns out to be an ℓc-group. For each subgroup K_1 we take into account the induced ℓ-cyclic order.

Let A be an ℓc-group with the ℓ-cyclic order C_1. Further, let B be a linearly ordered group; the corresponding ℓ-cyclic order on B (cf. 1.5) will be denoted by C_2. Next, let $A \times B$ be the cartesian product of the sets A and B with the group operation defined componentwise. For $(a_i, b_i) \in A \times B$ $(i = 1, 2, 3)$ we put

$$\big((a_1, b_1), (a_2, b_2), (a_3, b_3)\big) \in C$$

if some of the following conditions is valid:

(i) $(a_1, a_2, a_3) \in C_1$;

(ii) $a_1 = a_2 \neq a_3$ and $b_1 < b_2$;

(iii) $a_2 = a_3 \neq a_1$ and $b_2 < b_3$;

(iv) $a_3 = a_1 \neq a_2$ and $b_3 < b_1$;

(v) $a_1 = a_2 = a_3$ and $(b_1, b_2, b_3) \in C_2$.

Then $A \times B$ turns out to be an ℓc-group which will be denoted by $A \otimes B$.

3.4. Theorem (Swierczkowski [18]). *Let H be an ℓc-group and let K be as in 3.3. Then there exist an ℓc-subgroup A of K and a linearly ordered group B such that the ℓc-group H is isomorphic to the ℓc-group $A \otimes B$.*

3.5. Lemma. *Let A and B be as in 3.4, $H = A \otimes B$, $e \neq h \in H$, where e is the neutral element of A; the neutral elements of A and of B will be denoted by e_A or by e_B, respectively. The following conditions are equivalent:*

(i) *There is $b \in B$ with $b \neq e_B$ such that $h = (e_A, b)$.*

(ii) *Either*

(ii$_1$) $(e, h^{n_1}, h^{n_2}) \in C$ *for any* $n_1, n_2 \in \mathbb{N}$ *with* $n_1 < n_2$,

or

(ii$_2$) $(h^{n_2}, h^{n_1}, e) \in C$ *for any* $n_1, n_2 \in \mathbb{N}$ *with* $n_1 < n_2$.

Proof. Let (i) be valid. In view of the assumption, B is linearly ordered. If $e_B < b$, then (ii$_1$) holds. If $b < e_B$, then (ii$_2$) is satisfied.

Suppose that (i) fails to hold. Then we have $h = (a, b)$, $a \neq e_A$ (since the group operation in A is addition mod 1, we apply for this operation the additive notation).

a) Assume that there is $n \in \mathbb{N}$ with $na = e_A$. Hence $n > 1$.

Let $b = e_B$. Then $h^n = h^{2n} = e$, thus neither (e, h^n, h^{2n}) nor (h^{2n}, h^n, e) belong to C. Thus (ii) does not hold.

Suppose that $b > e_B$. Then we have

$$(e, h^n, h) \in C, \quad (e, h^n, h^{2n}) \in C.$$

Thus neither (ii$_1$) nor (ii$_2$) are satisfied and hence (ii) fails to hold. Similarly, if $b < e_B$, then (ii) is not valid.

b) Assume that $na \neq e_A$ for each $n \in \mathbb{N}$. Then $n_1 a \neq n_2 a$ whenever n_1 and n_2 are distinct elements of $\mathbb{N}$.

There exists the least $n \in \mathbb{N}$ with

$$\big((n+1)a, e_A, na\big) \in C_1.$$

Then we have

$$\big(e_A, (n+1)a, (n+2)a\big) \in C_1.$$

This yields

$$(h^{n+1}, e, h^n) \in C, \quad (e, h^{n+1}, h^{n+2}) \in C.$$

Therefore neither (ii$_1$) nor (ii$_2$) are valid. □

Now let G be a half ℓc-group. According to 3.4, without loss of generality we can assume that $G{\uparrow} = H$, where H is as in 3.4. Denote

$$H_1 = \{(a,b) \in H\colon\ a = 0\}.$$

Then H_1 is a normal subgroup of H.

Let $y \in G{\downarrow}$. If an element h of H satisfies the condition (ii$_1$), then the element yhy^{-1} satisfies the condition (ii$_2$); similarly, if h fulfils (ii$_2$), then (ii$_1$) holds for yhy^{-1}. Thus we have

3.6. Lemma. *H_1 is a normal subgroup of G.*

3.7. Lemma. *Let $y \in G{\downarrow}$, $y^2 = e$. Put $T = H_1 \cup H_1 y$. Then T is a subgroup of G.*

Proof. It suffices to apply the same steps as in the proof of Lemma 2.4 in [12]. □

3.8. Lemma. *Let y and T be as in 3.7 and let $\operatorname{card} H_1 > 1$. Then H_1 is a half ℓc-subgroup of G; moreover, T is a half linearly ordered group.*

Proof. From the relation $\operatorname{card} H_1 > 1$ and from 3.7 we infer that T is a half ℓc-subgroup of G. We have $T{\uparrow} = H_1$ and H_1 is linearly ordered; hence T is a half linearly ordered group. □

3.9. Lemma. *Let the assumptions of 3.8 be valid. Let $G{\downarrow} \neq \emptyset$. Then the group H_1 is abelian.*

Proof. This is a consequence of 3.8 and of [5], Proposition I.2.2. □

Proof of (A). (The statement (A) has been formulated in Introduction.)

Suppose that the assumptions of (A) are satisfied. Then we have $y^2 = e$ for each $y \in G{\downarrow}$. Thus from 3.9 we conclude that the group H_1 is abelian. Hence $H = A \otimes B$ is abelian, because B is isomorphic to H_1. Since $H = G{\uparrow}$, the group $G{\uparrow}$ is abelian. □

For a related result concerning half lattice ordered groups cf. [9].

In this section we introduce the notion of the lexicographic product of half cyclically ordered groups. The method is analogous to that of [11].

Let I be a linearly ordered set and for each $i \in I$ let G_i be a half cyclically ordered group such that there exists $y^{(i)} \in G_i\!\downarrow$ with $(y^{(i)})^2 = e$. We consider the element $y^{(i)}$ to be fixed. The relation of cyclic order on G_i is denoted by C_i.

Let G^1 be the cartesian product of the groups G_i $(i \in I)$. For $g \in G^1$ and $i \in I$ we denote by g_i the component of g in G_i. Next, let $y^{(1)}$ be the element of G^1 such that $(y^{(1)})_i = y^{(i)}$ for each $i \in I$.

If g and g' are elements of G^1, then we put

$$I(g, g') = \{i \in I \colon\ g_i \neq g'_i\}.$$

We denote by X the set of all $g \in G^1$ such that $g_i \in G_i\!\uparrow$ for each $i \in I$ and the set $I(g, e)$ is either empty or well-ordered.

Further, let $Y(y^{(1)})$ be the set of all $g \in G^1$ such that (i) $g_i \in G_i\!\downarrow$ for each $i \in I$, and (ii) either $g = y^{(1)}$ or the set $I(g, y^{(1)})$ is well-ordered. Then both the sets X and $Y(y^{(1)})$ are nonempty. Put

$$G^0(y^{(1)}) = X \cup Y(y^{(1)}).$$

By a method analogous to that in the proof of 2.1 in [11] we can verify

4.1. Lemma. *$G^0(y^{(1)})$ is a subgroup of the group G^1.*

4.2. Notation. We denote by C the set of all triples (a, b, c) of elements of $G^0(y^{(1)})$ such that there exists $i_1 \in I$ which has the following properties:

(i) $(a_{i_1}, b_{i_1}, c_{i_1}) \in C_{i_1}$;

(ii) if $i \in I$ and $i < i_1$, then $a_i = b_i = c_i$.

4.3. Lemma. *The group $G^0(y^{(1)})$ with the relation C is a half-cyclically ordered group.*

P r o o f. We have to verify that the conditions 1)–4) from 1.6 are valid.

1) Let $i \in I$. There exist $h^{(1)}$, $h^{(2)}$ and $h^{(3)}$ in $G_i\!\uparrow$ such that $(h^{(1)}, h^{(2)}, h^{(3)}) \in C_i$. For $j \in \{1, 2, 3\}$ let $\overline{h}^{(j)}$ be the element of $G^0(y^{(1)})$ such that for each $k \in I$,

$$\overline{h}^{(j)}_k = \begin{cases} h^{(j)} & \text{if } k = i, \\ e & \text{if } k \in I \setminus \{i\}. \end{cases}$$

Then $\overline{h}^{(j)} \in G^0(y^{(1)})$ for each $j = 1, 2, 3$. Moreover, according to the definition of C, the relation $(\overline{h}^{(1)}, \overline{h}^{(2)}, \overline{h}^{(3)}) \in C$ is valid. Hence $C \neq \emptyset$.

2) The validity of the condition 2) in 1.6 is an immediate consequence of the definition of C.

3) We have $G^0(y^{(1)})\uparrow = X$, $G^0(y^{(1)})\downarrow = Y(y^{(1)})$, whence the condition 3) of 1.6 is satisfied.

4) From 4.2 we conclude that the condition 4) of 1.6 holds. □

Let $y^{(2)}$ be an element of G^1 such that $y_i^{(2)} \in G_i\downarrow$ and $(y_i^{(2)})^2 = e$ for each $i \in I$. Then we can construct the half cyclically ordered group $G^0(y^{(2)})$ in the same way as we did above for $G^0(y^{(1)})$.

For each $b^{(1)} \in Y(y^{(1)})$ there is a uniquely determined element $a^{(1)} \in X$ such that

$$a^{(1)}y^{(1)} = b^{(1)}.$$

Let us put $\varphi(b^{(1)}) = a^{(1)}y^{(2)}$. Further, for each $a \in X$ we set $\varphi(a) = a$.

4.4. Proposition. *The mapping φ is an isomorphism of the half cyclically ordered group $G^0(y^{(1)})$ onto the half cyclically ordered group $G^0(y^{(2)})$.*

P r o o f. The same method as in the proof of 2.3 in [11] yields that φ is an isomorphism of the group $G^0(y^{(1)})$ onto the group $G^0(y^{(2)})$.

Let $C^{(2)}$ be the corresponding cyclic order on $G^0(y^{(2)})$. The definitions of C and $C^{(2)}$ imply

$$(x, y, z) \in C \Leftrightarrow \big(\varphi(x), \varphi(y), \varphi(z)\big) \in C^{(2)}.$$

□

In what follows we write G^0 and Y instead of $G^0(y^{(1)})$ or $Y(y^{(1)})$, respectively.

Under the assumptions as above we denote

$$G^0 = \Gamma_{i \in I} G_i;$$

G^0 is said to be the lexicographic product of half cyclically ordered groups G_i, and the structures G_i's are called lexicographic factors of G^0.

If the role of $y^{(1)}$ is to be emphasized, then we write also

$$G^0 = (y^{(1)})\Gamma_{i \in I} G_i.$$

Let G be a half cyclically ordered group and let

$$\varphi\colon\ G \longrightarrow \Gamma_{i \in I} G_i \tag{1}$$

be an isomorphism of G onto G^0. Then (1) is called a lexicographic product decomposition of G. For $g \in G$ and $i \in I$ we denote $g_i = (\varphi(g))_i$.

5. Condition (C_0)

Again, let G be a half cyclically ordered group with $G{\downarrow} \neq \emptyset$.

For the definition of the lexicographic product decomposition of a cyclically ordered group H cf. [10]. Below we will apply this definition to the case when $H = G{\uparrow}$.

We want to investigate the relations between the lexicographic product decompositions of the half cyclically ordered group G and the lexicographic product decompositions of the cyclically ordered group $G{\uparrow}$.

Let the element $y^{(1)}$ be as in Section 4.

As an immediate consequence of the definition of the lexicographic product decomposition of G we obtain

5.1. Lemma. *Let the relation* (1) *from Section* 4 *be valid. For each* $g \in G{\uparrow}$ *we put* $\varphi^0(g) = \varphi(g)$. *Then the relation*

$$\varphi^0\colon\ G{\uparrow} \to \prod_{i\in I} G_i{\uparrow} \tag{1'}$$

is a lexicographic product decomposition of the cyclically ordered group $G{\uparrow}$.

We say that the lexicographic product decomposition (1') is generated by (1). In such case we also say that the lexicographic product decomposition φ^0 of $G{\uparrow}$ can be extended onto G.

Under the assumption as in (1') let I_1 be a nonempty subset of I and $H = G{\uparrow}$. We put

$$H(I_1) = \{h \in H\colon\ h_j = e \quad \text{for each } j \in I \setminus I_1\}.$$

Further, for $i \in I$ we set $\overline{H}_i = H(I_1)$, where $I_1 = \{i_1\}$.

Let $i \in I$, and let $h^{(1)}$ be any element of $\overline{H}_i$. We put $\varphi_i(h^{(1)}) = \overline{h}^{(1)}$, where $\overline{h}^{(1)}$ is as in Section 4.

Then we obviously have

5.2. Lemma. $H(I_1)$ *is a subgroup of the group* H. *The mapping* φ_i *is an isomorphism of the cyclically ordered group* $\overline{H}_i$ *onto* $G_i{\uparrow}$.

5.3. Lemma. *Let* $\emptyset \neq I_1 \subseteq I$. *Assume that* (1) *holds. Then*

$$y^{(1)} H(I_1) y^{(1)} = H(I_1).$$

P r o o f. Let $z \in y^{(1)}H(I_1)y^{(1)}$. There is $t \in H(I_1)$ with $z = y^{(1)}ty^{(1)}$. Let $j \in I \setminus I_1$. Then $t_j = e$. Thus

$$z_j = y_j^{(1)} e y_j^{(1)} = (y_j^{(1)})^2 = e.$$

Therefore

$$(*) \qquad y^{(1)}H(I_1)y^{(1)} \subseteq H(I_1).$$

The relation $(*)$ yields

$$H(I_1) = (y^{(1)})^2 H(I_1)(y^{(1)})^2 \subseteq y^{(1)}H(I_1)y^{(1)}.$$

□

5.4. Corollary. *For each $i \in I$, $y^{(1)}\overline{H}_i y^{(1)} = \overline{H}_i$.*

Now let us assume that the cyclically ordered group $G\uparrow = H$ is represented as

$$(\beta_1) \qquad \varphi\colon H \to \Gamma_{i\in I} H_i.$$

For $\emptyset \neq I_1 \subseteq I$ let $H(I_1)$ be defined analogously as above. Consider the following condition (C_0) concerning the lexicographic product decomposition (β_1) of H:

(C_0) There exists $y^{(1)} \in G\downarrow$ such that $(y^{(1)})^2 = e$ and

$$y^{(1)}H(I_1)y^{(1)} = H(I_1)$$

whenever $\emptyset \neq I_1 \subseteq I$.

According to 5.3 we have

5.5. Lemma. *Assume that there exists a lexicographic product decomposition α_1 of G such that β_1 is generated by α_1. Then β_1 satisfies the condition (C_0).*

5.6. Lemma. *Let H_1 be a subgroup of the group $G\uparrow$ and let y be an element of $G\downarrow$ such that $y^2 = e$ and $yH_1 = H_1y$. Then $H_1 \cup H_1y$ is a subgroup of G.*

P r o o f. Let $h_1, h_2 \in H_1$. Then $h_1h_2 \in H_1$ and $h_1^{-1} \in H_1$. Further, there exists $h_2' \in H_1$ such that $yh_2 = h_2'y$. Thus

$$(h_1y)h_2 = h_1h_2'y \in H_1y, \quad h_2(h_1y) = (h_2h_1)y \in H_1y,$$
$$(h_1y)(h_2y) = h_1h_2'y^2 = h_1h_2' \in H_1.$$

Also, $(h_1y)^{-1} = yh_1^{-1} = h''y$ for some $y'' \in H_1$. □

Suppose that the condition (C_0) is valid for (β_1). We need some auxiliary results.

For each $i \in I$ we put

$$K_i = \overline{H}_i \cup \overline{H}_i y^{(1)},$$

where $y^{(1)}$ is as in (C_0).

In view of (C_0) and according to 5.6 we conclude that K_i is a subgroup of G. Moreover, under the induced cyclic order, K_i is a half cyclically ordered group with

$$K_i\!\uparrow = \overline{H}_i, \quad K_i\!\downarrow = \overline{H}_i y^{(1)}.$$

We define a mapping ψ_i of G into K_i as follows.

a) Let $g \in G\!\uparrow$. There exists $\overline{h}_i \in \overline{H}_i$ (under consideration of (β_1)) such that

$$(\overline{h}_i)_i = g_i.$$

We put $\psi_i(g) = \overline{h}_i$.

In fact, if we take β_1 instead of (1'), then under the notation as above we have

$$\psi_i(g) = (\varphi(g))_i.$$

b) Further, let $g' \in G\!\downarrow$. There exists a uniquely defined element $g \in G\!\uparrow$ with $g' = gy^{(1)}$. Then (under the notation as in a)) we set $\psi_i(gy^{(1)}) = \overline{h}_i y^{(1)}$.

5.7. Lemma. *For each $g \in G$, $\psi_i(gy^{(1)}) = \psi_i(g)y^{(1)}$.*

P r o o f. For $g \in G\!\uparrow$, this is a consequence of the definition of ψ_i. Let $g \in G\!\downarrow$. There is $g_1 \in G\!\uparrow$ with $g = g_1 y^{(1)}$. Then

$$\psi_i(gy^{(1)}) = \psi_i(g_1(y^{(1)})^2) = \psi_i(g_1),$$
$$\psi_i(g)y^{(1)} = \psi_i(g_1 y^{(1)})y^{(1)} = \psi_i(g_1)(y^{(1)})^2 = \psi_i(g_1).$$

□

5.8. Lemma. *Let $g \in G\!\uparrow$, $y^{(1)}gy^{(1)} = z$. Then $\psi_i(z) = y^{(1)}\psi_i(g)y^{(1)}$.*

P r o o f. In view of (C_0) we have

$$y^{(1)}\psi_i(g)y^{(1)} \in \overline{H}_i.$$

There exists $t \in G\!\uparrow$ such that

$$t_j = \begin{cases} e & \text{if } j = i, \\ g_j & \text{if } j \in I \setminus \{i\} \end{cases}$$

Hence $g = \overline{h}_i t$, where $\overline{h}_i = \psi_i(g)$. We have

$$z = y^{(1)}\overline{h}_i t y^{(1)} = (y^{(1)}\overline{h}_i y^{(1)})(y^{(1)} t y^{(1)}).$$

From the definition of t and from (C_0) we conclude that

$$(y^{(1)} t y^{(1)})_i = e,$$

whence

$$z_i = (y^{(1)}\overline{h}_i y^{(1)})_i = (y^{(1)}\psi_i(g) y^{(1)})_i.$$

Therefore $\psi_i(z) = y^{(1)}\psi_i(g)y^{(1)}$. □

5.9. Lemma. *ψ_i is a mapping of G onto K_i. Moreover, ψ_i is a homomorphism with respect to the group operation and $\psi_i(K_i) = K_i$ for each $i \in I$.*

Proof. Let h_i^0 be any element of $\overline{H}_i$. Then

$$\psi_i(\overline{h}_i^0) = \overline{h}_i^0, \quad \psi_i(h_i^0 y^{(1)}) = \overline{h}_i^0 y^{(1)}.$$

Thus ψ_i is surjective. Also $\psi_i(K_i) = K_i$ for each $i \in I$.

Let $g_1, g_2 \in G$, $g_1 g_2 = g$. We distinguish the following cases.

a$_1$) Let $g_1, g_2 \in G\uparrow$. Then under notation analogous to a) we have

$$\psi_i(g) = \overline{h}_i = \overline{h}_{1i}\overline{h}_{2i} = \psi_i(g_1)\psi_i(g_2).$$

a$_2$) Let $g_1 \in G\uparrow$, $g_2 \in G\downarrow$. There is $g_3 \in G\uparrow$ with $g_2 = g_3 y^{(1)}$. Then in view of 5.7 and a) we get

$$\psi_i(g) = \psi_i(g_1 g_3 y^{(1)}) = \psi_i(g_1 g_3) y^{(1)} = \psi_i(g_1)\psi_i(g_3) y^{(1)}$$
$$= \psi_i(g_1)\psi_i(g_3 y^{(1)}) = \psi_i(g_1)\psi_i(g_2).$$

a$_3$) Let $g_1 \in G\downarrow$, $g_2 \in G\uparrow$. There is $g_3 \in G\uparrow$ with $g_1 = g_3 y^{(1)}$. Further, there is $z \in G\uparrow$ such that $y^{(1)} g_2 = z y^{(1)}$. Then $y^{(1)} z y^{(1)} = g_2$, whence according to 5.8,

$$y^{(1)}\psi_i(z) y^{(1)} = \psi_i(g_2).$$

By applying 5.7 and a) we obtain

$$\psi_i(g) = \psi_i(g_3 y^{(1)} g_2) = \psi_i(g_3 z y^{(1)}) = \psi_i(g_3)\psi_i(z) y^{(1)}$$
$$= (\psi_i(g_3) y^{(1)})(y^{(1)}\psi_i(z) y^{(1)}) = \psi_i(g_3 y^{(1)})\psi_i(g_2) = \psi_i(g_1)\psi_i(g_2).$$

a$_4$) Let $g_1 \in G{\downarrow}$, $g_2 \in G{\downarrow}$. There is $g_3 \in G{\uparrow}$ with $g_2 = g_3 y^{(1)}$. Then a$_3$) and 2.7 yield

$$\psi_i(g) = \psi_i(g_1 g_3 y^{(1)}) = \psi_i(g_1 g_3) y^{(1)} = \psi_i(g_1)\psi_i(g_3) y^{(1)}$$
$$= \psi_i(g_1)\psi_i(g_3 y^{(1)}) = \psi_i(g_1)\psi_i(g_2).$$

□

We denote by K^0 the cartesian product of the groups K_i $(i \in I)$. For each $g \in G$ we put

$$\psi(g) = (\psi_i(g))_{i \in I}.$$

Thus ψ is a mapping of G into K^0; moreover, in view of 5.9, ψ is a homomorphism with respect to the group operation.

Denote $\psi(y^{(1)}) = y^{(01)}$.

5.10. Lemma. $(y^{(01)})^2 = e$.

Proof. We have to verify that $\psi_i((y^{(1)})^2) = e$ for each $i \in I$. Lemma 5.7 yields

$$\psi_i(y^{(1)}) = \psi_i(e y^{(1)}) = \psi_i(e) y^{(1)} = y^{(1)},$$
$$\psi_i((y^{(01)})^2) = \psi_i(y^{(1)} y^{(1)}) = \psi_i(y^{(1)}) y^{(1)} = y^{(1)} y^{(1)} = e.$$

□

According to 5.10 and in view of the definition of the lexicographic product of half cyclically ordered groups we can construct the lexicographic product

(2) $$K = (y^{(01)})\Gamma_{i \in I} K_i.$$

Then K is a subgroup of K^0.

The relation of cyclic order in K will be denoted by C_K.

We have already remarked above that $K_i{\uparrow} = \overline{H}_i$, $K_i{\downarrow} = \overline{H}_i y^{(1)}$ for each $i \in I$. From these relations and from the construction of K we conclude

5.11. Lemma.

(i) *$K{\uparrow}$ is the set of all elements k of K such that $k_i \in \overline{H}_i$ for each $i \in I$.*

(ii) *$K{\downarrow}$ is the set of all elements k' of K such that $k'_i \in \overline{H}_i y^{(1)}$ for each $i \in I$.*

5.12. Lemma. $\psi(G{\uparrow}) = K{\uparrow}$ *and* $\psi(G{\downarrow}) = K{\downarrow}$.

P r o o f. Let $g \in G{\uparrow}$. If $i \in I$, then in view of the definition of ψ_i we have $\psi_i(g) = \overline{h}_i$, $(\overline{h}_i)_i = g_i$ (under the notation as above). Moreover, according to (1') the set

$$I(g, e) = \{i \in I\colon\ g_i \neq e\}$$

is either empty or well-ordered. Since

$$(*) \qquad g_i \neq e \Leftrightarrow \overline{h}_i \neq e$$

we infer that $\psi(g)$ belongs to K and then, clearly, $\psi(g) \in K{\uparrow}$. By analogous steps we verify that if $\psi(g)$ is an element of $K{\uparrow}$, then g must belong to $G{\uparrow}$. Hence

$$\psi(G{\uparrow}) = K{\uparrow}.$$

From this relation and from

$$G{\downarrow} = G{\uparrow}y^{(1)}, \quad K{\downarrow} = K{\uparrow}y^{(1)}$$

we obtain (by applying 5.9 and 5.10) the relation $\psi(G{\downarrow}) = K{\downarrow}$. □

From 5.12 and from the relation $(*)$ we conclude

5.13. Lemma. *Let $g_1, g_2, g_3 \in G{\uparrow}$. Then*

$$(+) \qquad (g_1, g_2, g_3) \in C \Leftrightarrow \big(\psi(g_1), \psi(g_2), \psi(g_3)\big) \in C_K.$$

5.14. Lemma. *Let $g_1, g_2, g_3 \in G{\downarrow}$. Then the relation $(+)$ is valid.*

P r o o f. We have

$$(g_1, g_2, g_3) \in C \Leftrightarrow (g_1 y^{(1)}, g_2 y^{(1)}, g_3 y^{(1)}) \in C.$$

In view of 5.13,

$$(g_1 y^{(1)}, g_2 y^{(1)}, g_3 y^{(1)}) \in C \Leftrightarrow \big(\psi(g_1 y^{(1)}), \psi(g_2 y^{(1)}), \psi(g_3 y^{(1)})\big) \in C_K.$$

According to 5.7

$$\psi(g_i y^{(1)}) = \psi(g_i) y^{(01)} \quad (i = 1, 2, 3).$$

Therefore

$$\big(\psi(g_1 y^{(1)}), \psi(g_2 y^{(1)}), \psi(g_3 y^{(1)})\big) \in C_K$$
$$\Leftrightarrow \big(\psi(g_1) y^{(01)}, \psi(g_2) y^{(01)}, \psi(g_3) y^{(01)}\big) \in C_K \Leftrightarrow \big(\psi(g_1), \psi(g_2), \psi(g_3)\big) \in C_K.$$

□

5.15. Lemma. *ψ is an isomorphism with respect to the group operation.*

Proof. We have already remarked above that ψ is a homomorphism with respect to the group operation. Let $g \in G$, $\psi(g) = e$. Then in view of 5.12, $g \in G\uparrow$. We have $\psi_i(g) = e$ for each $i \in I$, whence $g_i = e$ for each $i \in I$, yielding that $g = e$. □

From 5.11–5.15 we conclude

5.16. Lemma. *ψ is an isomorphism of the half cyclically ordered group G onto the half cyclically ordered group K.*

From (2) we obtain

(2') $$K\uparrow = \Gamma_{i\in I} K_i.$$

The lexicographic product decomposition (2') is generated by (2). Thus (2') can be extended onto K.

Then in view of 5.15 and of the fact that ψ is constructed by means of the mappings ψ_i $(i \in I)$ (cf. also 5.12 and 5.9) we conclude

5.17. Lemma. *The lexicographic product β_1 of $G\uparrow$ can be extended onto G.*

5.18. Theorem. *Let G be a half cyclically ordered group with $G\downarrow \neq \emptyset$ and let β_1 be a lexicographic product decomposition of $G\uparrow$. Then the following conditions are equivalent:*

(i) *There exists a lexicographic product decomposition α_1 of G such that β_1 is generated by α_1.*

(ii) *β_1 satisfies the condition* (C_0).

Proof. This is a consequence of 5.5 and 5.17. □

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