Higher Structures

Mathieu Anel; Reid Barton
Choice Axioms and Postnikov Completeness

Higher Structures, Vol. 10 (2026), No. 1, 22–39

Persistent URL: http://dml.cz/dmlcz/153754

Terms of use:

© Institute of Mathematics, Czech Academy of Sciences, 2026

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Higher Structures 10(1):22–39, 2026.

HIGHER STRUCTURES

Choice axioms and Postnikov completeness

Mathieu Anel[a] **and Reid Barton**[a]

[a]*Carnegie Mellon University, 5000 Forbes Ave, 15213 PA, Pittsburgh, USA*

Abstract

We introduce homotopical variants of the axioms of countable and dependent choice for ∞-topoi and use them to give criteria for Postnikov completeness, revisiting a result of Mondal and Reinecke.

Communicated by: Denis-Charles Cisinski.
Received: 21st April, 2024. Accepted: 24th December, 2025.
MSC: 18N60; 03G30; 18G20.
Keywords: Choice axioms, Homotopical dimension, Postnikov towers, ∞-Topos.

1. Introduction

The axiom of choice classically says that every surjection between sets admits a section. An equivalent formulation says that a product of surjections between sets is again a surjection. When the surjections are assumed to be indexed by a countable set, one says the axiom of countable choice holds. This condition can be formulated in any category with products and a good notion of surjection, for example in a Grothendieck 1-topos or ∞-topos. An example of a 1-topos that does *not* satisfy countable choice is the topos of sheaves of sets on $[0,1] \subseteq \mathbb{R}$. However, by considering the ∞-topos of sheaves of spaces on $[0,1]$, we can formulate a weaker condition that does hold: a countable product of *connected* maps is a surjection.

More generally, we say that *countable choice of dimension* $\leq d$ (CC_d) holds in an n-topos $\mathcal{E}$ ($1 \leq n \leq \infty$) if countable products of $(d-1)$-connected maps are surjections in $\mathcal{E}$. We always have $\mathsf{CC}_d \Rightarrow \mathsf{CC}_{d+1}$; the strongest axiom CC_0 is the usual axiom of countable choice. For each $d \geq 1$, the ∞-topos of sheaves on $[0,1]^d$ satisfies CC_d but not CC_{d-1} (Example 2.26). It is also possible for CC_d to fail for every d, even $d = \infty$ (Proposition 4.2).

In connection with these axioms, we prove the following results.

Theorem 1.1. *If $\mathcal{E}$ is an ∞-topos where CC_d holds for some $d < \infty$, then the hypercompletion and the Postnikov completion of $\mathcal{E}$ coincide.*

Email addresses: mathieu.anel@protonmail.com (Anel)
barton2@andrew.cmu.edu (Barton)

© Anel and Barton, 2026, under a Creative Commons Attribution 4.0 International License.

DOI: 10.21136/HS.2026.02

Theorem 1.2. *If $\mathcal{E}_1$ is a 1-topos where CC_0 holds, then CC_0 also holds in its ∞-topos envelope $\mathcal{E}$.*

Theorem 1.1 says concretely that every formal Postnikov tower in $\mathcal{E}$ is the Postnikov tower of its limit (see Lemma A.5). Theorem 1.2 implies that this holds in the ∞-topos envelope of a 1-topos $\mathcal{E}_1$ with CC_0. In [15], answering a question of [3], the authors prove this last fact under the stronger condition that $\mathcal{E}_1$ is "replete", meaning that the composition of a countable tower of surjections is a surjection. This property is also known as the axiom of *dependent choice*. We say that the axiom of *dependent choice of dimension* $\leq d$ (DC_d) holds in an n-topos if the composition of a countable tower of $(d-1)$-connected maps is a surjection (see Definition 2.33). We always have $\mathsf{DC}_d \Rightarrow \mathsf{DC}_{d+1}$ and $\mathsf{DC}_d \Rightarrow \mathsf{CC}_d$. The strongest axiom DC_0 is the repleteness condition above. The relation $\mathsf{DC}_0 \Rightarrow \mathsf{CC}_0$ lets us reprove [15, Theorem A] in Corollary 3.13.

The axioms CC and DC have a tight relationship with the notion of *homotopy dimension* (Definition 2.1). Informally, an "operation" in a topos is of homotopy dimension $\leq d$ if it sends $(m+d)$-connected maps into m-connected maps for all m. The axiom CC_d says that countable products are of homotopy dimension $\leq d$ (Definition 2.19). We shall see that DC_d holds in any n-topos with enough objects of homotopy dimension $\leq d$ (see Definition 2.9 and Proposition 2.37).

As an application of Theorem 1.1, we give in Corollary 3.10 a new proof that an ∞-topos locally of homotopy dimension $\leq d$ is Postnikov complete [10, Proposition 7.2.1.10]. We also show that Theorem 1.1 is false for $d = \infty$ (Remark 3.7), and that the CC_d condition of Theorem 1.1 is not necessary for the hypercompletion and Postnikov completion to agree (Remark 3.8).

In an appendix, we summarize the various convergence properties for Postnikov towers and introduce some new terminology (Definition A.4).

Conventions. We set our conventions about categories, n-topoi and connectivity. We shall say simply *category* to refer to an $(\infty,1)$-category. By default, a category is not assumed small. An *n-category* is a category whose mapping spaces are all $(n-1)$-truncated. We denote the category of (small) spaces by $\mathcal{S}$. For $-1 \leq n \leq \infty$, we denote by $\mathcal{S}^{\leq n}$ the $(n+1)$-category of n-truncated spaces. For $n = 0$, $\mathcal{S}^{\leq 0} = \mathcal{S}\mathrm{et}$ is the 1-category of sets, and for $n = \infty$, we put $\mathcal{S}^{\leq \infty} = \mathcal{S}$ by convention. An n-category $\mathcal{E}$ is an *n-topos* if it is an (accessible) left-exact localization of $\mathrm{Fun}\left(C, \mathcal{S}^{\leq n-1}\right)$ for a small n-category C. (When $n < \infty$, all left-exact localizations of $\mathrm{Fun}\left(C, \mathcal{S}^{\leq n-1}\right)$ are accessible [10, Remark 6.4.1.2]. This is open for $n = \infty$.) An *algebraic morphism* (of n-topoi) $u^* : \mathcal{E} \to \mathcal{F}$ is a cocontinuous and left-exact functor. Such a functor always has a right adjoint u_*.

We shall say that a map in an n-topos is a *surjection*, or that it is *surjective*, if it is an effective epimorphism in the sense of [10, Section 6.2.3]. Recall that for a map $f : A \to B$, its diagonal is the map $\Delta(f) : A \to A \times_B A$. We put $\Delta^0(f) = f$ and $\Delta^{\ell+1}(f) = \Delta(\Delta^\ell(f))$. For $-2 \leq m \leq \infty$, we shall say that a map f is *m-connected* if, for every $0 \leq \ell \leq m+1$, the map $\Delta^\ell(f)$ is a surjection. Every map is (-2)-connected, and the (-1)-connected maps are the surjections. A map is m-connected if and only if it is $(m+1)$-connective in the sense of [10, Definition 6.5.1.10]. An object X of an n-topos is called *m-connected* if $X \to 1$ is m-connected. In the case $m = -1$, we shall also say that X is *inhabited*.

A map f is *m-truncated* if the diagonal $\Delta^{m+2}(f)$ is invertible. The (-2)-truncated maps are the isomorphisms, and the (-1)-truncated maps are the monomorphisms. For simplicity, we shall say that a map f is *∞-truncated* (rather than *hypercomplete*) if it is right orthogonal to ∞-connected maps. For each $-2 \leq m \leq \infty$, the classes of m-connected maps and m-truncated maps form a factorization system which is stable under base change (see [10, Example 5.2.8.16

and Remark 6.5.2.21], [1, Proposition 3.3.6], or [2, Lemma 2.2.22]). We denote the m-truncation of an object X by $X^{\leq m}$ ([10] uses the notation $\tau_{\leq m}X$).

For $1 \leq m \leq n \leq \infty$, the m-category of $(m-1)$-truncated objects in an n-topos is an m-topos [10, Proposition 6.4.5.4]. The *n-topos envelope* of an m-topos is the left adjoint to this truncation operation [10, Proposition 6.4.5.7]. The original m-topos is the full subcategory of $(m-1)$-truncated objects of its n-topos envelope. For $\mathcal{E}$ an ∞-topos, the *hypercompletion* $P_\infty : \mathcal{E} \to \mathcal{E}^{\leq\infty}$ is the localization inverting the class of ∞-connected maps [10, Section 6.5.2].

2. Choice axioms

2.1 Homotopy dimension We will formulate the axiom CC_d in terms of the notion of homotopy dimension.

Definition 2.1 (Homotopy dimension)**.** Let $u^* : \mathcal{E} \to \mathcal{F}$ be an algebraic morphism of n-topoi and u_* its right adjoint. For $-1 \leq d \leq \infty$, we shall say that u^* is of *homotopy dimension* $\leq d$ if for any $(d-1)$-connected map $A \to B$ in $\mathcal{F}$, the map $u_*A \to u_*B$ is a surjection in $\mathcal{E}$. In the case that $\mathcal{E} = \mathcal{S}^{\leq n-1}$ and $u^* : \mathcal{S}^{\leq n-1} \to \mathcal{F}$ is the unique algebraic morphism of n-topoi, we shall say simply that $\mathcal{F}$ is of homotopy dimension $\leq d$.

The following lemma shows that the previous definition is equivalent to [10, Definition 7.2.1.6].

Lemma 2.2. *The morphism $u^* : \mathcal{E} \to \mathcal{F}$ is of homotopy dimension $\leq d$ if and only if, for every $m \geq -2$ and every $(m+d)$-connected map $A \to B$ in $\mathcal{F}$, the map $u_*A \to u_*B$ is m-connected in $\mathcal{E}$.*

Proof. For the "if" direction, take $m = -1$. For the converse, a map f is $(m+d)$-connected if and only if $\Delta^\ell(f)$ is $(d-1)$-connected for every $0 \leq \ell \leq m+1$. Then $u_*(\Delta^\ell(f))$ is a surjection for $0 \leq \ell \leq m+1$. Using that $u_*\Delta(f) = \Delta(u_*f)$, we deduce that u_*f is m-connected. □

Examples 2.3.
a) Recall from [10, Theorem 7.2.3.6] that the ∞-topos of sheaves on a topological space of covering dimension $\leq d$ is of homotopy dimension $\leq d$. In particular, the ∞-topos of sheaves on the cube $[0,1]^d$ is of homotopy dimension $\leq d$.
b) In an n-topos for $n < \infty$, every $(n-1)$-connected map is invertible, and so all n-topoi and algebraic morphisms between n-topoi are of homotopy dimension $\leq n$. (Beware the m-topos envelope of an n-topos, for $m > n$, may have a strictly larger homotopy dimension.)
c) If an ∞-topos is of homotopy dimension $\leq \infty$, then for every ∞-connected object X, the space of global sections of X is m-connected for every m by Lemma 2.2, and therefore contractible. This condition is in fact an equivalence by Lemma 2.5 below. We shall give examples of ∞-topoi not of homotopy dimension $\leq \infty$ in Section 4.

Definition 2.4. Let $\mathcal{E}$ be an n-topos. We say that an object $X \in \mathcal{E}$ is *(externally) of homotopy dimension* $\leq d$ if the topos $\mathcal{E}_{/X}$ is of homotopy dimension $\leq d$.

Recall that a morphism of $\mathcal{E}_{/X}$ is m-connected if and only if its underlying morphism in $\mathcal{E}$ is [10, Proposition 6.5.1.19]. The following lemma is essentially [10, Lemma 7.2.1.7].

Lemma 2.5. *Let $\mathcal{E}$ be an n-topos and X an object of $\mathcal{E}$. The following are equivalent:*

1. *X is of homotopy dimension $\leq d$.*
2. *Every $(d-1)$-connected map $A \to X$ of $\mathcal{E}$ admits a section.*
3. *For every $(d-1)$-connected map $A \to B$ of $\mathcal{E}$, any map $X \to B$ admits a lift to A.*

4. *For every $m \geq -2$, the space of sections of any $(m+d)$-connected map of $\mathcal{E}$ is m-connected.*

Proof. (1) ⇒ (2): A $(d-1)$-connected map $f : A \to X$ of $\mathcal{E}$ amounts to a $(d-1)$-connected object of $\mathcal{E}_{/X}$, and a section of f amounts to a global section of this object. Then the claim follows from the definition of homotopy dimension (with B the terminal object of $\mathcal{E}_{/X}$). (2) ⇒ (3) because the pullback of $A \to B$ along $X \to B$ is again $(d-1)$-connected, and a section of this pullback is the same as a lift of the map $X \to B$ to A. (3) ⇒ (1): Let $A \to B$ be a $(d-1)$-connected map of $\mathcal{E}_{/X}$. By (3), any global section of B (i.e. section in $\mathcal{E}$ of the structural map $B \to X$) lifts to some global section of A. Finally, (2) ⇔ (4) by Lemma 2.2. □

In particular, in a 1-topos, an object is of homotopy dimension ≤ 0 if and only if it is (externally) projective in the usual sense, as the (-1)-connected maps are the surjections. (In [3], projective objects are called "weakly contractible".)

Remark 2.6. Definition 2.4 has a corresponding internal notion: an object X of an n-topos $\mathcal{E}$ is *internally of homotopy dimension* $\leq d$ if the algebraic morphism $X \times - : \mathcal{E} \to \mathcal{E}_{/X}$ is of homotopy dimension $\leq d$. Informally, this means (using Lemma 2.2) that X-indexed dependent products in $\mathcal{E}$ reduce connectivity by at most d. When $d = 0$ and $n = 1$, the two conditions of being (internally) of homotopy dimension $\leq d$ reduce to the conditions of being (internally) projective (see [12, Exercises IV.15–16]).

In the proof of Theorem 1.1, we will use the following auxiliary definition. Let K be an n-category (in particular, K could be a set) and $\mathcal{E}$ an n-topos. The K-limit functor $\lim_K : \mathcal{E}^K \to \mathcal{E}$ is the right adjoint to an algebraic morphism $\Delta_K : \mathcal{E} \to \mathcal{E}^K$ (the constant diagram functor).

Definition 2.7 (Homotopy $\mathcal{E}$-dimension)**.** We shall say that K is of *homotopy $\mathcal{E}$-dimension* $\leq d$ if the algebraic morphism $\Delta_K : \mathcal{E} \to \mathcal{E}^K$ is of homotopy dimension $\leq d$.

Below, a "K-limit of maps" refers to the limit of a K-indexed diagram in the arrow category.

Lemma 2.8. *The following statements are equivalent:*

1. *K is of homotopy $\mathcal{E}$-dimension $\leq d$.*
2. *In $\mathcal{E}$, K-limits of $(d-1)$-connected maps are surjections.*
3. *In $\mathcal{E}$, for every $m \geq -2$, K-limits of $(m+d)$-connected maps are m-connected maps.*
4. *For every E in $\mathcal{E}$, K is of homotopy $\mathcal{E}_{/E}$-dimension $\leq d$.*
5. *There exists a surjection $E \twoheadrightarrow 1$ such that K is of homotopy $\mathcal{E}_{/E}$-dimension $\leq d$.*
6. *In every slice $\mathcal{E}_{/E}$, K-limits of $(d-1)$-connected objects are inhabited.*
7. *In every slice $\mathcal{E}_{/E}$, for every $m \geq -2$, K-limits of $(m+d)$-connected objects are m-connected in $\mathcal{E}$.*

Proof. (1)⇔(2) by definition. We show the remaining statements are equivalent to (2). (2)⇔(3) is a consequence of Lemma 2.2. (4)⇒(2) is clear; let us see the converse. Let E be an object in $\mathcal{E}$ and $A_k \to B_k \to E$ be a K-indexed diagram of $(d-1)$-connected maps in $\mathcal{E}_{/E}$. We denote by $\lim_{/E} A_k \to \lim_{/E} B_k$ their limit in $\mathcal{E}_{/E}$. This limit can be computed as a base change of the limit in $\mathcal{E}$:

$$\begin{array}{ccccc} \lim_{/E} A_k & \longrightarrow & \lim_{/E} B_k & \longrightarrow & E \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow{\scriptstyle \Delta} \\ \lim A_k & \longrightarrow & \lim B_k & \longrightarrow & \lim E = E^{|K|} \end{array}$$

where $|K|$ is the groupoidification of K. If $\lim A_k \to \lim B_k$ is a surjection, then so is $\lim_{/E} A_k \to \lim_{/E} B_k$ by base change. This shows (2)⇒(4). (2)⇒(5) is clear, and (5)⇒(2) holds because $E\times - : \mathcal{E} \to \mathcal{E}_{/E}$ preserves limits and preserves and reflects connectivity of maps. (4)⇒(7) follows from Lemma 2.2, and (7)⇒(6) is clear, so it remains to show (6)⇒(2). Let $A_k \to B_k$ be a K-indexed diagram of $(d-1)$-connected maps in $\mathcal{E}$. We put $A = \lim A_k$, $B = \lim B_k$ and $A'_k = A_k \times_{B_k} B$. The maps $A'_k \to B$ are $(d-1)$-connected by base change. By assumption on $\mathcal{E}_{/B}$, the object $\lim_{/B} A'_k \to B$ is inhabited. The conclusion follows from the remark that $A = \lim_{/B} A'_k$. □

The rest of this section contains results that will be useful to verify that a topos satisfies the choice axioms CC_d or DC_d.

Definition 2.9 (Enough objects)**.** An n-topos has *enough objects of homotopy dimension* $\leq d$ if every object X admits a covering family $(X_i \to X)$ where each X_i is of homotopy dimension $\leq d$. Recall from [10, Definition 7.2.1.8] that an n-topos is *locally of homotopy dimension* $\leq d$ if it is generated under colimits by its subcategory of objects of homotopy dimension $\leq d$.

Example 2.10. A 1-topos has enough objects of homotopy dimension ≤ 0 if and only if it has enough (externally) projective objects in the usual sense.

Remark 2.11. If an n-topos is locally of homotopy dimension $\leq d$, then it has enough objects of homotopy dimension $\leq d$. For $n < \infty$, the converse is also true. For every object X, the existence of enough objects of homotopy dimension $\leq d$ lets us build a (semi-simplicial) hypercover $X_\bullet$ of X by objects of homotopy dimension $\leq d$. By the analogue of [11, Corollary A.5.3.3] for n-topoi, the colimit of this hypercover gives an isomorphism $|X_\bullet| = X$, so the objects of homotopy dimension $\leq d$ generate the topos under colimits.

For $n = \infty$, however, the converse is false. In the ∞-topos of parametrized spectra $\mathcal{PS}\mathrm{p}$, any set equipped with the zero spectrum is of homotopy dimension ≤ 0. Every object of $\mathcal{PS}\mathrm{p}$ admits a surjection from such an object, so $\mathcal{PS}\mathrm{p}$ has enough objects of homotopy dimension ≤ 0. This also means (by Lemma 2.5) that *any* object of homotopy dimension ≤ 0 must be a retract of such an object, hence itself a set equipped with the zero spectrum. These objects only generate the subcategory $\mathcal{S} \subseteq \mathcal{PS}\mathrm{p}$ under colimits.

Warning 2.12. An ∞-topos of homotopy dimension $\leq d$ need not be locally of homotopy dimension $\leq d$. By the previous remark, the ∞-topos of parameterized spectra is a counterexample (with $d = 0$).

Lemma 2.13. *Let $\mathcal{E}_n$ be an n-topos and $\mathcal{E}$ its ∞-topos envelope. Then the inclusion functor $\mathcal{E}_n \hookrightarrow \mathcal{E}$ preserves and reflects surjections.*

Proof. Recall that $\mathcal{E}$ can be constructed as the category of higher sheaves on a small full subcategory $\mathcal{C} \subseteq \mathcal{E}_n$ for the canonical topology. We may choose $\mathcal{C}$ large enough to contain any particular surjection $f : A \to B$ of $\mathcal{E}_n$. (See the proofs of [10, Propositions 6.4.3.6 and 6.4.5.7].) Then the image of f in $\mathcal{E}$ is also a surjection, by the definition of the canonical topology. On the other hand, both in $\mathcal{E}_n$ and in $\mathcal{E}$, the surjections are characterized as maps that are the colimit of their Čech nerve [10, Corollary 6.2.3.5]. As $\mathcal{E}_n \subseteq \mathcal{E}$ is the full subcategory of $(n-1)$-connected objects, it is closed under finite limits and therefore this embedding commutes with the construction of the Čech nerve. So if $f : A \to B$ in $\mathcal{E}_n$ is such that B is the colimit of the Čech nerve of f in $\mathcal{E}$, then this must also be the case in $\mathcal{E}_n$. □

A similar argument also shows that every object X in $\mathcal{E}$ admits a surjection $Y \to X$ with Y in $\mathcal{E}_n$, because $\mathcal{E}_n \subseteq \mathcal{E}$ is closed under arbitrary coproducts.

Lemma 2.14. *If an n-topos $\mathcal{E}_n$ has enough objects of homotopy dimension ≤ 0, then the ∞-topos envelope $\mathcal{E}$ of $\mathcal{E}_n$ also has enough objects of homotopy dimension ≤ 0.*

Proof. Choose a surjection $Y \to X$ in $\mathcal{E}$ with Y in $\mathcal{E}_n$, and then a surjection $P \to Y$ in $\mathcal{E}_n$, where P is an object of homotopy dimension ≤ 0. The composition $P \to Y \to X$ is a surjection in $\mathcal{E}$; we have to verify that the object P is still of homotopy dimension ≤ 0 in $\mathcal{E}$. Let $f : Z \to P$ be a surjection in $\mathcal{E}$. Choose a surjection $T \to Z$ with T in $\mathcal{E}_n$. The composition $T \to Z \to P$ is a surjection in $\mathcal{E}$ between objects of $\mathcal{E}_n$. By Lemma 2.13, it is a surjection in $\mathcal{E}_n$. Since P is of homotopy dimension ≤ 0 in $\mathcal{E}_n$, the map $T \to P$ (and therefore the map $Z \to P$) admits a section. This shows that P is of homotopy dimension ≤ 0 in $\mathcal{E}$. □

Remark 2.15. For $0 < d < n$, we do not know if the ∞-topos envelope of an n-topos with enough objects of homotopy dimension $\leq d$ has enough objects of homotopy dimension $\leq d$ in general.

Lemma 2.16. *Any ∞-topos $\mathcal{E}$ locally of homotopy dimension $\leq d$ is hypercomplete.*

Proof. Let $\mathcal{D} \subseteq \mathcal{E}$ be the full subcategory of objects of homotopy dimension $\leq d$. By assumption, the nerve functor $N : \mathcal{E} \to \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$ is conservative. (If $f : X \to Y$ in $\mathcal{E}$ is sent to an equivalence in $\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$, then the class of objects $A \in \mathcal{E}$ such that f induces an equivalence $\mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ contains $\mathcal{D}$, and is closed under colimits.) Since $\mathcal{D}$ consists of objects of homotopy dimension $\leq d$, N sends $(m+d)$-connected maps to m-connected maps for all m. In particular, it sends ∞-connected maps to ∞-connected maps, that is, to isomorphisms. Since N is conservative, every ∞-connected map in $\mathcal{E}$ is invertible. □

Remark 2.17. It is not necessarily the case that an ∞-topos with enough objects of homotopy dimension $\leq d$ is hypercomplete. A counterexample for $d = 0$ is given by the ∞-topos of parametrized spectra.

Remark 2.18. It is not true that the ∞-topos envelope of an n-topos locally of homotopy dimension $\leq d$ is itself locally of homotopy dimension $\leq d$, because this envelope need not be hypercomplete. A counterexample for $d = 0$ is given in [5, Example A.10], see [15, Example 3.29].

2.2 Countable choice Given an n-topos $\mathcal{E}$, we denote by $\mathbb{N}$ the constant sheaf $\coprod_{n \in \mathbb{N}} 1 \in \mathcal{E}$ on the set of natural numbers. By extensivity of sums in $\mathcal{E}$, we have a canonical equivalence $\mathcal{E}_{/\mathbb{N}} = \mathcal{E}^{\mathbb{N}}$, and the right adjoint to $\mathbb{N} \times - : \mathcal{E} \to \mathcal{E}_{/\mathbb{N}}$ coincides with the product functor $\prod : \mathcal{E}^{\mathbb{N}} \to \mathcal{E}$.

Definition 2.19 (Countable choice)**.** Let $\mathcal{E}$ be an n-topos (for $1 \leq n \leq \infty$). For $-1 \leq d \leq \infty$, we say that the axiom of *countable choice of dimension $\leq d$* (CC_d) holds in $\mathcal{E}$ if $\mathbb{N}$ is of homotopy $\mathcal{E}$-dimension $\leq d$, that is, if countable products of $(d-1)$-connected maps are surjections in $\mathcal{E}$ (or if any of the equivalent characterizations of Lemma 2.8 holds).

Examples 2.20.
a) We always have $\mathsf{CC}_{-1} \Rightarrow \mathsf{CC}_0 \Rightarrow \mathsf{CC}_1 \Rightarrow \mathsf{CC}_2 \Rightarrow \cdots \Rightarrow \mathsf{CC}_\infty$.
b) By Lemma 2.8 (4), if CC_d holds in $\mathcal{E}$, then it holds in every slice of $\mathcal{E}$.
c) The axiom CC_{-1} says that a countable product of arbitrary maps is a surjection. This holds if and only if $\mathcal{E} = 1$.

d) The axiom CC_0 says that countable products of surjections are surjections. In a 1-topos, this is equivalent to the usual statement of countable choice expressed in the internal language [13, Proposition 3.42]. For $1 \le n \le \infty$, CC_0 holds in the n-topos $\mathcal{S}^{\le n-1}$ and in all diagram categories $\mathrm{Fun}\left(C, \mathcal{S}^{\le n-1}\right)$ (in particular, in slices of $\mathcal{S}^{\le n-1}$).

e) For $d \ge n$, the axiom CC_d holds trivially in every n-topos $\mathcal{E}$, since the $(n-1)$-connected maps coincide with the isomorphisms. In a 1-topos, then, the only non-trivial axiom is CC_0.

f) The axiom CC_∞ says that countable products of ∞-connected maps are surjections. By Lemma 2.2 this is equivalent to saying that countable products of ∞-connected maps are ∞-connected maps. The axiom CC_∞ holds for free in any hypercomplete ∞-topos. An example where it fails (and thus any CC_d also) is given in Proposition 4.2.

g) The ∞-topos of sheaves on the interval $[0,1]$ does not satisfy CC_0. For each $k \ge 1$, choose a cover $(U_{k,\alpha})$ of $[0,1]$ by open sets of diameter less than $1/k$, and set $A_k = \coprod_\alpha U_{k,\alpha}$. The product $\prod A_k$ is empty, since there is no nonempty open set $U \subseteq [0,1]$ small enough such that all of the objects A_k have sections over U. On the other hand, we will see below that this ∞-topos does satisfy CC_1. More generally, for $0 \le d < \infty$, the ∞-topos of sheaves on the d-cube $[0,1]^d$ satisfies CC_d but not CC_{d-1}; see Example 2.26.

h) The ∞-topos of sheaves on the space $\coprod_d [0,1]^d$ satisfies CC_∞, but does not satisfy CC_d for any $d < \infty$.

i) We will see in Proposition 2.27 that if an ∞-topos $\mathcal{E}$ has CC_d, then so does the n-topos $\mathcal{E}^{\le n-1}$ for any n.

j) The ∞-topos of $\mathcal{PS}\mathrm{p}$ of parametrized spectra satisfies CC_0 without being hypercomplete. Its hypercompletion is the functor $b : \mathcal{PS}\mathrm{p} \to \mathcal{S}$ extracting the base space. The axiom CC_0 holds in $\mathcal{S}$ and it holds in $\mathcal{PS}\mathrm{p}$ because b preserves countable products and reflects surjections.

Remark 2.21. Let us explain intuitively the geometric meaning of the axiom CC_d for $d = 0, 1$. The failure of CC_0 can be understood as follows. Let X be a topological space, I a set, and (A_i) a family of inhabited sheaves on X (i.e. whose stalks are all nonempty). A local section of $\prod A_i$ around a point x is a neighborhood U of x and a section of every A_i over U. Every A_i has local sections around x, but because the neighborhoods of x are only closed under finite intersections, it might not be possible to find a section of all the A_i over the same neighborhood of x. For this reason, the stalk of $\prod A_i$ can be empty if the set I is infinite, preventing the map $\prod A_i \to 1$ from being a surjection. Of course, this does not happen if x has enough neighborhoods where surjective sheaves always have sections (i.e. if X is locally of homotopy dimension ≤ 0 around x, like an Alexandrov space, or the Cantor space).

The axiom CC_1 says that when the A_i are 0-connected sheaves (i.e., gerbes), then the product $\prod A_i$ always has local sections around x. However, the space of such sections might not be connected anymore. This happens if X is locally of covering dimension ≤ 1 around x, for example in $X = [0,1]$. The key remark is that given any neighborhood U of x, we can always find a section of a sheaf A on U when A is 0-connected. Essentially, one starts with local sections on a cover U_α of U, and because A is connected, one can find homotopies between these sections on the $U_{\alpha\beta}$, and use them to construct a global section on U. Because X is of covering dimension ≤ 1, there are no triple intersections to consider. (If X had been of covering dimension ≤ 2, we would have needed coherence homotopies on triple intersections. Such homotopies always exist if A is a 1-connected sheaf.) Given a family of connected sheaves A_k, we see that we can construct a section of each of A_k on the same neighborhood of x. Thus the stalk of $\prod A_i$ at x is not empty.

Remark 2.22. We now discuss the logical meaning of the axiom CC_d. Recall first that, in

ordinary logic, the class of surjections is used to provide semantics for *existential conditions*. Using the notation of [7], a proposition $\exists x.P(x)$ is interpreted as the image of the map $p : \Sigma_{x:X} P(x) \to 1$, that is as the maximal subterminal object over which the map p is a surjection. The class of isomorphisms is similarly used for *unique existential conditions*. The proposition $\exists! x.P(x)$ is interpreted as the condition that the map $p : \Sigma_{x:X} P(x) \to 1$ is an isomorphism, i.e., the maximal subterminal object over which the map p is an isomorphism.

Between these two conditions, the classes of n-connected maps provide semantics for an (external) hierarchy of intermediate "uniqueness levels" for existential conditions. Let us introduce the notation $\exists^n x.P(x)$ to mean that $\Sigma_{x:X} P(x)$ is an n-connected object. Informally, for $n \geq 0$, $\exists^n x.P(x)$ means that there exists x satisfying $P(x)$ and for any two such x, there exists a path (or equality) between them in the sense of $\exists^{n-1}$. Semantically, $\exists^n x.P(x)$ corresponds to the maximal subterminal object over which the map p is n-connected. When $n = -1$, this recovers the quantifier $\exists$. When $n = \infty$, this does *not* recover the quantifier $\exists!$, since the semantics of $\exists^\infty x.P(x)$ is that the object $\Sigma_{x:X} P(x)$ is ∞-connected rather than contractible.

$$\exists! \;\Rightarrow\; \exists^\infty \;\Rightarrow\; \dots \;\Rightarrow\; \exists^0 \;\Rightarrow\; \exists^{-1} = \exists\,.$$

With these notions in mind, any operation sending $(m+d)$-connected maps to m-connected maps (like $\prod_{\mathbb{N}}$ in CC_d) can be thought of as an operation "weakening the uniqueness level" of existential conditions.

Remark 2.23. Using the terminology of Remark 2.6, we can say that CC_d holds in $\mathcal{E}$ if and only if the object $\mathbb{N} \in \mathcal{E}$ is internally of homotopy dimension $\leq d$. This condition can also be expressed in the language of homotopy type theory, for instance as

$$\mathsf{CC}^{\mathsf{int}} : \mathbb{N} \to \mathsf{Prop}\,, \qquad \mathsf{CC}^{\mathsf{int}}_d := \prod_{X:\mathbb{N}\to\mathcal{U}} \Big(\prod_{n:\mathbb{N}} \mathsf{isConn}_{d-1}(X(n))\Big) \to \mathsf{isConn}_{-1}\Big(\prod_{n:\mathbb{N}} X(n)\Big)\,.$$

Here $\mathsf{isConn}_m(A)$ is the proposition that a type A is m-connected, i.e., that the m-truncation $\|A\|_m$ is contractible. We set $\mathsf{isConn}_\infty(A) := \prod_{m:\mathbb{N}} \mathsf{isConn}_m(A)$. The statement $\mathsf{CC}^{\mathsf{int}}_d$ is the appropriate formulation of CC_d in an elementary ∞-topos [16, 17]. (In this context, the external statement of Definition 2.19 may not even make sense since elementary topoi may not have external countable products when their natural number object is not standard. For the same reason, the internal definition of isConn_∞ may be stronger than its external counterpart.)

Remark 2.24. Throughout this section we could replace $\mathbb{N}$ by an arbitrary infinite set I, obtaining an axiom of "I-indexed choice of dimension $\leq d$", whose strength generally increases with the cardinality of I. We could also impose this axiom for every set I, meaning that arbitrary products of $(d-1)$-connected maps in $\mathcal{E}$ are surjections. Note that, even for a 1-topos $\mathcal{E}$ and for $d = 0$, this last axiom is weaker than the internal axiom of choice in $\mathcal{E}$, since it only concerns families of maps of $\mathcal{E}$ indexed by an external set (equivalently, internal families in $\mathcal{E}$ indexed on a constant sheaf).

In [19], the author introduces the axiom AB4*-d for a Grothendieck abelian category $\mathcal{A}$, which states that for every set I, the derived functors of the I-ary product functor $\prod_I : \mathcal{A}^I \to \mathcal{A}$ vanish in degrees higher than d. When $d = 0$, this axiom reduces to Grothendieck's axiom AB4* (products are exact). The axiom CC_d is a non-abelian analogue of AB4*-d, except that we consider only countable products.

The following result will be a consequence of Proposition 2.37 proved in the next section.

Proposition 2.25. *Any n-topos with enough objects of homotopy dimension $\leq d$ satisfies CC_d.*

Example 2.26 (The d-cube)**.** Let $\mathcal{E}$ denote the ∞-topos of sheaves on the d-cube $X = [0,1]^d$. By classical dimension theory, every subset of $\mathbb{R}^d$ has covering dimension $\leq d$ [6, Chapter 1]. In particular, every open subset $U \subseteq X$ has covering dimension $\leq d$ so by [10, Theorem 7.2.3.6], $\mathcal{E}_{/U}$ has homotopy dimension $\leq d$. Then $\mathcal{E}$ satisfies CC_d by Proposition 2.25.

We show that $\mathcal{E}$ does not have CC_{d-1}. We already checked this for $d = 1$ in Example 2.20 (g), so assume $d \geq 2$. We take $A \in \mathcal{E}$ to be the Eilenberg–Mac Lane object $K(F,d)$ for a sheaf of abelian groups F on X constructed below in such a way that $\pi_0(A^{\mathbb{N}})$ is not the terminal object of $\mathcal{E}$. Then A will be $(d-1)$-connected, but $A^{\mathbb{N}}$ will not be 0-connected, so $\mathcal{E}$ does not satisfy CC_{d-1}.

We can compute the stalk of the sheaf $\pi_0(A^{\mathbb{N}})$ at a point $x \in X$ as

$$\pi_0(A^{\mathbb{N}})_x = \pi_0((A^{\mathbb{N}})_x) = \pi_0(\operatorname*{colim}_{U \ni x} A^{\mathbb{N}}(U)) = \operatorname*{colim}_{U \ni x} \pi_0(\prod_{\mathbb{N}} A(U)) = \operatorname*{colim}_{U \ni x} \prod_{\mathbb{N}} \pi_0(A(U))$$
$$= \operatorname*{colim}_{U \ni x} \prod_{\mathbb{N}} H^d(U, F|_U).$$

(Compare [19, Proposition 1.6].) Here, H^d denotes ordinary sheaf cohomology. The first step used [10, Remark 6.5.1.4], and the last step used [10, Remark 7.2.2.17]. In this colimit, we may restrict attention to the cofinal family of contractible open neighborhoods U of x, as X is locally contractible.

Let $x = (\frac{1}{2}, \ldots, \frac{1}{2})$ be the center of X and let $Z \subseteq X$ be the (closed) union of $\{x\}$ and the spheres S_k of radius $1/k$ centered at x for all $k \geq 3$. Let $j : X \smallsetminus Z \to X$ denote the inclusion of the open complement of Z, and set $F = j_!\mathbb{Z}$. For U a contractible open neighborhood of x, the localization sequence $0 \to (j_!j^*\mathbb{Z})|_U \to \mathbb{Z}|_U \to (i_*i^*\mathbb{Z})|_U \to 0$ yields an isomorphism $H^d(U, F|_U) \cong H^{d-1}(U \cap Z, \mathbb{Z})$, as $H^{d-1}(U,\mathbb{Z}) = H^d(U,\mathbb{Z}) = 0$. Moreover, these isomorphisms are compatible with restriction to a smaller $U' \subseteq U$.

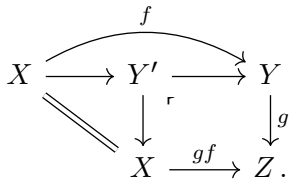
The space Z has $S_k \cup \{x\}$ as a retract for each k, so there is an element $\alpha \in \prod_{\mathbb{N}} H^{d-1}(Z,\mathbb{Z})$ such that, for each $k \geq 3$, α_k restricts to a generator of $H^{d-1}(S_k,\mathbb{Z})$. Any contractible open neighborhood U of x contains S_k for some $k \geq 3$, and then the restriction of α to $\prod_{\mathbb{N}} H^{d-1}(U \cap Z,\mathbb{Z})$ is nonzero in the kth component. Hence, α represents a nontrivial element of $\operatorname{colim}_{U \ni x} \prod_{\mathbb{N}} H^d(U, F|_U) = \pi_0(A^{\mathbb{N}})_x$.

Proposition 2.27. *If CC_d holds in $\mathcal{E}$, then it also holds in every $(n+1)$-topos $\mathcal{E}^{\leq n}$ for $0 \leq n \leq \infty$ (in particular, in the hypercompletion $\mathcal{E}^{\leq \infty}$).*

The proof of Proposition 2.27 will need a few lemmas about connected maps. In this proof, we write $P_k = (-)^{\leq k} : \mathcal{E} \to \mathcal{E}^{\leq k}$ for the left adjoint to the inclusion $\mathcal{E}^{\leq k} \subseteq \mathcal{E}$ for any $0 \leq k \leq \infty$; $P_\infty : \mathcal{E} \to \mathcal{E}^{\leq \infty}$ is the hypercompletion functor.

Lemma 2.28. *If $f : X \to Y$ and $g : Y \to Z$ are two maps such that g is $(k+1)$-connected and gf is k-connected, then f is k-connected.*

Proof. We consider the pullback of g along gf



The map $Y' \to Y$ is k-connected since it is a base change of gf. The map $Y' \to X$ is $(k+1)$-connected since it is a base change of g. The map $X \to Y'$ is k-connected since it is a section of a $(k+1)$-connected map (see [10, Proposition 6.5.1.20] or [7, Lemma 7.5.11]). The map f is the composition of two k-connected maps, thus k-connected. □

Lemma 2.29. *If a map $X \to Y$ is sent to an isomorphism by $P_{k+1} : \mathcal{E} \to \mathcal{E}^{\leq k+1}$, then it is k-connected.*

Proof. Apply Lemma 2.28 to the commutative triangle $X \to Y \to P_{k+1}X = P_{k+1}Y$. □

Lemma 2.30. *For $k+1 \leq n \leq \infty$, a map $X \to Y$ in $\mathcal{E}$ is k-connected if and only if the map $P_nX \to P_nY$ is k-connected in $\mathcal{E}^{\leq n}$. (In other words, the functor $P_n : \mathcal{E} \to \mathcal{E}^{\leq n}$ preserves and reflects k-connected maps. Note that we may have $k = n = \infty$.)*

Proof. The factorization system (k-connected, k-truncated) is defined both in $\mathcal{E}$ and in $\mathcal{E}^{\leq n}$. The inclusion $\mathcal{E}^{\leq n} \subseteq \mathcal{E}$ preserves diagonals and isomorphisms, thus k-truncated maps. (Since it is moreover a conservative functor, it reflects k-truncated maps; we will use this below.) Consequently, its left adjoint $P_n : \mathcal{E} \to \mathcal{E}^{\leq n}$ preserves k-connected maps for every k.

We assume that the map $P_nX \to P_nY$ is k-connected in $\mathcal{E}^{\leq n}$. Let us show that it is also k-connected when viewed as a map in $\mathcal{E}$. We consider its (k-connected, k-truncated) factorization $P_nX \to Z \to P_nY$ in $\mathcal{E}$. The map $Z \to P_nY$ is k-truncated in $\mathcal{E}$, thus n-truncated since $k < n$. This shows that Z is an object of $\mathcal{E}^{\leq n}$. Since the inclusion $\mathcal{E}^{\leq n} \subseteq \mathcal{E}$ reflects k-truncated maps, the map $Z \to P_nY$ is also k-truncated in $\mathcal{E}^{\leq n}$. Since Z is in $\mathcal{E}^{\leq n}$, the image of $P_nX \to Z$ by P_n is itself. Since it is k-connected in $\mathcal{E}$, it is k-connected in $\mathcal{E}^{\leq n}$ by the argument above. Since $P_nX \to Z$ and $P_nX \to P_nY$ are k-connected in $\mathcal{E}^{\leq n}$, so is $Z \to P_nY$ by cancellation. This shows that $Z \to P_nY$ is both k-connected and k-truncated in $\mathcal{E}^{\leq n}$, thus an isomorphism. Hence we have shown the map $P_nX \to P_nY$ is k-connected in $\mathcal{E}$.

Conversely, we now consider the canonical square in $\mathcal{E}$

$$\begin{array}{ccc} X & \longrightarrow & P_nX \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P_nY\,. \end{array}$$

The map $Y \to P_nY$ is n-connected, thus $(k+1)$-connected. The map $X \to P_nX$ is n-connected, thus k-connected. The map $P_nX \to P_nY$ is k-connected, so the composition $X \to P_nX \to P_nY$ is k-connected. Then the map $X \to Y$ is k-connected by Lemma 2.28. □

Remark 2.31. For $n < \infty$, since not every map between n-connected objects is an n-connected map, the functor $P_n : \mathcal{E} \to \mathcal{E}^{\leq n}$ inverts some maps which are not n-connected. Since the n-connected maps in $\mathcal{E}^{\leq n}$ are exactly the isomorphisms, this shows that Lemma 2.30 is false when $k+1 > n$.

Remark 2.32. For $n = 0$, Lemma 2.30 recovers that $X \to Y$ is a surjection if and only if the map $P_0X \to P_0Y$ is a surjection. For $n = \infty$, the result is [11, Proposition A.4.2.1 and Example A.4.2.4] or [2, Proposition 4.3.2].

Proof of Proposition 2.27. If $n < \infty$, we have seen that CC_d holds in the $(n+1)$-topos $\mathcal{E}^{\leq n}$ for $d \geq n+1$. So, we may assume that $d \leq n$.

Let $(X_k \to Y_k)$ be a countable family of $(d-1)$-connected maps in $\mathcal{E}^{\leq n}$, and let $p : X \to Y$ be their product in $\mathcal{E}^{\leq n}$. We denote by ι_n the inclusion $\mathcal{E}^{\leq n} \subseteq \mathcal{E}$. The map $\iota_n p : \iota_n X \to \iota_n Y$ is the product of the maps $\iota_n X_k \to \iota_n Y_k$ in $\mathcal{E}$, since ι_n preserves products. Using $P_n \iota_n = id$ and the reflection of $(d-1)$-connected maps of $P_n : \mathcal{E} \to \mathcal{E}^{\leq n}$ (Lemma 2.30), we deduce that each map $\iota_n X_k \to \iota_n Y_k$ is $(d-1)$-connected in $\mathcal{E}$. By assumption on $\mathcal{E}$, the map $\iota_n p : \iota_n X \to \iota_n Y$ is surjective in $\mathcal{E}$. Thus the map $P_n \iota_n p = p$ is surjective in $\mathcal{E}^{\leq n}$. □

A counterexample to the converse of Proposition 2.27 for $n = \infty$ will be given in Remark 4.3.

2.3 Dependent choice In this section, we introduce the stronger axiom of dependent choice. We denote by $\overleftarrow{\mathbb{N}}$ the poset $\cdots \to 2 \to 1 \to 0$. A diagram $X_\bullet : \overleftarrow{\mathbb{N}} \to \mathcal{E}$ is called a *tower*. The *composition* of the tower $X_\bullet$ is the map $\lim X_\bullet \to X_0$.

Definition 2.33 (Dependent choice)**.** Let $\mathcal{E}$ be an n-topos (for $1 \leq n \leq \infty$). For $-1 \leq d \leq \infty$, we say that the axiom of *dependent choice of dimension* $\leq d$ (DC_d) holds in $\mathcal{E}$ if the composition of a tower of $(d-1)$-connected maps is a surjection.

The condition DC_0 is called "repleteness" in [3, 15]. An argument similar to that of Lemma 2.2 would show that DC_d holds if and only if the limit of every tower of $(m+d)$-connected maps is an m-connected map for all $m \geq -2$. By reindexing the tower, the axiom DC_d holds if and only if, for every tower $X_\bullet$ of $(d-1)$-connected maps, all the maps $\lim X_\bullet \to X_k$ are surjections.

Lemma 2.34. *The axiom* DC_d *implies* CC_d.

Proof. For a countable family of $(d-1)$-connected maps $A_k \to B_k$, their product $A \to B$ is the limit of a tower of $(d-1)$-connected maps in $\mathcal{E}_{/B}$

$$\cdots \to (A_0 \times A_1 \times A_2) \times_{B_0 \times B_1 \times B_2} B \to (A_0 \times A_1) \times_{B_0 \times B_1} B \to A_0 \times_{B_0} B \to B\,. \qquad \square$$

Examples 2.35.
a) We always have $\mathsf{DC}_{-1} \Rightarrow \mathsf{DC}_0 \Rightarrow \mathsf{DC}_1 \Rightarrow \cdots \Rightarrow \mathsf{DC}_\infty$.
b) The axiom DC_{-1} says that the composition of an arbitrary tower is a surjection. This holds if and only if $\mathcal{E} = 1$.
c) The axiom DC_0 says that the composition of a tower of surjections is a surjection. In a 1-topos, this is equivalent to the usual statement of dependent choice expressed in the internal language [13, Proposition 3.35].
Jensen proved [9] that there is a model of Zermelo–Fraenkel set theory that satisfies countable choice but not dependent choice [8, Theorem 8.12]. An example of a 1-topos where CC_0 holds but not DC_0 is given in [13, Section 4.3].
d) The d-cube $[0,1]^d$ of Example 2.26 has enough objects of homotopy dimension $\leq d$ and satisfies DC_d by Proposition 2.37 below. However, it does not satisfy DC_{d-1}, since we saw in Example 2.26 that it does not satisfy CC_{d-1}.
e) An argument similar to that of Proposition 2.27 would show that if DC_d holds in an ∞-topos $\mathcal{E}$, then it holds in its hypercompletion $\mathcal{E}^{\leq \infty}$. The ∞-topos of Remark 4.3 gives a counterexample to the converse.

Question 2.36. For $d > 0$, does there exist an n-topos which satisfies CC_d, but not DC_d?

Proposition 2.37. *An n-topos with enough objects of homotopy dimension $\leq d$ satisfies* DC_d, *hence also* CC_d.

Proof. Let $\cdots \to A_1 \to A_0$ be a tower of $(d-1)$-connected maps. We want to show that the composition $\lim A_\bullet \to A_0$ is a surjection. By assumption on $\mathcal{E}$, there exists a covering family $(q_i : X_i \to A_0)$ where the X_i are objects of homotopy dimension $\leq d$. For each i, using the fact that the maps $A_{k+1} \to A_k$ are $(d-1)$-connected and Lemma 2.5, we can construct for every $k \geq 0$ in turn a lift $X_i \to A_{k+1}$ of the previous map $X_i \to A_k$, and therefore a map $X_i \to \lim A_\bullet$ whose composition with $\lim A_\bullet \to A_0$ is the original map q_i. Since the map $\coprod X_i \to A_0$ is a surjection and admits a factorization $\coprod X_i \to \lim A_\bullet \to A_0$, the map $\lim A_\bullet \to A_0$ must be a surjection (see [10, Corollary 6.2.3.12 (2)]). The last assertion is Lemma 2.34. □

Example 2.38. The pro-étale 1-topoi have enough objects of homotopy dimension ≤ 0 and therefore satisfy DC_0 [3, Proposition 4.2.8]. In particular, the 1-topos of condensed sets satisfies DC_0. In these examples, we assume that the sites of definition are cut off at an uncountable strong limit cardinal, as described in [3, Remark 4.1.2], so as to obtain a Grothendieck topos with enough projective objects.

The *light* condensed sets [4, Video 2/24] form the 1-topos of sheaves on the category of light profinite sets, those with only countably many clopen subsets, for the topology generated by finite jointly surjective families. This 1-topos does not have enough objects of homotopy dimension ≤ 0, but still satisfies DC_0 [4, Video 3/24, 49'].

3. Proofs of the main theorems

This section will use the terminology and notations of Section A, which the reader is advised to read first.

Theorems 1.1 and 1.2 will follow from the following lemmas. We fix an ∞-topos $\mathcal{E}$. Recall that an n-category K is of homotopy $\mathcal{E}$-dimension $\leq d$ if it satisfies the conditions of Lemma 2.8.

Lemma 3.1. *If $\mathbb{N}$ is of homotopy $\mathcal{E}$-dimension $\leq d$, then the poset $\overleftarrow{\mathbb{N}}$ is of homotopy $\mathcal{E}$-dimension $\leq d+1$.*

Proof. By Lemma 2.8 (4) for $K = \mathbb{N}$, $\mathbb{N}$ is also of homotopy $\mathcal{E}_{/E}$-dimension $\leq d$ for every $E \in \mathcal{E}$. We will check Lemma 2.8 (7) for $K = \overleftarrow{\mathbb{N}}$. We show that, in any slice $\mathcal{E}_{/E}$, if countable products of $(m+d+1)$-connected objects are $(m+1)$-connected, then $\overleftarrow{\mathbb{N}}$-limits of $(m+d+1)$-connected objects are m-connected.

Recall that $\overleftarrow{\mathbb{N}}$ is the free category on the graph $s, i : \mathbb{N} \rightrightarrows \mathbb{N}$, where i is the identity and s the successor map. Then the limit of a diagram $\cdots \to X_1 \to X_0$ in $\mathcal{E}_{/E}$ is equivalent to the equalizer of the pair $s, i : \prod X_k \rightrightarrows \prod X_k$, where i is the identity map and s is induced by the maps $X_{k+1} \to X_k$. Equivalently, this equalizer is the pullback of the diagram

$$\begin{array}{ccc} \lim X_k & \longrightarrow & \prod X_k \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \Delta} \\ \prod X_k & \xrightarrow{(s,i)} & \prod X_k \times \prod X_k \,. \end{array}$$

When each X_k is an $(m+d+1)$-connected object, their product $\prod X_k$ is an $(m+1)$-connected object by assumption on $\mathbb{N}$, and the diagonal of $\prod X_k$ is an m-connected map. The map $\lim X_k \to \prod X_k$ is m-connected by base change, so the map $\lim X_k \to 1$ is m-connected by composition. □

Recall that we denote the Postnikov tower of an object X by $\cdots \to X^{\leq 1} \to X^{\leq 0}$.

Lemma 3.2. *If $\overleftarrow{\mathbb{N}}$ has finite homotopy $\mathcal{E}$-dimension, then, for every object X in $\mathcal{E}$, the map $X \to \lim X^{\leq k}$ is the hypercompletion of X.*

Proof. Let d be the homotopy $\mathcal{E}$-dimension of $\overleftarrow{\mathbb{N}}$. We put $Y = \lim X^{\leq k}$. For a fixed $n \geq d$, all the maps $X \to X^{\leq k+n}$ are n-connected. The map $(X \to Y) = \lim (X \to X^{\leq k+n})$ is $(n-d)$-connected by assumption on $\overleftarrow{\mathbb{N}}$. Since n was arbitrary, this shows the map is ∞-connected. Let $\mathcal{E}^{\leq \infty} \subseteq \mathcal{E}$ be the hypercompletion of $\mathcal{E}$. All truncated objects are in $\mathcal{E}^{\leq \infty}$ and $\mathcal{E}^{\leq \infty}$ is closed under limits, so $Y = \lim X^{\leq k}$ is in $\mathcal{E}^{\leq \infty}$. This shows that the map $X \to Y$ is the hypercompletion of X. □

Remark 3.3. Putting together Lemmas 3.1 and 3.2, we see that CC_d implies that the hypercompletion of an object is given by the limit of its Postnikov tower. This is a result interesting on its own. It is equivalent to the convergence of Postnikov towers in $\mathcal{E}^{\leq \infty}$, which is apparently weaker than the Postnikov completeness of $\mathcal{E}^{\leq \infty}$ (i.e. weaker than Theorem 1.1) though we do not know any counterexample (see Section A). The following lemma shows that CC_d implies in fact a stronger result, of which Theorem 1.1 is a mere reformulation.

Lemma 3.4. *If $\overleftarrow{\mathbb{N}}$ has finite homotopy $\mathcal{E}$-dimension, then every Postnikov tower in $\mathcal{E}$ is effective.*

Proof. Suppose $\overleftarrow{\mathbb{N}}$ has homotopy $\mathcal{E}$-dimension $\leq d$, with $d < \infty$. Let $\cdots \to X_1 \to X_0$ be a formal Postnikov tower in $\mathcal{E}$, and put $X = \lim X_k$. For a fixed $n \geq d$, all the maps $X_{k+n} \to X_n$ are n-connected. The map $(X \to X_n) = \lim (X_{k+n} \to X_n)$ is $(n-d)$-connected by assumption on $\overleftarrow{\mathbb{N}}$. In particular, it induces an equivalence $X^{\leq n-d} = (X_n)^{\leq n-d} = X_{n-d}$. Since n was arbitrary, this shows the tower is effective. □

Remark 3.5. Putting together Lemmas 3.1 and 3.4, we deduce that "if CC_d holds for some finite d, then Postnikov towers are effective". This statement can be formulated and proven in homotopy type theory in essentially the same way. An implementation in Cubical Agda has been done by Owen Milner [14].

We can now prove the two results of the introduction.

Proof of Theorem 1.1. Let $\mathcal{E}$ be an ∞-topos. The Postnikov completion $(-)^{\leq \bullet} : \mathcal{E} \to \mathcal{P}\mathrm{ost}(\mathcal{E})$ and the hypercompletion $P_\infty : \mathcal{E} \to \mathcal{E}^{\leq \infty}$ invert the same class of maps, namely the ∞-connected maps. If all Postnikov towers are effective and so $(-)^{\leq \bullet} : \mathcal{E} \to \mathcal{P}\mathrm{ost}(\mathcal{E})$ is a localization, then the canonical morphism $\mathcal{E}^{\leq \infty} \to \mathcal{P}\mathrm{ost}(\mathcal{E})$ is an equivalence. When $\mathcal{E}$ satisfies CC_d for some finite d, this holds by Lemmas 3.1 and 3.4. □

Remark 3.6. By Theorem 1.1 and Proposition 2.27, if CC_d holds in $\mathcal{E}$, it holds also in $\mathcal{P}\mathrm{ost}(\mathcal{E})$.

Remark 3.7. Recall from Example 2.20 (f) that every hypercomplete topos satisfies CC_∞. Since there exist hypercomplete topoi which are not Postnikov complete, this shows that Theorem 1.1 is false for $d = \infty$.

Remark 3.8. Recall from [11, Corollary A.7.2.6] that the subcategory of Postnikov complete ∞-topoi is closed under colimits of geometric morphisms. The ∞-topos of sheaves on the cube $[0,1]^d$ is Postnikov complete because it is locally of homotopy dimension $\leq d$ [10, Proposition 7.2.1.10]. The ∞-topos $\coprod_d [0,1]^d$ of Example 2.20 (h) is Postnikov complete as a colimit of Postnikov complete ∞-topoi. This shows that the converse of Theorem 1.1 is false. Another counterexample is given by the ∞-topos of Proposition 4.2.

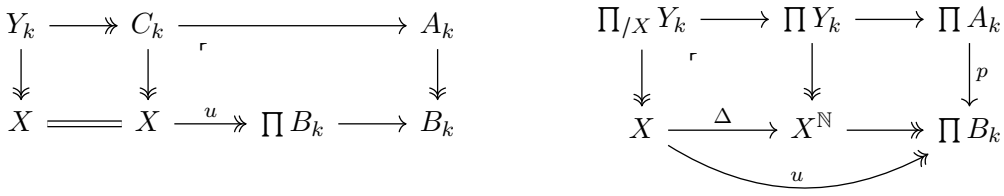
Remark 3.9. By Example 2.20 (b), the conclusion of Theorem 1.1 holds for all slices of $\mathcal{E}$. We do not know if the slices of Postnikov complete topoi are always Postnikov complete.

We can apply Theorem 1.1 to get a proof of [10, Proposition 7.2.1.10].

Corollary 3.10. *Any ∞-topos locally of homotopy dimension $\leq d$ is Postnikov complete.*

Proof. Let $\mathcal{E}$ be such an ∞-topos. By Lemma 2.16 we know that $\mathcal{E}$ is hypercomplete. By Proposition 2.25 and Theorem 1.1, $\mathcal{E}$ is in fact Postnikov complete. □

Proof of Theorem 1.2. Let $\mathcal{E}_1$ be a 1-topos satisfying CC_0; we must show its ∞-topos envelope $\mathcal{E}$ also satisfies CC_0. Let $(A_k \to B_k)$ be a countable family of surjections in $\mathcal{E}$. We want to show that $p : \prod A_k \to \prod B_k$ is a surjection. By construction of $\mathcal{E}$, there exists a surjection $X \to \prod B_k$ by some object X in $\mathcal{E}_1$. We put $C_k = X \times_{B_k} A_k$. The map $C_k \to X$ is a surjection by base change. Let $Y_k \to C_k$ be a cover by an object of $\mathcal{E}_1$.



The composite map $Y_k \to C_k \to X$ is a surjection in $\mathcal{E}$ between objects of $\mathcal{E}_1$. It is also a surjection in $\mathcal{E}_1$ by Lemma 2.13. The object $\prod_{/X} Y_k$ is in $\mathcal{E}_1$ since discrete objects are closed under limits. By the hypothesis on $\mathcal{E}_1$, the product map $\prod Y_k \to X^{\mathbb{N}}$ is a surjection in $\mathcal{E}_1$, and so is the map $\prod_{/X} Y_k \to X$ by base change along $X \to X^{\mathbb{N}}$. It is also a surjection in $\mathcal{E}$, since the embedding $\mathcal{E}_1 \to \mathcal{E}$ preserves surjections. The map $\prod_{/X} Y_k \to X \to \prod B_k$ is a surjection as a composition of surjections. Since this composition factors through $p : \prod A_k \to \prod B_k$, p is also a surjection. □

Remark 3.11. Using a similar argument, it is possible to show that if $\mathcal{E}_1$ is a 1-topos where DC_0 holds, then DC_0 also holds in its ∞-topos envelope $\mathcal{E}$.

Question 3.12. Let $\mathcal{E}$ be the ∞-topos envelope of a $(d+1)$-topos $\mathcal{E}_d$ satisfying CC_d (or DC_d) for $d > 0$. Does $\mathcal{E}$ satisfy CC_d (or DC_d)?

By putting together Theorems 1.1 and 1.2 with Example 2.35 (c), we get a proof of [15, Theorem A].

Corollary 3.13. *If a 1-topos $\mathcal{E}_1$ satisfies CC_0 (a fortiori DC_0), then the hypercompletion of the ∞-topos envelope of $\mathcal{E}_1$ is Postnikov complete.*

4. A topos without CC_∞

This section gives an example of an ∞-topos where the axiom CC_d fails for every $d \leq \infty$ (Proposition 4.2). As a technical step, we start by giving an example of an ∞-connected object with no global section. We shall use the object introduced in [5, Example A.10], but following the exposition of [18, 11.3]. We denote by $\mathcal{R}$ the ∞-topos constructed there and by R the distinguished ∞-connected object. (In the notations of [18, 11.3], $\mathcal{R} = \mathbf{E}$ and $R = aX$.) Rezk proves that the space of global sections of R is (noncanonically) equivalent to the zeroth space of the sphere

spectrum, $\Gamma(R) \simeq QS^0$. Let $a, b : 1 \to R$ be two global sections in the classes of 0 and 1 in $\pi_0(\Gamma(R)) = \mathbb{Z}$. We denote by $\Omega_{a,b}R$ the corresponding path object (fiber product) in $\mathcal{R}$. Because R is ∞-connected, so is $\Omega_{a,b}R$, but its space of global sections is now empty:

$$\Gamma(\Omega_{a,b}R) \;=\; \Omega_{0,1}\Gamma(R) \;=\; \Omega_{0,1}QS^0 \;=\; \varnothing\,.$$

More generally, such ∞-connected objects can be built from any non-trivial ∞-connected object. Let $\mathcal{E}$ be an ∞-topos with a non-trivial ∞-connected object E. There exists $X \in \mathcal{E}$ such that $\mathrm{Map}(X, E)$ is not contractible. By replacing $\mathcal{E}$ by $\mathcal{E}_{/X}$ and E by $E \times X \to X$, we may assume $X = 1$. If the space of global sections $\Gamma(E)$ is already empty, then E is the sought object. Otherwise, choose a global section to serve as basepoint of E. There exists an $0 \le n < \infty$ such that $\Gamma(E)$ is $(n-1)$-connected but not n-connected. In other words, there exist two pointed maps $a, b : S^n \to E$ which lie in distinct components of $\Gamma(\Omega^n E)$. The object we are looking for is the path space $\Omega_{a,b}(\Omega^n E)$. (This is really Lemma 2.5 (2) $\Rightarrow$ (4) with $m = d = \infty$, $X = 1$.)

Let $\mathcal{F}\mathrm{in} \subseteq \mathcal{S}$ be the full subcategory of finite spaces (the spaces that can be obtained from the point by finite colimits). The algebraically free ∞-topos on one generator is $\mathcal{S}[X] = \mathrm{Fun}\,(\mathcal{F}\mathrm{in}, \mathcal{S})$ and the universal object X is the canonical inclusion $\mathcal{F}\mathrm{in} \to \mathcal{S}$ [2, Definition 2.2.6]. For any ∞-topos $\mathcal{E}$, the category of algebraic morphisms $\mathcal{S}[X] \to \mathcal{E}$ is equivalent to $\mathcal{E}$. There exists a topological localization $\mathcal{S}[X] \to \mathcal{S}[X^{(\infty)}]$ forcing the universal object X to become ∞-connected [2, Example 3.1.3 (c) and Example 4.1.6 (d)]. This means that for any ∞-topos $\mathcal{E}$, the category of algebraic morphisms $\mathcal{S}[X^{(\infty)}] \to \mathcal{E}$ is equivalent to the full subcategory of $\mathcal{E}$ spanned by the ∞-connected objects.

Lemma 4.1. *The universal ∞-connected object $X^{(\infty)} : \mathcal{F}\mathrm{in} \to \mathcal{S}$ satisfies $X^{(\infty)}(\varnothing) = \varnothing$.*

Proof. The value $X^{(\infty)}(\varnothing)$ is the space of global sections, since $\varnothing$ is terminal as an object of $\mathcal{F}\mathrm{in}^{op}$. If this space was non-empty, every ∞-connected object of every ∞-topos would have at least one global section. Since we have seen that there exist ∞-connected objects with no global sections, we must have $X^{(\infty)}(\varnothing) = \varnothing$. □

Let $\mathcal{F}\mathrm{in}^{(\mathbb{N})} \subseteq \mathcal{F}\mathrm{in}^{\mathbb{N}}$ be the full subcategory of sequences of finite spaces whose values at all but finitely many indices are the empty space. The algebraically free ∞-topos $\mathcal{S}[X_0, X_1, \dots]$ on a countable set of generators is $\mathrm{Fun}\left(\mathcal{F}\mathrm{in}^{(\mathbb{N})}, \mathcal{S}\right)$. We consider its localization $\mathcal{E} = \mathcal{S}[X_0^{(\infty)}, X_1^{(\infty)}, \dots]$ forcing all the X_k to become ∞-connected. The object $X_k^{(\infty)}$ is the functor $X^{(\infty)} \circ p_k : \mathcal{F}\mathrm{in}^{(\mathbb{N})} \to \mathcal{F}\mathrm{in} \to \mathcal{S}$ where p_k is the projection to the kth component.

Proposition 4.2. *The axiom CC_∞ (and thus any CC_d or DC_d) is false in $\mathcal{E}$.*

Proof. In fact, we show that the countable product $\prod X_k^{(\infty)}$ is the initial object of $\mathcal{E}$. This product can be computed in the category of functors $\mathcal{F}\mathrm{in}^{(\mathbb{N})} \to \mathcal{S}$. Let $A_\bullet$ be an object in $\mathcal{F}\mathrm{in}^{(\mathbb{N})}$. There exists some index n such that $A_n = \varnothing$, so $(\prod X_k^{(\infty)})(A_\bullet) = \prod X^{(\infty)}(A_k) = \varnothing$. Since $A_\bullet$ was arbitrary, this shows that the functor $\prod X_k^{(\infty)} : \mathcal{F}\mathrm{in}^{(\mathbb{N})} \to \mathcal{S}$ is the initial functor, so it must also be the initial object of $\mathcal{E}$. □

Remark 4.3. The hypercompletion of $\mathcal{S}[X^{(\infty)}]$ is $\mathcal{S}$: the algebraic morphism $\mathcal{S}[X^{(\infty)}] \to \mathcal{S}$ sending $X^{(\infty)}$ to 1 is a cotopological localization since it is generated by inverting the ∞-connected map $X^{(\infty)} \to 1$, and because $\mathcal{S}$ is hypercomplete, it must be the maximal cotopological localization, that is, the hypercompletion of $\mathcal{S}[X^{(\infty)}]$. A similar argument shows that the hypercompletion of $\mathcal{E}$ is also $\mathcal{S}$. Since $\mathcal{S}$ is Postnikov complete, this shows that the converse of Theorem 1.1 is false. This also provides a counterexample to the converse of Proposition 2.27.

Appendix A: Postnikov separation, convergence, effectivity, and completeness

This appendix summarizes the various conditions of convergence on Postnikov towers and introduces some terminology. The original material is in [10, Section 5.5.6].

The conventions on truncation are at the beginning of the paper. We fix a presentable category $\mathcal{C}$. The *Postnikov completion* of $\mathcal{C}$ is the limit $\mathcal{P}\mathrm{ost}(\mathcal{C})$ of the tower $\cdots \to \mathcal{C}^{\leq 1} \to \mathcal{C}^{\leq 0}$, computed in the category of categories, with transition functors given by truncation. The category $\mathcal{P}\mathrm{ost}(\mathcal{C})$ is presentable. Its objects are towers $X_\bullet : \overleftarrow{\mathbb{N}} \to \mathcal{C}$ such that the map $X_{k+1} \to X_k$ identifies X_k with the k-truncation of X_{k+1}. We shall call such a tower a *formal Postnikov tower* ([10] calls it a Postnikov pretower).

The *Postnikov tower* $X^{\leq \bullet}$ of an object X is defined as the tower of truncations $\cdots \to X^{\leq 1} \to X^{\leq 0}$. This defines a functor $(-)^{\leq \bullet} : \mathcal{C} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$ whose right adjoint sends a tower to its limit. For any object X in $\mathcal{C}$, the unit of the adjunction $(-)^{\leq \bullet} : \mathcal{C} \rightleftarrows \mathcal{P}\mathrm{ost}(\mathcal{C}) : \lim$ is the canonical map $X \to \lim_n X^{\leq n}$, while for any formal Postnikov tower $X_\bullet$ with limit X, the counit is the canonical morphism of towers $X^{\leq \bullet} \to X_\bullet$.

A map in $\mathcal{C}$ is a *Postnikov equivalence* if it is inverted by $\mathcal{C} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$. We shall say that an object is ∞*-truncated* if it is local for Postnikov equivalences. The *Postnikov separation* of $\mathcal{C}$ is defined as the localization $\mathcal{C} \to \mathcal{C}^{\leq \infty}$ with respect to Postnikov equivalences. The category $\mathcal{C}^{\leq \infty}$ is equivalent to the full subcategory spanned by ∞-truncated objects. This gives a factorization

$$\mathcal{C} \longrightarrow \mathcal{C}^{\leq \infty} \longrightarrow \mathcal{P}\mathrm{ost}(\mathcal{C}) = \lim \mathcal{C}^{\leq n},$$

where the first functor is a localization and the second functor is conservative. The truncations of an object X and their limit fit into a diagram

$$X \longrightarrow X^{\leq \infty} \longrightarrow \lim_n X^{\leq n} \longrightarrow \ldots \longrightarrow X^{\leq 1} \longrightarrow X^{\leq 0} \longrightarrow X^{\leq -1} \longrightarrow X^{\leq -2} = 1.$$

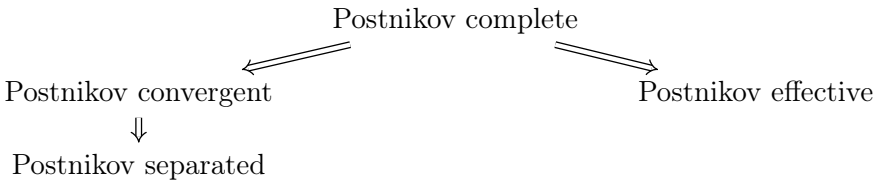
Remark A.1. The factorization $\mathcal{C} \to \mathcal{C}^{\leq \infty} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$ is typical of separation–completion processes. Other examples include the separation–completion $A \to A/\bigcap I^n \to \lim A/I^n$ of a ring A at an ideal I, or the separation–completion $X \to X/(d = 0) \to \overline{X}$ of a pseudometric space (X, d). For this reason, we suggest replacing the terminology *hypercompletion* by *Postnikov separation*. A similar terminology is used for prestable categories in [11, Definition C.1.2.12].

Remark A.2. When $\mathcal{C} = \mathcal{E}$ is an ∞-topos, for every $-2 \leq n \leq \infty$, the localization $\mathcal{E} \to \mathcal{E}^{\leq n}$ is produced by inverting all n-connected maps. In particular, the Postnikov separation $\mathcal{E}^{\leq \infty}$ coincides with the hypercompletion of $\mathcal{E}$ and is an ∞-topos. By [11, Theorem A.7.2.4 and Proposition A.7.3.4], the category $\mathcal{P}\mathrm{ost}(\mathcal{E})$ is an ∞-topos and the functor $(-)^{\leq \bullet} : \mathcal{E} \to \mathcal{P}\mathrm{ost}(\mathcal{E})$ is an algebraic morphism of ∞-topoi. Moreover, the two functors of the factorization $\mathcal{E} \to \mathcal{E}^{\leq \infty} \to \mathcal{P}\mathrm{ost}(\mathcal{E})$ are algebraic morphisms of ∞-topoi.

Definition A.3.
1. We say an object $X \in \mathcal{C}$ *has a convergent Postnikov tower* if the unit map $X \to \lim_n X^{\leq n}$ is invertible, i.e., if X is the limit of its Postnikov tower.
2. We say a formal Postnikov tower $X_\bullet$ is *effective* if the counit map $(\lim_n X_n)^{\leq \bullet} \to X_\bullet$ is invertible, i.e., if $X_\bullet$ coincides with the Postnikov tower of its limit.

Definition A.4. A presentable category $\mathcal{C}$ is said to be
1. *Postnikov separated* if $\mathcal{C} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$ is conservative;

2. *Postnikov convergent* if $\mathcal{C} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$ is fully faithful;
3. *Postnikov effective* if $\mathcal{C} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$ is a localization (i.e. if $\lim : \mathcal{P}\mathrm{ost}(\mathcal{C}) \to \mathcal{C}$ is fully faithful);
4. *Postnikov complete* if $\mathcal{C} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$ is an equivalence.



Lemma A.5. *A presentable category $\mathcal{C}$ is:*

1. *Postnikov separated if and only if it is hypercomplete ($X = X^{\leq\infty}$, for every object X);*
2. *Postnikov convergent if and only if every object has a convergent Postnikov tower ($X = \lim X^{\leq n}$, for every object X);*
3. *Postnikov effective if and only if every formal Postnikov tower is effective, if and only if $\mathcal{C}^{\leq\infty}$ is Postnikov effective, if and only if $\mathcal{C}^{\leq\infty}$ is Postnikov complete, if and only if $\mathcal{C}^{\leq\infty} = \mathcal{P}\mathrm{ost}(\mathcal{C})$;*
4. *Postnikov complete if and only if it is Postnikov convergent and Postnikov effective, if and only if it is Postnikov separated and Postnikov effective.*

Proof. Most of the statements are reformulations of Definitions A.3 and A.4. We only give an argument for (4). If formal Postnikov towers are effective, the functor $\mathcal{C} \to \mathcal{P}\mathrm{ost}(\mathcal{C})$ is a localization and is therefore an equivalence if and only if it is fully faithful or conservative. □

Remark A.6. The first equivalence of Lemma A.5 (4) is Lurie's criteria for Postnikov completeness in [10, Proposition 5.5.6.26]. With the notations therein, the implication (1) ⇒ (2) is what we call "Postnikov convergent", and the implication (2) ⇒ (1) is what we call "Postnikov effective".

Remark A.7. Lemma A.5 (3) implies in particular that if a category $\mathcal{C}$ is Postnikov effective, the hypercompletion of any object X is the limit of its Postnikov tower: $X^{\leq\infty} = \lim X^{\leq\bullet}$.

Remark A.8. An example of an ∞-topos which is hypercomplete (Postnikov separated) but not Postnikov convergent is detailed in [15, Example 3.17]. By Lemma A.5 (4), this is also an example of an ∞-topos which is not Postnikov effective. An example of an ∞-topos which is Postnikov effective but not hypercomplete is the ∞-topos $\mathcal{S}[X^{(\infty)}]$ classifying ∞-connected objects (see Remark 4.3). We do not know an example of an ∞-topos or even a presentable category which is Postnikov convergent but not Postnikov effective.

Acknowledgements

The authors thank Mike Shulman for his comments on an earlier draft of the paper. This material is based upon work supported by the Air Force Office of Scientific Research under award numbers FA9550-21-1-0009 and FA9550-23-1-0434.

References

[1] M. Anel, G. Biedermann, E. Finster, and A. Joyal, *A generalized Blakers–Massey theorem*, Journal of Topology **13** (2020), no. 4, 1521–1553, doi:10.1112/topo.12163.

[2] M. Anel, G. Biedermann, E. Finster, and A. Joyal, *Left-exact localizations of ∞-topoi II: Grothendieck topologies*, Journal of Pure and Applied Algebra **228** (2024), no. 3, 107472, doi:10.1016/j.jpaa.2023.107472.

[3] B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2015), 99–201.

[4] D. Clausen and P. Scholze, *Analytic Stacks*, Lecture Course (2023–2024). Available on YouTube.

[5] D. Dugger, S. Hollander, and D. C. Isaksen, *Hypercovers and simplicial presheaves*, Mathematical Proceedings of the Cambridge Philosophical Society **136** (2004), no. 1, 9–51.

[6] R. Engelking, *Dimension theory*, Mathematical Studies, North-Holland Publishing Company, 1978.

[7] The Univalent Foundations Program, *Homotopy type theory: Univalent foundations of mathematics*, Institute for Advanced Study, 2013, on the HoTT website.

[8] T. J. Jech, *The axiom of choice*, Studies in Logic and the Foundations of Mathematics, Vol. 75, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[9] R. B. Jensen, *Consistency results for ZF*, Notices of the Amer. Math. Soc. **14** (1967), no. 1, 137.

[10] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[11] J. Lurie, *Spectral algebraic geometry*, version of February 3, 2018, Online book.

[12] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Universitext, Springer New York, 1994.

[13] S. Mejak, *Topos models of set-theoretic principles*, Master's thesis, University of Ljubljana, 2019. https://www.researchgate.net/publication/334737639_TOPOS_MODELS_OF_SET-THEORETIC_PRINCIPLES

[14] O. Milner, https://github.com/owen-milner/choicepostnikov

[15] S. Mondal and E. Reinecke, *On Postnikov completeness for replete topoi*, arXiv:2210.14146.

[16] nLab authors, *Elementary (∞,1)-topos*, *n*Lab page, June 2021.

[17] N. Rasekh, *A Theory of Elementary Higher Toposes*, arXiv:1805.03805.

[18] C. Rezk, *Toposes and Homotopy Toposes*, Note, available on Rezk's homepage, 2010.

[19] J.-E. Roos, *Derived Functors of Inverse Limits Revisited*, Journal of the London Mathematical Society **73** (2006), no. 1, 65–83.