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Borel and weakly Borel sets in Banach spaces

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The following interesting theorem was proved by Edgar.

Theorem: If a Banach space B admits an equivalent locally uniformly rotund norm then weak and strong Borel sets in B coincide.

Since it does not seem to be generally known it may be worthwhile to sketch here its proof. For $A \subset B$ define $\alpha(A) = \inf\{\varepsilon > 0 ; x \in A \Rightarrow B(x,\varepsilon) \cap A$ is a weak neighbourhood of x in $A\}$. If, for any natural n, the space B can be covered by countably many weakly Borel sets A_i^n with $\alpha(A_i^n) < \frac{1}{n}$ then any norm closed set F equals $\bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} A_i^n \cap \overline{(F \cap A_i^n)}^w$.

Suppose now that B has a LUR norm; it implies that on any sphere weak and strong topology coincide. Let $S(u,v) = \{x ; u < \|x\| \le v\}$ (u, v rational numbers) and let $S^\varepsilon(u,v)$ be the union of all weakly open subsets of $S(u,v)$ with diameter less then ε. Since $\alpha(S^\varepsilon(u,v)) < \varepsilon$, it is sufficient to prove that for each $\varepsilon > 0$ the sets $S^\varepsilon(u,v)$ cover B. If $x \in B$ then $B(x,\frac{\varepsilon}{4}) \supset \{y ; \|y\| = \|x\| , |\langle x_i', (y-x)\rangle| < \delta\}$ for some $x_i' \in B'$, $\|x_i'\| = 1$, $\delta > 0$. Choosing $u < \|x\| < v$ sufficiently close to $\|x\|$ we find that $B(x,\frac{\varepsilon}{2})$ is a weak neighbourhood of x in $S(u,v)$, thus $x \in S^\varepsilon(u,v)$.

Let us remark that B admits a LUR norm if it is weakly compactly generated or, more generally, weakly analytic (as was recently shown by Vašák). Let us also remark that recently Talagrand proved that in ℓ^∞ weakly and strongly Borel sets do not coincide.