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Some remarks on Caratheodory construction of measures in metric spaces

### U. Peiste, Greifswald

If we have a metric space  $\mathcal{X} = (X, \mathbf{e})$  and A<X, then by Halmos (2) p.53

 $H^{p}(A) := \sup \inf \{ \sum_{i=1}^{n} (A_{i}) / A_{i} \in R(X) \land \bigcup_{i=1}^{n} A_{i} \land A \circ (A_{i}) \notin \ell \}$  is called the p-dimensional Hausdorff measure of A, where  $p \in R_{i}(0), R(X)$  is the set of all subsets of X and o(B) denotes the diameter of BCX. General considerations on such a definition are given in the book of Federer [7] 169-171. I will start with these considerations. Let o(B) be a family of subsets of X and o(B) a function on o(B). A sequence o(B) is called an allowed o(B) covering of A with respect to o(B), iff

- 1. F. 67 for all ien
- 2. 💆 F<sub>i</sub> > A
- 3. 8(F1) SE

If we define

 $i_{\ell}(A) = \inf \left\{ \sum_{i=1}^{n} 5(F_i) / (F_i)_{i \in \mathbb{N}} \text{ is allowed } \ell\text{-covering of } A \text{ with respect to } \mathcal{F} \right\}$ 

so we obtain

- a)  $i_{\ell}(A) \ge i_{\ell}(A)$  for  $\ell \le \ell'$
- b)  $i_{\ell}(A \vee B) = i_{\ell}(A) + i_{\ell}(B)$  whenever  $f(A,B) \geq 2 \ell > 0$ The validity of a? is obviously.

To b7: Let  $(F_1^A)_{1\in\mathbb{N}}$  and  $(F_1^B)_{1\in\mathbb{N}}$  be allowed  $\boldsymbol{\mathcal{E}}$ -coverings of A,B respectively, then  $F_1^A$ ,  $F_1^B$ ,  $F_2^A$ ,  $F_2^B$ , ... is an allowed  $\boldsymbol{\mathcal{E}}$ -covering of A $\boldsymbol{\mathcal{U}}$ B. Hence it holds

(\*)  $i_{\varepsilon}(AvB) \leq i_{\varepsilon}(A) \neq i_{\varepsilon}(B)$ 

On the other hand let be  $g(A,B) > 2\mathcal{E}$ , then every allowed  $\mathcal{E}$ -covering  $(F_i^{A \cup B})_{i \in \mathbb{N}}$  of AuB consists of tow disjoint allowed  $\mathcal{E}$ -coverings  $(F_i^A)_{i \in \mathbb{N}}$  and  $(F_i^B)_{i \in \mathbb{N}}$  of A,B respectively, that means

(xx)  $i_{\xi}(A \cup B) \ge i_{\xi}(A) + i_{\xi}(B)$  whehever  $g(A,B) > 2 \ell$ .

- (\*) and (\*\*) is the proof for b).
- a) implies

$$\psi(A) = \lim_{\ell \to C^*} i_{\ell}(A) = \sup_{\ell \to C} i_{\ell}(A)$$
 for all  $A \in X$ 

 $\psi: \mathcal{R}(X) \longrightarrow \overline{R}$  is an outer measure (i.e.  $0 \le \psi(A) \le \infty$ ,  $\psi(\emptyset) = 0$ ,  $\psi(A) \le \psi(B)$  for A<B,  $\psi(\bigcup_{i=1}^{N} A_i) \le \sum_{i=1}^{N} \psi(A_i)$ )

c)  $\psi: \mathcal{R}(X) \longrightarrow \overline{R}$  is a metric outer measure,

i.e.  $\psi(A \cup B) = \psi(A) + \psi(B)$  whenever f(A, B) > 0

The proof of c) is a conclusion of b), namely g(A,B) > 0 implies the existence of G>0 such that g(A,B) > G. Then we get  $i_{\mathcal{E}}(A \cup B) - i_{\mathcal{E}}(A) + i_{\mathcal{E}}(B)$  for all  $\mathcal{E} < \mathcal{E}$ , and that means  $g(A \cup B) = g(A) + g(B)$  by definition of g.

Let us denote by  $\mathcal{A}_{\xi}$  the  $\mathfrak{T}$ -field of  $\mathfrak{T}$ -measurable sets (AeX is called  $\mathfrak{T}$ -measurable, iff  $\mathfrak{T}(E) = \mathfrak{T}(EnA) + \mathfrak{T}(EnA')$  for all EcX). For every metric outer measure  $\mathfrak{P}: \mathfrak{F}(X) \longrightarrow \overline{R}_{\xi}$  holds the following

Lemma: (Federer [1] p.75, Halmos [2] p.48)

Let  $\phi: \mathbf{g}(X) \longrightarrow \overline{\mathbf{g}}$  be an outer measure on X, then  $\mathcal{A}_{\phi} > \mathcal{B}(X)$  Iff  $\phi$  is a metric outer measure, where  $\mathbf{B}(X)$  denotes the  $\mathbf{G}$ -field of Borelsets of X.

By this lemma it holds  $A_{\mathfrak{P}}$   $\mathfrak{F}(X)$ .  $\mathfrak{P}:A_{\mathfrak{P}}\longrightarrow \overline{\mathbb{R}}$ , is called the Caratheodory measure on X with respect to F and  $F:F\longrightarrow \overline{\mathbb{R}}_{\bullet}$ .

# Examples for Caratheodory measures:

- 1)  $5(F) := \delta(F)$  for all  $F \in \mathcal{F}$ 
  - a)  $\mathcal{F} \{X\}$ If  $\{X\} = 1$ , then  $\{x \in \mathbb{N}\}$ . In the case  $\{X\} = 1$  it holds  $\{x \in \mathbb{N}\}$  for all A < X
  - b)  $\mathbf{F} = \{F/ \text{ FcXa } |F| = 1\}$ , then

$$\psi(A) = \begin{cases} 0 & \text{for } |A| \leq 2, \\ \infty & \text{for } |A| > \infty. \end{cases}$$

- c)  $X = \{x/x=0 \ v \ x=\frac{1}{n}, n \in \mathbb{N} \}, g(a,b) = |a-b|, \overline{f} = \{F/F \in X \land |F| \ge 2\}$ then  $\psi(A) = \begin{cases} 0 & \text{for } A = \{0\} \\ \infty & \text{otherwise} \end{cases}$
- d) (X,5) = (R,5),  $\mathcal{F}=\{F/F=\{a,b(Aa,beR\}\}$ , then  $\Psi$  is the Lebesgue measure on R.
- 2) 5(F) = S(F) For all FeF and peRe 101
  - a)  $\mathcal{F} = \mathcal{R}(\mathbf{X})$ , Then  $\mathbf{\Psi}$  corresponds to the p-dim. Hausdorff measure  $\mathbf{H}^{\mathbf{p}}$ .

The 1-dim. Hausdorff measure, the 1-dim. spherical measure and the set of between points

Let us start with the definition of betneen points

 $x \in X$  is called between  $a, b \in X$ ,  $a \neq x$ ,  $b \neq x$ , iff g(a,b) = g(a,x) + g(x,b).

Let B(a,b) be the set if all between points of  $a,b \in X$  and  $B^{*}(a,b) = B(a,b) \cup \{a,b\}$ , then it holds for the reals  $g(a,b) = \sigma(B^{*}(a,b)) = R(B^{*}(a,b)) = H^{1}(B(a,b)) = S^{1}(B(a,b))$ , where  $a,b \in R$ , g denotes the euclidian metric on the reals, R the 1-dim. Lebesgue measure and R the 1-dim. spherical measure. In my lecture in Warnemünde (in autumn 1977) "A special property of 1-dimensional Hausdorff measure". I asked for the validity of the equation

 $g(a,b) = G(B(a,b)) = H^{1}(B(a,b))$  in an arbitrary metric space X. The main result was the following 1 Theorem.

- Let (X, ) be a complete and convex metric space (convex in he sense of Menger) and a,b X, hen the following conditions a e quivalent:
  - i  $g(a,b) g'(a,b) = H^{1}(B(,b))$ is posile to connec a d wi ui hort t rc.
  - 3 B (a b) is n arc, i.e ho eo orphic to 0 1].
    . nere is a unique metric segm nt (a b) c nne ting a and b.
- 5 If p,qeB\*(,b) with p+q, then peB\*(aq) o eB\*(q.b) Remarks:

# 1) An arc connecing a,b & X is a homeomorphism f: [0,1] into X such that f(0)=a and f(1)=b.

A shortest arc connecting  $a,b \in X$  is an arc f:[0,1] into X connecting a,b such that  $l(f) \le l(g)$  for all arcs g:[0,1] into X connecting a,b, where l(f) denotes the length of the arc f.

- 2) (a,b) is called a metric segm nt connecting a,b iff
  - 1. (a,b) < X and a,b €(a,b) s
  - 2. (a,b)<sub>s</sub> is congruent to an interval [x,y] ⊂ R, i.e. there is an intervall [x,y] and a metrical isomorphism f: [x,y] → (a,b)<sub>s</sub>, such that f(x)=a and f(y)=b.

Now let (X, | | | ) be a n-dimensional normed vector space and  $K_{\mathbf{r}}(\mathbf{x}) := \{y/y \in \mathbb{X} \land \|\mathbf{x}-y\| \le r\}$ 

the r-ball with the centre  $x \in X$ , then  $K_r(x)$  is a convex and symmetrical set. A point y Kn(x) is called an extreme point of Kr(x), iff there is not a finite line gcK\_(x), containing y in the relative interior of g. For example let us consider R2 having the following two norms

- 1)  $\|x\|_{1} = (x_1^2 + x_2^2)^{\frac{3}{2}}$ , where  $x = (x_1, x_2)$ In this case every point  $y \in FrK_n(x)$  is an extreme point.
- 2)  $|x| := \sup \{|x_1|, |x_2|\}$ Then Frk,(x) contains exactly four extreme points.

We obtain the following theorem as a conclusion of the theorem above.

### 2. Theorem:

Let (X, 1) be a n-dimensional normed vector space, then the following two conditions are equivalent:

- 1.  $\varphi(x,y) = \delta(B^{k}(x,y)) = H^{1}(B^{k}(x,y))$  for each  $x,y \in X$
- 2. Every point y belonging to Frk (x) is an extreme point of  $K_n(x)$  for each  $x \in X$  and r > 0.

We suppose  $y_0 \in FrK_1(0)$  and  $y_0$  is not extreme point of  $K_1(0)$ , hence there is a finite line (a,b] < K1(0), such that y is the centre of [a,b], i.e.  $y_0 = \frac{4}{2}(a + b)$ . The definition  $c := a - y_0$  implies the following equations

$$| x | = | y_0 + c | = 1$$
  
 $| x | = | y_0 - c | = 1$ 



 $\|2y_0 - a\| = \|2y_0 - y_0 - c\| = \|y_0 - c\| = 1$ . On the other hand it holds: | 2y | = 2|y | = 2 . Hence the arcs [0,2y0] and [0,a]v[a,2y0] are two shortest arcs connecting 0 and 2yo. That is a contradiction to theorem 1 condition 2. (If  $y_0 \in FrK_p(x)$  and  $y_0$  is not extreme point, so we get a contradiction in the same way)

. On the supposition that 1. does not hold there are two points  $x,y \in X$  and tow shortest arcs  $f:[0,1] \longrightarrow X$ ,  $g:[0,1] \longrightarrow X$ connecting x and y, such that  $f \neq g$  and l(f)=l(g)=g(x,y). It is possible to find a,b & X , such that

$$a \in g([0,1]) \land a \notin f([0,1])$$

$$0 < \tilde{r} := 1([x,a]_g) = 1([x,b]_f) < g(x,y),$$

where l((x,a), denotes the length of g from x to a and

l([x,b]<sub>f</sub>) denotes the length of f from x to b. We ask for the distance  $g(x,y_0)$  and  $g(y,y_0)$ , where  $y_0 = \frac{1}{2}(a+b)$ .  $g(x,y_0) = |x-y_0| \le \frac{1}{2}(|x-a| + |x-b|) = \frac{1}{2}(|g(x,a) + g(x,b)) = \frac{1}{2}(|r+r|) = r$   $g(y,y_0) = |y-y_0| \le \frac{1}{2}(|g(y,a) + g(y,b)) = \frac{1}{2}(|g(x,y) - r| + g(x,y) - r) = e(x,y) - r$ 

In the case  $\rho(x,y_0) < \hat{r}$  we obtain for the length of the arc  $[x,y_0]v[y_0,y]$  connecting x and y

$$1([x,y_0]\cup[y_0,y])=1([x,y_0])+1([y_0,y])=\rho(x,y_0)+\rho(y_0,y)$$

$$<\tilde{r}+\rho(x,y)-\tilde{r}=\rho(x,y), \text{ i.e.}$$

 $1([x,y]u[y_0,y]) < g(x,y).$ 

That is impossible, hence  $f(x,y_0) = \tilde{r}$ . That means  $y_0$  belongs to  $FrK_{\tilde{r}}(x)$  and so we get a contradiction to 2.(because  $y_0$  is not extreme point).

Now we consider some relations between the Hausdorff measure and the spherical measure on a metric space  $\mathcal K$ . The following properties are well known (see Federer [1])

- 1.  $H^{p}(A) \leq S^{p}(A)$  for all  $A \subset X$  and  $p \in \mathbb{R} \setminus \{0\}$
- 2. If there is a real  $c \ge 1$  for every subset  $A \subset X$ , such that A is contained in a closed ball having the diameter smaller or equal  $c \cdot \mathcal{O}(A)$ , then  $S^p(A) \le c^p$   $H^p(A)$ . (c must be independent of A)
- c = 2 fulfils the condition above. We obtain such a real number c in the n-dim. Euclidian space  $\mathbb{R}^n$  by Jungs theorem: (Federer 2.10.41)

If  $A \subset \mathbb{R}^n$  and  $0 < \delta(A) < \infty$ , then A is contained in a unique closed ball with minimal diameter, which does not exceed  $(2n/n+1)^{\frac{1}{2}} \cdot \delta(A) .$ 

For example if we consider a equilateral triangle  $\Delta$  in  $\mathbb{R}^2$ , then the smallest closed ball containing  $\Delta$  has the diameter 2r, where  $r = \frac{1}{\sqrt{2}} \cdot \mathcal{O}(\Delta)$ 

3. If  $A \subset X$  is congruent to a closed interval (x,y), then  $S^{1}(A) = (x - y)$ .

Now it is easy to prove the following theorem.

## 3. Theorems

For every complete and convex metric space  $\mathcal{Z}$  holds:  $H^{1}(B^{*}(a,b)) = g(a,b) \iff S^{1}(B^{*}(a,b)) = g(a,b)$ 

for arbitrary a,b & X.

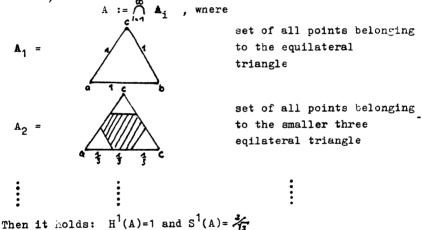
Proof:  $\Rightarrow$  By theorem 1  $H^1(B^*(a,b)) = g(a,b)$  implies:  $B^*(a,b)$  is

congruent to a closed interval [x,y] such that g(a,b)=|x-y|. Now we use property 3. and obtain  $S^1(B^x(a,b))=|x-y|=g(a,b)$ .

By property 1. we have  $H^1(B^*(a,b)) \leq S^1(B^*(a,b)) = g(a,b)$ . On the other hand it holds  $g(a,b) \leq H^1(B^*(a,b))$  in every complete and convex metric space.

#### Remarks:

- 17 This theorem implies the validity of theorem 1 for the 1-dim. spherical measure.
- 2) By theorem 3 it holds  $H^1(B^*(a,b)) = \rho(a,b)$  implies  $H^1(B^*(a,b)) = S^1(B^*(a,b))$  for every complete and convex metric space. But in general it does not hold  $H^1(A) = S^1(A)$  for  $A \subset X$  and  $(X, \rho)$  complete and convex metric space. For example: Let  $(X, \rho)$  be the Euclidian plain  $R^2$ . We define



# References:

- .[1] H. Federer: Geometric Measure Theory, Berlin 1969
  - [2] P. R. Halmos: Measure Theory, New York 1950
  - [3] W. Rinow: Die inner Geometrie der metrischen Räume, Berlin 1961