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On some problems in βN

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I have introduced a new type of infinite numbers.

Definition. Let E be a set and R a reflexive symmetric transitive relation between subsets of E denoted by .

$a_1 \sim_R a_2$ or $a_1 = a_2$ mod R (for $a_1, a_2 \subseteq E$). Let us denote

$a_1 \leq a_2$ mod R $\iff \exists\, a_2' \subseteq a_2$ such that $a_1 = a_2'$ mod R,

$a_1 < a_2$ mod R $\iff$ ($a_1 \leq a_2$ mod R and $a_1 \neq a_2$ mod R).

(E, R) is an ∞-number if

(i) $a_1, a_2 \subseteq E$, $|a_1|, |a_2| < \aleph_0 \Rightarrow (a_1 = a_2$ mod R $\iff |a_1| = |a_2|)$,

(ii) $a_1, a_2 \subseteq E \Rightarrow a_1 \leq a_2$ mod R or $a_1 \geq a_2$ mod R (comparability),

(iii) $a_1 \subset a_2$ (i.e. $a_1 \neq a_2$) $\Rightarrow a_1 < a_2$ mod R

- i.e. "<" is the finest relation possible - for $x \in a$, $a \setminus \{x\}$ is strictly smaller than a,

(iv) $a_1 \sim a_1'$, $a_2 \sim a_2'$, $a_1 \cap a_2 = a_1' \cap a_2' = \emptyset \Rightarrow (a_1 \cup a_2) \sim (a_1' \cup a_2')$.

Cardinal and ordinal numbers are not very fine measures with respect to the inclusion ($\omega_0 - n = \omega_0$) - ∞-numbers are the finest possible ones in this sense.

Definition. Let $\alpha_i = (E_i, R_i)$, i=1,2 be two ∞-numbers. Then

$(E_1,R_1) = (E_2,R_2)$ iff they are isomorphic,

$(E_1,R_1) \leq (E_2,R_2)$ iff $\exists\, E \subseteq E_2$: $(E_1,R_1) = (E, R_{2|E})$,

where $R_{2|E}$ denotes restriction of equivalence R_2 to subsets of E,

$(E_1,R_1)+(E_2,R_2)=(E,R)$ iff $\exists\, E = E_1' \cup E_2'$, $E_1' \cap E_2' = \emptyset$ such that

$$(E_i,R_i) = (E_i', R_{|E_i'}),\ i=1,2.$$

Basic properties of ∞-numbers we summarize in

Theorem 1. Let (E,R) be an ∞-number. Then the relation defined for $a_1, a_2 \subseteq E$ by $(a_1, R_{|a_1}) = (a_2, R_{|a_2})$ coincides with $a_1 \sim_R a_2$.

Especially $E_1 \subset E$ (i.e. $E_1 \neq E$) $\Rightarrow (E_1, R_{|E_1}) \lneq (E,R)$.

<u>Theorem 2</u>. $\alpha_1 + \alpha_2$ is uniquelly determined and $\alpha_1 + \alpha_2$ exists iff $\alpha_1 \gtrless \alpha_2$ ($\Leftrightarrow$ $\alpha_1 \leq \alpha_2$ or $\alpha_1 \geq \alpha_2$). Moreover $\sum_{x \in (E,R)} n_x$ may be uniquelly defined as ∞-number if $|n_x| \leq K$, $\forall x$, (K finite number).

In fact, there is a more subtile structure in (E,R).

<u>Definition</u>. $\mathcal{P} = \{$all partitions of $E\}$;

$\mathcal{P}_0 = \{P = \{P^\alpha\} \in \mathcal{P} \mid |P^\alpha| < \aleph_0, \ \forall \alpha\}$;

$\mathcal{P}_{00} = \{P = \{P^\alpha\} \mid |P^\alpha| > 1$ only for finite number of α's$\}$;

$P_1 \leq P_2 \Longleftrightarrow \forall P_1^{\alpha_1} \ \exists P_2^{\alpha_2} : P_1^{\alpha_1} \subseteq P_2^{\alpha_2}$;

$P_1 \vee P_2 = \inf \{P \mid P \geq P_1, \ P \geq P_2\}$;

$\mathcal{J} \subset \mathcal{P}_0$ is an ideal $\Longleftrightarrow$ ($P_1 \leq P \in \mathcal{J} \Rightarrow P_1 \in \mathcal{J}$; $P_1, P_2 \in \mathcal{J} \Rightarrow P_1 \vee P_2 \in \mathcal{J}$);

$\mathcal{F} = \{f : E \to \mathbb{Z}\}$, $\mathbb{Z}$ = integers ;

$\mathcal{F}_b = \{f : E \to \mathbb{Z} \mid f$ is bounded$\}$;

For $f \in \mathcal{F}$, $P \in \mathcal{P}_0$ we define $f_{/P}$ on $P = \{P^\alpha\}$ by $f_{/P}(P^\alpha) = \sum_{x \in P^\alpha} f(x)$.

<u>Theorem 3</u>. Let (E,R) be an ∞-number. Then there exists an ideal $\mathcal{J} \subset \mathcal{P}_0$, $\mathcal{J} \supset \mathcal{P}_{00}$ such that

(1) $\forall f \in \mathcal{F}_b \ \exists P \in \mathcal{J} : f_{/P} \geq 0$ or $f_{/P} \leq 0$ on P,

(2) $a_1 = a_2$ mod R $\Longleftrightarrow \exists P = \{P^\alpha\} \in \mathcal{J} : |P^\alpha \cap a_1| = |P^\alpha \cap a_2|$, $\forall \alpha$.

<u>Note</u>. Opposite is obvious - R is defined by (2) and (ii) follows from (1).

<u>Theorem 4</u>. Suppose that there is a selective ultrafilter in $\beta\mathcal{N}$ ($\mathcal{N}$ = natural numbers). Then there is an ∞-number $(\mathcal{N}, R)$.

<u>Construction</u> : Let ϕ be a selective ultrafilter on $\mathcal{N}$ (i.e. if $\{I^\alpha\}$ is a partition of $\mathcal{N}$, $I^\alpha \notin \phi \ \forall \alpha \Rightarrow \exists u \in \phi : |u \cap I^\alpha| = 1 \ \forall \alpha$).

The existence of ϕ follows from the continuum hypothesis. For $u = \{u_1, u_2, \dots\} \in \phi$ we define $I[u] = \{(1,u_1), (u_1+1, u_2), (u_2+1, u_3), \dots\}$ where (a,b) means an interval in the natural ordering of $\mathcal{N}$.

<u>Lemma</u>. $\forall f : \mathcal{N} \to \mathbb{Z}$, $\exists u \in \phi$ such that f is non-decreasing on u or f is non-increasing on u.

Lemma ⇒ Th.4 : let us denote $\mathcal{J} = \{P[u] \mid u \in \phi\}$ and $\forall f : \mathcal{N} \to \mathbb{Z}$ denote $\tilde{f}(n) = \sum_{k=1}^{n} f(k)$, $n \in \mathcal{N}$. For each f there is $u \in \phi$, such that $\tilde{f}$ is (for example) non-decreasing on u and then $\forall P^\alpha = (u_{\alpha-1}+1, u_\alpha) \in P[u]$ we obtain

$$f_{/P[u]}(P^\alpha) = \sum_{k \in P^\alpha} f(k) = \tilde{f}(u_\alpha) - \tilde{f}(u_{\alpha-1}) \geq 0 .$$

Let us denote $\mathcal{A}$ the set of all ∞-numbers $(\mathcal{N}, R)$ obtainable from some selective ultrafilter through this construction.

Problems.

Problem 1. Does $\alpha_1 \gtrless \alpha_2$ hold ? Especially for $\alpha_1, \alpha_2 \in \mathcal{A}$? If the answer is negative, is there a reasonable subclass of comparable ∞-numbers ?

Problem 2. Assuming the generalized continuum hypothesis, are there (E,R) with uncountable E ?

Problem 3. To find the explicite condition for $\alpha \in \mathcal{A}$. To clarify the relation between $\mathcal{A}$ and the set of selective ultrafilters in $\beta\mathcal{N}$ - our construction (from ϕ to α) depends clearly on the natural ordering of $\mathcal{N}$. To study properties of

$\mathcal{A}_\phi = \{\alpha$ obtatainable from ϕ by some ω_0-ordering of $E\}$,

$\Phi_\alpha = \{\phi \mid \alpha$ can be obtained from ϕ using some ω_0-ordening of $E\}$.

Problem 4. May the product $\alpha_1 \cdot \alpha_2$ be reasonably defined ? Probably the definition of ∞-numbers must be refined.

Problem 5. The set $\{\alpha \mid \alpha \leq (E,R)\} \cong 2^E/R$ is linearly ordered by the inclusion. May properties of (E,R) (or relations between them) be translated into language of such ordenings ? To characterize all ordenings of this type.