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Stable Banach spaces

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A separable Banach space E is called stable if for every bounded sequences (x_n) and (y_m) in E and every ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$, we have:

$$\lim_{\substack{n\\ \mathcal{U}}} \lim_{\substack{m\\ \mathcal{V}}} \| x_n+y_m \| = \lim_{\substack{m\\ \mathcal{V}}} \lim_{\substack{n\\ \mathcal{U}}} \| x_n+y_m \| .$$

This notion was first introduced by T.L. Krivine and B. Maurey ([5]) to extend a result of D. Aldous ([1]). These theorems are the following:

<u>Theorem 1.</u> ([1])

Every subspace of L^1 contains an l^p-space, $1\le p<+\infty$

<u>Theorem 2.</u> ([5])

Every stable Banach space contains an l^p-space, $1\le p<+\infty$

<u>Examples.</u>

(1) Hilbert spaces, l^p- and L^p-spaces $(1\le p<+\infty)$ Orlicz spaces $l^\emptyset$ and $L^\emptyset$ ($\emptyset$ having the s_2-condition), Lorentz spaces $L^{p,\emptyset}$ are stable ([6]).

(2) c_0, the Tsirelson spaces T and T', the James space J are not stable.

<u>Property 1.</u> ([5])

If E is stable, then every subspace of E and the spaces $l^p(E)$ and $L^p(E)$ with $1\le p<+\infty$ are stable

<u>Open problem</u>

If E is stable and reflexive, are E' and every quotient space of E stable ?

<u>Theorem 3.</u> ([4])

Every stable Banach space is weakly sequentially complete

<u>Corollary</u>

If E is stable then E is reflexive if and only if E

does not contain l^1

Sketch of the proof of theorem 3

We have to define some notions:

σ is a type on E if $\exists (a_n) \subset E$, $\exists \mathcal{U}$ ultrafilter on $\mathbb{N}$ such that: $\forall x \in E$, $\sigma(x) = \lim\limits_{\substack{n \\ \mathcal{U}}} \| x + a_n \|$.

The type $\lambda\sigma$ is defined by:

$$\forall x \in E, \quad \lambda\sigma(x) = |\lambda| \sigma\left(\frac{x}{\lambda}\right) = \lim_{\substack{n \\ \mathcal{U}}} \| x + \lambda a_n \|$$

The type $\sigma * \tau$ is defined by:

$$\forall x \in E, \quad \sigma * \tau(x) = \lim_{\substack{n \\ \mathcal{U}}} \lim_{\substack{m \\ \mathcal{V}}} \| x + a_n + b_m \|$$

if

$$\tau(x) = \lim_{\substack{m \\ \mathcal{V}}} \| x + b_m \|$$

If (x_n) is a bounded sequence in E and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$, we define the spreading-model associated to (x_n) and $\mathcal{U}$ by the completion of $E \times R^{(\mathbb{N})}$ under the semi-norm:

$$\left| x + \sum_{i=1}^{k} \lambda_i 1_i \right| = \lim_{\substack{n_1 \\ \mathcal{U}}} \cdots \lim_{\substack{n_k \\ \mathcal{U}}} \left\| x + \sum_{i=1}^{k} \lambda_i x_{n_i} \right\|$$

(See [2] or [3] for more details).

If (x_n) has no Cauchy subsequences, then this is a norm. In [2] it is proved that every sequence (x_n) has a "good subsequence" (x'_n) which means:

$$\forall \varepsilon > 0, \quad \forall k \in \mathbb{N}, \quad \forall (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k, \quad \exists \nu \in \mathbb{N}$$

such that:

$$\nu < n_1 \ldots < n_k \Rightarrow \forall x \in E,$$

$$\left| \left| x + \sum_{i=1}^{k} \alpha_i e_i \right| - \left\| x + \sum_{i=1}^{k} \alpha_i x'_{n_i} \right\| \right| \le \varepsilon$$

(x_n) will be called a good sequence if it has the property of the subsequence (x'_n) above - (e_n) will be called the fundamental sequence of the spreading model

Relations between types and spreading models in stable Banach spaces

If σ is a type on E defined by (x_n) and $\mathcal{U}$, the spreading model associated to (x_n) and $\mathcal{U}$ is given by:

$$\left| x + \sum_{i=1}^{k} \alpha_i e_i \right| = \lim_{\substack{n_1 \\ \mathcal{U}}} \dots \lim_{\substack{n_k \\ \mathcal{U}}} \left\| x + \sum_{i=1}^{k} \alpha_i x_{n_i} \right\|$$

$$= (\alpha_1 \sigma * \dots * \alpha_k \sigma)(x) .$$

On the other hand, if (e_n) is the fundamental sequence of a spreading model associated to (x_n) and $\mathcal{U}$, then the type σ is given by: $\sigma(x) = | x+e_1 | = \lim_{\substack{n \\ \mathcal{U}}} \| x+x_n \|$

Property 2.

If E is stable then every fundamental sequence (e_n) is symmetric

$$\left(\text{i.e.: } \left| x + \sum_{i=1}^{n} \alpha_i e_{\sigma(i)} \right| = \left| x + \sum_{i=1}^{n} \alpha_i e_i \right| \quad \text{where } \sigma\right.$$

is a permutation of $\mathbb{N}$).

We now give the proof of the theorem 3

Suppose (x_n) is a "good sequence", weakly Cauchy and not convergent in E. Let (e_n) be the fundamental sequence of the spreading model associated to (x_n). It is easy to see that (e_n) is of "type l_1^+", symmetric and basic, so it is equivalent to the unit vector basis of l^1. Let $y_n = x_{2n+1} - x_{2n}$. Then (y_n) converges weakly to 0 and the fundamental sequence (f_n) of the spreading model associated to (y_n) is defined by $f_n = e_{2n+1} - e_{2n}$ and so is also equivalent to the unit vector basis of l^1.

Let σ be the type defined by (y_n) $\left[\text{i.e.: } \sigma(x) = \lim_n \| x + y_n \| = | x + f_1 | \right]$ and K be the closure under the pointwise topology of $\left\{ \tau,\ \tau = \alpha_1 \sigma * \dots * \alpha_k \sigma ,\ (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^{(\mathbb{N})},\ \tau(0) = 1 \right\}$.

We can show that if $\tau \in K$, then the spreading model associated with τ is equivalent to l^1. We know from [5], that K contains an l^p-type τ_0

[i.e.: $\alpha_1 \tau_0 * \ldots * \alpha_k \tau_0(x) = (|\alpha_1|^p + \ldots + |\alpha_k|^p)^{1/p} \tau_0(x)$].

So we must have $p=1$. It is easy to see, by a diagonal argument that τ_0 is defined by a sequence of convex blocks $(\mathcal{U}_n)$ on (y_n). The sequence $(\mathcal{U}_n)$ converges weakly to 0 [because (y_n) does] and by [5] contains a sequence equivalent to the unit vector basis of l^1.

This is a contradiction and proves the theorem 3. We give some more results on stable Banach spaces. The proof of the theorem 4 is very close to the proof of theorem 3.

Theorem 4. ([4])

Every spreading model of a stable Banach space E is stable.

If a spreading model of a stable Banach space E contains an l^p-space $(1 \leq p < +\infty)$, then E itself contains l^p.

No spreading model of a stable Banach space E contains c_o.

Open problem

Find an "isomorphic" characterization of stable Banach spaces.

References

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[2] A, Brunel, L. Sucheston: On B-convex Banach spaces, Meth. System theory - Vol. 7, N°. 4 (1973)

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