THE INTEGRAL COHOMOLOGY RINGS OF REAL INFINITE DIMENSIONAL FLAG MANIFOLDS

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In 1981 E.H. Brown [2] and M. Feshbach [4] described the integral cohomology rings of real infinite Grassmannians $BO(n)$ and $BSO(n)$ in terms of generators and relations. The main difficulty of the computation is the description of a suitable basis for the image of the first Steenrod square $Sq^1(H^*(BG;Z_2))$. We extend the method used in [2] for the computation of the integral cohomology rings of real infinite dimensional flag manifolds. We shall use the fact that those spaces are the classifying spaces for suitable groups.

1. Statement of results

Let $k_1,\dots,k_m$ and n be integers with $k_1+\dots+k_m \leqslant n$. Let us denote by $F^u_{k_1,\dots,k_m}(R^n)$ the manifold of flags $(X^1,\dots,X^m)$ where each X^i is an unoriented k_i-dimensional linear subspace in the n-dimensional euclidean space R^n and X^i is orthogonal to X^j for $i\neq j$, $1\leqslant i,j\leqslant m$. Take $F^u_{k_1,\dots,k_m}$ as the direct limit of the spaces $F^u_{k_1,\dots,k_m}(R^n)$ and similarly denote by $F^o_{k_1,\dots,k_m}$ the direct limit of the manifolds of oriented flags. We claim that $F^u_{k_1,\dots,k_m}$ is equivalent to the classifying space $B(O(k_1)\times\dots\times O(k_m)) \cong BO(k_1)\times\dots\times BO(k_m)$ of the group $O(k_1)\times \times\dots\times O(k_m)$. Indeed, we have an isomorphism of $F^u_{k_1,\dots,k_m}$ and $O(n)/(O(n-k_1-\dots-k_m)\times O(k_1)\times\dots\times O(k_m)) \cong V^{u,n}_{k_1+\dots+k_m}/(O(k_1)\times\dots\times O(k_m))$ where $V^{u,n}_{k_1+\dots+k_m} \cong O(n)/O(n-k_1-\dots-k_m)$ denotes the Stiefel variety of $(k_1+\dots+k_m)$-frames in R^n. Because the limit of $V^{u,n}_{k_1+\dots+k_m}$ is the total space of the universal fibration for the group $O(k_1+\dots+k_m)$ and, consequently, it is the total space of the universal fibration for the group $O(k_1)\times\dots\times O(k_m)$, we obtain our statement. So, we shall identify

$$F^u_{k_1,\dots,k_m} \quad \text{and} \quad BO(k_1)\times\dots\times BO(k_m)$$

and similarly in the oriented case

$$F^o_{k_1,\dots,k_m} \quad \text{and} \quad BSO(k_1)\times\dots\times BSO(k_m).$$

Because the infinite flag manifolds are presented as the product of infinite Grassmannians, their cohomology could be expressed in terms of the cohomology of Grassmannians using the Küneth formula. But the Küneth formula gives no information about the ring structure and the computation related with it is rather unmanageable.

Let $F_{k_1,\dots,k_m}$ be $F^u_{k_1,\dots,k_m}$ or $F^o_{k_1,\dots,k_m}$ and let G_{k_j} be $BO(k_j)$ or $BSO(k_j)$, $1\leq j\leq m$. Let $\gamma_j \to G_{k_j}$ be the canonical vector bundle over the Grassmannian G_{k_j}. We have the projections $q_j : F_{k_1,\dots,k_m} \to G_{k_j}$ and we shall identify the characteristic classes of the bundle γ_j with their images under q_j^* in the group $H^*(F_{k_1,\dots,k_m})$. Let us write $K = (k_1,\dots,k_m)$ and introduce the following notation:

$p_{i,j} = p_i(\gamma_j)$, the i-th Pontrjagin class, $1 \leq j \leq j$,

$w_{i,j} = w_i(\gamma_j)$, the i-th Stiefel-Whitney class, $1\leq j\leq m$,

$X_j \;=\; e(\gamma^o_j)$, the Euler class (orientable case), $1\leq j\leq m$.

<u>Remark</u>. Let for $1\leq j\leq m$ be $\omega_j \to F_{k_1,\dots,k_m}$ the fibration whose fiber over the flag $(X^1,\dots,X^m)$ consists of points of X^j, $1\leq j\leq m$. We show that the characteristic classes of the bundle ω_j correspond to the characteristic classes of the bundle γ_j. We prove the statement for m=2, the proof in the general case is similar.

Let us define $f_{1,n} : R^n \to R^{2n}$ and $f_{2,n} : R^n \to R^{2n}$ by $f_{1,n}(x_1,\dots,x_n) = (x_1,0,x_2,0,\dots,x_n,0)$ and $f_{2,n}(x_1,\dots,x_n) = (0,x_1,0,x_2,\dots,0,x_n)$. Let f_1 and f_2 be the induced endomorphisms of R^∞ and let $g_i : G_k \to G_k$ be the endomorphisms of the Grassmannian G_k induced by the maps f_i, $i = 1,2$. It is not hard to show that g_i is a weak homotopy equivalence. As the equivalense between $G_{k_1}\times G_{k_2}$ and F_{k_1,k_2} can be taken the map $h : G_{k_1}\times G_{k_2} \to F_{k_1,k_2}$ defined by $h(X\times Y) = (g_1(X), g_2(Y))$. The projection $\pi_i : G_{k_1}\times G_{k_2} \to G_{k_i}$ to the i-th factor is the classifying map for the fibration γ_i, $i = 1,2$ and the map $\lambda_i : F_{k_1,k_2} \to G_{k_i}$ defined by $\lambda_i(X^1,X^2) = X^i$ can be taken as the classifying map for the fibration ω_i, $i = 1,2$. It is clear that the following diagram

$$\begin{array}{ccc} G_{k_1}\times G_{k_2} & \xrightarrow{\quad h \quad} & F_{k_1,k_2} \\ & {\scriptstyle \pi_i}\searrow \quad \swarrow{\scriptstyle \lambda_i} & \\ & G_{k_i} & \end{array}$$

commutes up to g_i so it homotopy commutes. The statement of the remark now follows from the naturality of the characteristic classes.

Now, let ϱ and δ be homomorphisms in the long exact sequence

$$(1.1)\quad H^q(X;Z) \xrightarrow{\cdot 2} H^q(X;Z) \xrightarrow{\varrho} H^q(X;Z_2) \xrightarrow{\delta} H^{q+1}(X;Z) \longrightarrow \cdots$$

coming from the short exact sequence $0 \to Z \to Z \to Z_2 \to 0$ of coefficients. Write $(i,j) < (p,q)$ if $j < q$ or $j=q$ and $i < p$. Denote by $D(A,B)$ the symmetric difference $A\cup B \setminus A\cap B$ of A and B. The main theorems of the paper read as follows.

__1.2. Theorem__. For given $K = (k_1,\dots,k_m)$ let M_K^o be the system of all finite sequences $I = \{(i_1,j_1),\dots,(i_g,j_g)\}$ of ordered pairs of natural numbers with $1 \le j_s \le m$, $1 \le i_s \le [(k_{j_s}-1)/2]$ and with $(i_1,j_1) < \dots < (i_g,j_g)$. Let $w(I) = w_{2i_1,j_1}\cdots w_{2i_g,j_g}$ and $p(I) = p_{i_1,j_1}\cdots p_{i_g,j_g}$. Then there exists an isomorphism $H^*(F_K^o;Z) \cong R_K^o/I_K^o$ of graded rings where R_K^o is the polynomial ring

$$R_K^o = Z[\{p_{i,j}\}_{1\le j\le m,\ 1\le i\le [(k_j-1)/2]}, x_1,\dots,x_j, \{\delta w(I)\}_{I\in M_K^o}]$$

and the ideal I_K^o is generated by the following relations:

(i) $2\delta w(I) = 0$, $I \in M_K^o$

(ii) $x_j = \delta w_{2k,j}$ for $k_j = 2k+1$,

(iii) $$\delta w(I)\,\delta w(J) = \sum_{(i,j)\in I} \delta w_{2i,j}\,\delta w(D(I\setminus\{(i,j)\},J)) \cdot p((I\setminus\{(i,j)\})\cap J),\quad I,J\in M_K^o.$$

__1.3. Theorem__. For given $K = (k_1,\dots,k_m)$ let M_K^u be the system of all finite sequences $I = \{(i_1,j_1),\dots,(i_g,j_g)\}$ with $j_s = 1,\dots,m$, $i_s = 1,\dots,[k_{j_s}/2]$ or $i_s = 1/2$ and $(i_1,j_1) < \dots < (i_g,j_g)$. Let $w(I) = w_{2i_1,j_1}\cdots w_{2i_g,j_g}$ and $p(I) = p_{i_1,j_1}\cdots p_{i_g,j_g}$ with the convention that $p_{1/2,i} = \delta w_{1,i}$, $1\le i\le m$. Then there exists an isomorphism $H^*(F_K^u;Z) \cong R_K^u/I_K^u$ of graded rings where R_K^u is the polynomial ring

$$R_K^u = Z[\{p_{i,j}\}_{1\leq j\leq m,\ 1\leq i\leq [k_j/2]}, \{\delta w(I)\}_{I\in M_K^u}]$$

and the ideal I_K^u is generated by the following relations:

(i) $2\delta w(I) = 0, \quad I\in M_K^u$,

(ii) $\delta w(I) = \delta w_{2k,j}\delta w(I\setminus\{(1/2,j),(k,j)\})$ if $k_j = 2k$, $(1/2,j)\in I$, $(k,j)\in I$, $I\in M_K^u$, $1\leq j\leq m$,

$\delta w(I)\,\delta w_{2k,j} = p_{k,j}\,\delta w((I\setminus\{(k,j)\})\cup\{(1/2,j)\})$ if $k_j = 2k$, $(1/2,j)\notin I$, $(k,j)\in I$, $I\in M_K^u$, $1\leq j\leq m$,

(iii) the relations for $\delta w(I)\delta w(J)$ given by the formulas in the theorem 1.2 , $I,J\in M_K^u$.

The following examples ilustrate the previous theorems. We shall write for the convenience $w_{1,1} = w_1$, $w_{1,2} = w_1'$, $p_{i,1} = p_i$ and $p_{i,2} = p_i'$.

Example A. The graded ring $H^*(F^o_{5,3};Z)$ is isomorphic to the ring $R^o_{5,3}/I^o_{5,3}$ where $R^o_{5,3}$ is the polynomial ring

$$R^o_{5,3} = Z[p_1,p_2,p_1',\delta w_2,\delta w_4,\delta w_2',\delta(w_2w_4),\delta(w_{2}w_2'),\delta(w_4w_2'),\delta(w_2w_4w_2')]$$

and the ideal $I^o_{5,3}$ is generated by the following relations:

$$2\delta w(I) = 0,\ I\in M^o_{5,3},$$
$$\delta(w_2w_4)\,\delta w_2' = \delta w_2\,\delta(w_4w_2') + \delta w_4\,\delta(w_2w_2'),$$
$$\delta(w_2w_2')\,\delta(w_2w_2') = (\delta w_2)^2p_1' + (\delta w_2')^2p_1,$$
$$\delta(w_2w_4)\,\delta(w_2w_4) = (\delta w_2)^2p_2 + (\delta w_4)^2p_1,$$
$$\delta(w_4w_2')\,\delta(w_4w_2') = (\delta w_4)^2p_1' + (\delta w_2')^2p_2,$$
$$\delta(w_2w_4)\,\delta(w_4w_2') = \delta w_2\delta w_2'.\,p_2 + \delta w_4\,\delta(w_2w_4w_2'),$$
$$\delta(w_2w_4)\,\delta(w_2w_2') = \delta w_4\delta w_2'.\,p_1 + \delta w_2\,\delta(w_2w_4w_2'),$$
$$\delta(w_2w_2')\,\delta(w_4w_2') = \delta w_2\delta w_4.\,p_1' + \delta w_2'\,\delta(w_2w_4w_2'),$$
$$\delta(w_2w_4w_2')\,\delta(w_2w_4) = \delta w_2\,\delta(w_2w_2').p_2 + \delta w_4\,\delta(w_4w_2').p_1,$$
$$\delta(w_2w_4w_2')\,\delta(w_2w_2') = \delta w_2\,\delta(w_2w_4).p_1' + \delta w_2'\,\delta(w_4w_2').p_1,$$
$$\delta(w_2w_4w_2')\,\delta(w_4w_2') = \delta w_4\,\delta(w_2w_4).p_1' + \delta w_2'\,\delta(w_2w_2').p_2,$$
$$\delta(w_2w_4w_2')\,\delta(w_2w_4w_2') = (\delta w_2)^2.p_2.p_1' + (\delta w_4)^2.p_1.p_1' + (\delta w_2')^2.p_1.p_2.$$

Example B. There is an isomorphism $H^*(F^u_{3,1};Z) \cong R^u_{3,1}/I^u_{3,1}$ of graded rings where $R^u_{3,1}$ is the polynomial ring generated by the elements p_1, $\delta w_1,\delta w_2,\delta w_1',\delta(w_1w_2),\delta(w_1w_1'),\delta(w_2w_1'),\delta(w_1w_2w_1')$ and the ideal $I^u_{3,1}$ can be obtained from the relations in $I^{o}_{5,3}$ in the example A writing w_1, w_2, w_2' instead of w_2, w_4, w_2' and δw_1, p_1, $\delta w_1'$ instead of p_1, p_2, p_1'.

Example C. The graded ring $H^*(F^u_{2,1};Z)$ is isomorphic to the ring $R^u_{2,1}/I^u_{2,1}$ where the polynomial ring $R^u_{2,1}$ has the same generators as the ring $R^u_{3,1}$ in the example B and the ideal $I^u_{2,1}$ is generated by the same relations as the ideal $I^u_{3,1}$ in the previous example and by the relations:

$$\delta(w_1w_2) = 0,$$
$$\delta(w_1w_2w_1') = \delta w_2\,\delta w_1',$$
$$\delta w_2 \delta w_2 = p_1 \delta w_1,$$
$$\delta(w_2w_1')\,\delta w_2 = p_1\,\delta(w_1w_1').$$

2. Proofs

The method described in this paragraph is a modification of that in [2]. By [1, IV.24] the torsion subgroups of the groups $H^*(BSO(k);Z)$ and $H^*(BO(k);Z)$ form a Z_2-vector space. Because flag manifolds are product of Grassmannians, the Küneth formula together with the fact that $Tor_Z(Z,Z) = 0$ and $Tor_Z(Z_2,Z) \cong Z_2$ say that the torsion subgroups of the integral cohomology rings of flag manifolds form a Z_2-vector space, too.

It is an immediate consequence of the exactness of (1.1) that the torsion subgroup T^*_K of the group $H^*(F_K;Z)$ is equal to $\delta H^*(F_K;Z_2)$, so we shall need the explicit description of the last group given in the following lemma. The lemma can be proved using the Küneth formula [6] and the computation in [5, theorems 7.1, 19.1 and exercises].

2.1. Lemma. There are the following isomorphisms of graded rings

$$H^*(F^o_{k_1,\dots,k_m};Z_2) \cong Z_2[w_{i,j}] \quad 1\leq j\leq m,\ 2\leq i\leq k_j,$$
$$H^*(F^u_{k_1,\dots,k_m};Z_2) \cong Z_2[w_{i,j}] \quad 1\leq j\leq m,\ 1\leq i\leq k_j,$$

and, if k is an integral domain containing 1/2 (for example $Z[1/2]$) then

$$H^*(F^u_{k_1,\dots,k_m};k) \cong k[p_{i,j}] \quad 1\leq j\leq m,\ 1\leq i\leq [k_j/2],$$
$$H^*(F^o_{k_1,\dots,k_m};k) \cong k[p_{i,j},\ x_s] \quad 1\leq j\leq m,\ 1\leq i\leq [(k_j-1)/2]. \quad 1\leq s\leq m \ \&\ k_s \text{ even}$$

There is the natural map $h:Z[p_{i,j}]\longrightarrow H^*(F^u_K;Z)$. The map $h\otimes Z[1/2]$ is by the previous lemma and by the Küneth formula an isomorphism so the map h is a monomorphism and $\mathrm{Image}(h)\cap T_K^* = \{0\}$.

We claim that for each $f\in H^*(F_K^u;Z)$ there exists a polynomial $g\in Z[p_{i,j}]$ with $(f-h(g))\in T_K^*$. Indeed, there exist $a\in Z[p_{i,j}]$ and a natural number d with $h(a) = 2^d.f$ (recall that $h\otimes Z[1/2]$ is an epimorphism). We have $\varrho(h(a)) = 0$ and, because $\varrho(p_{i,j}) = w_{2i,j}^2$, there exists $b\in Z[p_{i,j}]$ with $a = 2b$. The induction gives $g\in Z[p_{i,j}]$ with $h(2^d g) = 2^d.h(g) = 2^d.f$, so $(f-h(g))\in T_K^*$. The oriented case can be discussed similarly, so we have proved the following lemma.

2.2. Lemma. There are the following isomorphisms of graded groups:

$$H^*(F^o_{k_1,\dots,k_m};Z) \cong Z[p_{i,j}, X_s] \oplus \delta H^*(F^o_{k_1,\dots,k_m};Z_2),$$
$$H^*(F^u_{k_1,\dots,k_m};Z) \cong Z[p_{i,j}] \oplus \delta H^*(F^u_{k_1,\dots,k_m};Z_2).$$

Now, we start to prove our theorems. The generators of the polynomial rings R_K are the elements of the corresponding cohomology groups so we have the natural homomorphisms $S_K^o:R_K^o\longrightarrow H^*(F_K^o;Z)$ and $S_K^u:R_K^u\longrightarrow H^*(F_K^u;Z)$. Using the Wu formula for the action of Sq^1 on the Stiefel-Whitney classes, the fact that $\varrho\delta = Sq^1$ and the obvious relations between characteristic classes we can show that $\varrho S_K(I_K) = 0$ (compare the computation in [2]). By [2, lemma 2,2] the map $\varrho|T_K^*$ is a monomorphism, hence $S_K(I_K) = 0$. So the map S_K factors to a homomorphism $\Phi_K:R_K/I_K\longrightarrow H^*(F_K;Z)$. Clearly, the group $Z[p_{i,j}, X_s]$ can be taken as the free part of the group R_K^o/I_K^o and the group $Z[p_{i,j}]$ can be taken as the free part of the group R_K^u/I_K^u. By the lemma 2.2 the map Φ_K induces an isomorphism of the free parts, so it remains to show that $\varrho\circ\Phi_K$ is an isomorphism of the torsion part of R_K/I_K to the group $Sq^1H^*(F_K;Z_2)$. Consider the orientable case and introduce the following notation:

$$h_i = [(k_i-1)/2],\ 1\leq i\leq m,$$

$$\begin{array}{llll} w'_2 = w_{2,1}, & w'_{2(h_1+1)} = w_{2,2}, & \dots, & w'_{2(h_1+\dots+h_{m-1}+1)} = w_{2,m} \\ w'_4 = w_{4,1}, & w'_{2(h_1+2)} = w_{4,2}, & \dots, & w'_{2(h_1+\dots+h_{m-1}+2)} = w_{4,m} \\ \vdots & \vdots & & \vdots \\ w'_{2h_1} = w_{2h_1,1}, & w'_{2(h_1+h_2)} = w_{2h_2,2}, & \dots, & w'_{2(h_1+\dots+h_m)} = w_{2h_m,m} \end{array}$$

and, following this pattern,

$$\begin{array}{llll} p'_1 = p_{1,1}, & p'_{h_1+1} = p_{1,2}, & \dots, & p'_{h_1+\dots+h_{m-1}+1} = p_{1,m}, \\ \vdots & \vdots & & \vdots \\ p'_{h_1} = p_{h_1,1}, & p'_{h_1+h_2} = p_{h_2,2}, & \dots, & p'_{h_1+\dots+h_m} = p_{h_m,m}. \end{array}$$

Let $s_1,\dots,s_m,t_1,\dots,t_m,f_2,f_3,\dots,f_{2(h_1+\dots+h_m)+1}$ be natural numbers and decompose $s_i = 2a_i+\alpha_i$, $t_i = 2b_i+\beta_i$, $f_{2i} = 2m_i+\epsilon_i$ where a_i, b_i,m_i are natural numbers and α_i, β_i, ϵ_i are 0 or 1. Let us denote

$$w^{s,t,f} = \prod_i (w_{1,i}^{s_i}.w_{k_i,i}^{t_i}) \prod_j (w'_{2j})^{f_{2j}} \prod_j (\delta w'_{2j})^{f_{2j+1}},$$

$$z^{s,t,f} = \prod_i (p_{k_i/2,i}^{b_i}) \prod_j (p'_j)^{m_j} \prod_i (\delta w_{1,i})^{a_i} \prod_j (\delta w'_{2j})^{f_{2j+1}}.$$

$$.\ \delta(\prod_i (w_{1,i}^{\alpha_i}.\ w_{k_i,i}^{\beta_i}) \prod_j (w'_{2j})^{\epsilon_j})$$

with the convention that $w_{k_i,i} = 1$ and $p_{k_i/2,i} = 1$ for k_i odd and $1\leqslant i\leqslant m$. Clearly $\varrho\phi(z^{s,t,f}) = Sq^1(w^{s,t,f})$. Let U be the set of all $(s,t,f) = (s_1,\dots,s_m,t_1,\dots,t_m,f_2,f_3,\dots,f_{2(h_1+\dots+h_m)+1}$ with $t_i = 0$ for k_i odd, $1\leqslant i\leqslant m$, such that

either there exists j with f_{2j} odd and, if j_o is the largest j with this property, then $f_{2j+1} = 0$ for $j>j_o$,

or f_{2j} is even and $f_{2j+1} = 0$ for all j and there is i, $1\leqslant i\leqslant m$, such that s_i+t_i is odd.

It can be shown, using tools of elementary mathematics, that the set $\{z^{s,t,f}\}_{(s,t,f)\in U}$ spans the torsion subgroup of R^u_K/I^u_K and that $\{Sq^1(w^{s,t,f})\}_{(s,t,f)\in U}$ forms a basis for $Sq^1(H^*(F^u_K;Z_2))$ (compare the computation in [2]). So the theorem 1.3 is proved.

The statement of the theorem 1.2 for the orientable case can be proved by the similar way and we shal not give the proof here. The computation developed in the paper can be easily modified for the description of the integral cohomology rings of the spaces $G_{k_1}\times\dots\times G_{k_m}$ where G_{k_i} is BSO(k_i) or BO(k_i), $1\leqslant i\leqslant m$.

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