Barbara Opozda

Remark on mixed foliate generic submanifolds

In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 9. pp. [199]–204.

Persistent URL: http://dml.cz/dmlcz/701400

REMARK ON MIXED FOLIATE GENERIC SUBMANIFOLDS

Barbara Opozda

This paper is in final form and no version of it will be submitted for publication elsewhere.

0. Let M' be a Kählerian manifold with a complex structure J' and a Hermitian metric $(\, , \,)$. Let M be a real submanifold in M'. $(\, , \,)$ will mean also the induced metric tensor field on M. The norm defined by $(\, , \,)$ will be denoted by $\| \ \|$. We set

$\mathcal{N}$ - the normal bundle of TM in $TM'_{|M}$,

p - the projection onto TM in $TM'_{|M} = TM \oplus \mathcal{N}$,

n - the projection onto $\mathcal{N}$ in $TM'_{|M} = TM \oplus \mathcal{N}$,

$P = p \circ J'_{|TM}$, $\psi = n \circ J'_{|TM}$,

$\mathcal{D}_x = T_xM \cap J'T_xM$ for $x \in M$,

$\mathcal{H}_x = T_xM + J'T_xM$ for $x \in M$,

$\mathcal{D}^{\perp}_x$ - the orthogonal complement to $\mathcal{D}_x$ in T_xM ,

$\mathcal{D}_{0\,x}$ - the orthogonal complement to T_xM in $\mathcal{H}_x$,

$\mathcal{N}\mathcal{H}_x$ - the orthogonal complement to $\mathcal{H}_x$ in T_xM',

∇', ∇ - the Riemannian connections on M' and M respectively,

D - the normal connection, i.e. the connection in $\mathcal{N}$ induced by ∇',

α , A - the second fundamental form and the second fundamental tensor respectively for M in M',

R', R - the curvature tensors (of type (1.3) as well as of type (0,4)) associated with ∇' and ∇ respectively,

$h(X,Y) = n\, J'\alpha(X,Y) - \alpha(X,PY)$.

Since M' is Kählerian $h(X,Y) = (\bar{\nabla}_X \psi)Y = \psi \nabla_X Y - D_X \psi Y$.

Recall the equations of Gauss and Codazzi :

$$(0.1)\quad R'(W,Z,X,Y) = R(W,Z,X,Y) + (\alpha(X,Z), \alpha(Y,W)) - (\alpha(Y,Z), \alpha(X,W)),$$

$$(0.2)\quad (R'(X,Y)Z)^{\perp} = (\bar{\nabla}_X \alpha)(Y,Z) - (\bar{\nabla}_Y \alpha)(X,Z)$$

for $X,Y,Z,W \in T_xM$, $x \in M$, where $^{\perp}$ denotes the normal part of a vector tangent to M'.

If p and p' are J'- invariant planes in T_xM', then the holomorphic bisectional curvature by p and p' is given by

$$H'_B(p,p') = R'(X,Y,X,Y) + R'(J'X,Y,J'X,Y),$$

where X and Y are unit vectors in p and p' respectively. If X,Y are arbitrary vectors tangent to M' at a point x, then we shall denote $R'(X,Y,X,Y) + R'(J'X,Y,J'X,Y)$ by $H'_B(X,Y)$.

A real submanifold of M is called generic if $\dim \mathfrak{D}_x$ is constant on M. If M is generic, then we set $\mathfrak{D} = \bigcup_{x\in M} \mathfrak{D}_x$,

$$\mathfrak{D}^{\perp} = \bigcup_{x\in M} \mathfrak{D}^{\perp}_x, \quad \mathcal{H} = \bigcup_{x\in M} \mathcal{H}_x, \quad \mathcal{N}\mathcal{H} = \bigcup_{x\in M} \mathcal{N}\mathcal{H}_x, \quad \mathfrak{D}_0 = \bigcup_{x\in M} \mathfrak{D}_{0x}.$$

$\mathfrak{D}, \mathfrak{D}^{\perp}, \mathcal{H}, \mathcal{N}\mathcal{H}, \mathfrak{D}_0$ are vector bundles over M. The distribution $\mathfrak{D}$ is called the holomorphic distribution. A real submanifold M of M' is called a CR - submanifold if $J'\mathfrak{D}^{\perp} \subset \mathfrak{D}_0$. A CR - submanifold is a generic submanifold, [4]. A generic submanifold is called purely real (resp. holomorphic) if $\mathfrak{D} = \{0\}$ (resp. $\mathfrak{D}^{\perp} = \{0\}$) A generic submanifold is said to be proper if it is neither purely real nor holomorphic. A purely real CR - submanifold is called totally real. If M is a generic submanifold of M, then the induced f - structure on M is defined by

$$f(X) = \begin{cases} 0 & \text{for } X \in \mathfrak{D}^{\perp} \\ J'X & \text{for } X \in \mathfrak{D} \end{cases}$$

By a generic product we mean a generic submanifold for which the almost product structure $(\mathfrak{D}, \mathfrak{D}^{\perp})$ is parallel. Of course, it is equivalent to the fact, that M is locally the Riemannian product of a holomorphic submanifold of M' and a purely real submanifold of M'. Since M' is Kählerian the parallelity of f is equivalent to the parallelity of $(\mathfrak{D}, \mathfrak{D}^{\perp})$. In the next we shall use

Proposition 0.1 ,[4]. $\nabla f = 0$ if and only if $\alpha(X,Y) \in J\mathcal{H}$ provided X or Y belongs to $\mathfrak{D}$.

A generic submanifold is said to be mixed totally geodesic if $\alpha(X,Y) = 0$ for $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^{\perp}$, [2]. By a generic mixed foliate submanifold we shall mean a generic submanifold which is mixed totally geodesic and the holomorphic distribution $\mathfrak{D}$ is integrable.

This definition is analogous to the definition of a mixed foliate submanifold in the case of a CR - submanifold ,[2]. A CR submanifold is mixed foliate if and only if the tensor field h is symmetric ,[5].

1. B-Y Chen and S.Montiel proved in [3] the following theorems which generalize some earlier theorems.

Theorem 1.1. A generic submanifold in $\mathbb{C}^n$ is a generic product if and only if it is mixed foliate

Theorem 1.2. Let M be a generic submanifold of a complex-space-form with positive holomorphic sectional curvature. If M is mixed foliate, then M is holomorphic or purely real.

We shall prove

Theorem 1.3. Let M be a generic mixed foliate submanifold of a Kählerian manifold M'. If the holomorphic bisectional curvature of M is non-negative, then M is a generic product. If M passes through a point of M' in which M' has positive holomorphic bisectional curvature, then M is holomorphic or purely real.

Proof. Suppose that M is a proper generic submanifold. Let $X \in \mathfrak{D}_x$ and $Y \in \mathfrak{D}^{\perp}_x$. Using the fact that $J'Y = PY + \psi Y$ we find

$$(1.1) \quad 2R'(J'X,Y,X,\psi Y) = R'(X,PY,X,PY) - R'(X,\psi Y,X,\psi Y) - R'(X,J'Y,X,J'Y).$$

On the other hand

$$2R'(J'X,Y,X,\psi Y) = -2R'(Y,X,J'X,\psi Y) - 2R'(X,J'X,Y,\psi Y) = -2R'(J'^2X,Y,J'X,\psi Y) - 2R'(X,J'X,Y,\psi Y).$$

Using the formula (1.1) for $R'(J'(J'X),Y,J'X,\psi Y)$, we obtain.

$$(1.2) \quad 2R'(J'X,Y,X,\psi Y) = -R'(J'X,PY,J'X,PY) + R'(J'X,\psi Y,J'X,\psi Y) + R'(X,Y,X,Y) - 2R'(X,J'X,Y,\psi Y)$$

Since M is mixed totally geodesic $A_{\psi\mathfrak{D}^{\perp}}\mathfrak{D} \subset \mathfrak{D}$. If follows that $A_{\psi Y}X = A^{T}_{\psi Y}X$, where A^{T} denotes the second fundamental tensor for a leaf of $\mathfrak{D}$ in M'. By virtue of this fact, the facts that M is mixed totally geodesic and $\mathfrak{D}$ is involutive and by the equation of Codazzi, we have (comp.(6.6) and (6.7) in [3])

$$
\begin{aligned}
(1.3)\quad & -R'(X,J'X,Y,\psi Y) = (R'(X,J'X)Y,\psi Y) \\
&= (\alpha(X,\nabla_{J'X}Y),\psi Y) - (\alpha(J'X,\nabla_X Y),\psi Y) \\
&\qquad + (\alpha(\nabla_{J'X}X - \nabla_X J'X, Y),\psi Y) \\
&= (A_{\psi Y}X,\nabla_{J'X}Y) - (A_{\psi Y}J'X,\nabla_X Y) \\
&= (A_{\psi Y}X,\nabla'_{J'X}Y) - (A_{\psi Y}J'X,\nabla'_X Y) \\
&= -(A^{T}_{\psi Y}X, A^{T}_{Y}J'X) + (A^{T}_{\psi Y}J'X, A^{T}_{Y}X) \\
&= 2(A^{T}_{\psi Y}X, A^{T}_{J'Y}X) \\
&= 2\|A^{T}_{\psi Y}X\|^2 + 2(A^{T}_{\psi Y}X, A^{T}_{PY}X) \\
&= 2\|A^{T}_{\psi Y}X\|^2 + \|A^{T}_{J'Y}X\|^2 - \|A^{T}_{PY}X\|^2 - \|A^{T}_{\psi Y}X\|^2 \\
&= \|A^{T}_{\psi Y}X\|^2 + \|A^{T}_{J'Y}X\|^2 - \|A^{T}_{PY}X\|^2 = \|A^{T}_{\psi Y}X\|^2 + \|A^{T}_{Y}X\|^2 \\
&\qquad - \|A^{T}_{PY}X\|^2
\end{aligned}
$$

Combining this with (1.1) and (1.2), we obtain

$$
(1.4)\quad 2\|A^{T}_{\psi Y}X\|^2 + 2\|A^{T}_{Y}X\|^2 - 2\|A^{T}_{PY}X\|^2 = H'_B(X,PY) - H'_B(X,\psi Y) - H'_B(X,Y).
$$

If we use this formula for PY instead of Y, we get

$$
(1.5)\quad 2\|A^{T}_{\psi PY}X\|^2 + 2\|A^{T}_{PY}X\|^2 - 2\|A^{T}_{P^2Y}X\|^2 = H'_B(X,P^2Y) - H'_B(X,\psi PY) - H'_B(X,PY)
$$

By virtue of (1.4) and (1.5) we have

$$
(1.6)\quad 2\|A^{T}_{\psi PY}X\|^2 + 2\|A^{T}_{Y}X\|^2 + 2\|A^{T}_{\psi Y}X\|^2 - 2\|A^{T}_{P^2Y}X\|^2 = H'_B(X,P^2Y) - H'_B(X,\psi Y) - H'_B(X,Y) - H'_B(X,\psi PY)
$$

The tensor field P is skew-symmetric. In fact, $(Z,PW) = (Z,J'W) = -(J'Z,W) = -(PZ,W)$. Therefore P^2 is symmetric. Of course

$P^2(\mathfrak{D}^\perp) \subset \mathfrak{D}^\perp$. Let $Y_1, \dots, Y_k$ be an ortonormal basis of $\mathfrak{D}^\perp_x$ consisting of eigenvectors of $P^2|_{\mathfrak{D}^\perp_x}$. Let $P^2(Y_i) = \lambda_i Y_i$ for $i=1,\dots,k$. Since $\|P^2 Y_i\| \leqslant \|Y_i\|$, $\lambda_i^2 < 1$ for every $i=1,\dots,k$. The formula (1.6) used for $Y = Y_i$ has the form.

$$(1.7)\quad 2\,\|A^T_{\psi Y_i} X\|^2 + 2\,\|A^T_{\psi PY_i} X\|^2 + (1-\lambda_i^2)\, 2\,\|A^T_{Y_i} X\|^2 =$$

$$= -(1-\lambda_i^2)\, H'_B(X, Y_i) - H'_B(X, \psi Y_i) - H'_B(X, \psi PY_i)\,.$$

The left hand side of this equality is non-negative. If there is $x \in M$, $X \in \mathfrak{D}_x$ and $i \in \{1,\dots,k\}$ such that the right hand side is negative, then we have a contradiction so M must be purely real or holomorphic. It holds, for instance, in the case where M passes through a point of M' in which the holomorphic bisectional curvature is positive. Now, suppose that the holomorphic bisectional curvature of M' is non-negative. If for every $x \in M$, $X \in \mathfrak{D}_x$ and i 1,...,k the right hand side of (1.7) is zero, then $A^T_{\psi Y} X = 0$ for any $X \in \mathfrak{D}_x$ and $Y \in \mathfrak{D}^\perp_x$. If means that $A_{\psi Y} X = 0$ for every $Y \in \mathfrak{D}^\perp_x$, $X \in \mathfrak{D}_x$ and $x \in M$. Since $\psi|_{\mathfrak{D}^\perp}$ is an epimorphism onto $\mathfrak{D}_0$ and $(A_{\psi Y} X, W) = (\alpha(X,W), \psi Y)$ for any $W \in TM$, $\alpha(X,W) \in \mathcal{N}\mathcal{H}$ for any $X \in \mathfrak{D}$ and $W \in TM$. By Proposition (0.1) $\nabla f = 0$, i.e. M is a generic product. The proof is completed.

Suppose now that M is a mixed foliate proper CR - submanifold . Then (1.4) reduces to the following

$$(1.8)\quad 2\,\|A^T_Y X\|^2 = -H'_B(X,Y)$$

for $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$.

For a mixed foliate submanifold the equation of Gauss implies

$$(1.9)\quad H'_B(X,Y) = R(X,Y,X,Y) + R(JX,Y,JX,Y)$$

for $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$.

In fact, if $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$, $\alpha(X,Y) = \alpha(J'X,Y) = 0$. Hence

$$(1.10)\quad \begin{cases} R'(X,Y,X,Y) = R(X,Y,X,Y) - (\alpha(X,X)\,,\ \alpha(Y,Y)) \\ R'(J'X,Y,J'X,Y) = R(J'X,Y,J'X,Y) - (\alpha(J'X,J'X),\ \alpha(Y,Y)) \end{cases}$$

The holomorphic distribution $\mathfrak{D}$ is integrable, so $\alpha(J'X,J'X) = (J'^2X,X) = -\alpha(X,X)$, (see, for instance [1]).

Therefore (1.10) implies (1.9).

Consequently

(1.11) $\quad 2\,\| A^T_Y X\|^2 = -R(X,Y,X,Y) - R(J'X,Y,J'X,Y)$

for $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$

If there are $x \in M$, $X \in \mathfrak{D}_x$ and $Y \in \mathfrak{D}^\perp_x$ such that the right hand side of (1.11) is negative then we have a contradiction, hence M is holomorphic or purely real. If the right hand side is zero for any $x \in M$, $X \in \mathfrak{D}_x$ and $Y \in \mathfrak{D}^\perp_x$, then $A^T_Y X = 0$, i.e. $A^T_{J'Y} X = 0$. In manner as in the previous case we conclude from this that $\nabla f = 0$. Therefore we have proved

Theorem 1.4. Let M be a mixed foliate CR - submanifold of a Kählerian manifold. If the Riemannian sectional curvature of M is non-negative, then M is a generic product. If at a point of M the Riemannian sectional curvature of M is positive, then M is holomorphic or totally real.

REFERENCES

[1] Chen B.Y. : Differential Geometry of Real Submanifolds in a Kähler Manifold. Mn.Math. 91 (1981), 257-274.

[2] Chen B.Y. : CR - submanifolds of a Kaehler manifold I,II. J.Diff.Geom. 16 (1981), 305-322, 493-509.

[3] Chen B.Y. , Montiel S. : Real submanifolds of a Kaehler manifold, Algebras, Groups and Geometries. 1 (1984), 176-212.

[4] Opozda B. : Metric polynomial structures, to appear

[5] Opozda B. : Generic submanifolds in almost Hermitian manifolds

Barbara Opozda

Instytut Matematyki

Uniwersytet Jagielloński

30-050 Kraków, Poland