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MULTISYMPLECTIC FORMS OF DEGREE THREE IN DIMENSION SEVEN

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ABSTRACT. The group $Gl(n)$ operates naturally on the space $\Lambda^3\mathbb{R}^{n*}$ of 3-forms on $\mathbb{R}^n$. We say that two 3-forms are of the same algebraic type if they belong to the same orbit. Multisymplectic 3-structure on an n-dimensional manifold M is given by a closed smooth 3-form ω of maximal rank on M which is of the same algebraic type at each point of M. This means that for each point $x \in M$ the form ω_x is isomorphic with a chosen canonical 3-form on $\mathbb{R}^n$. From the geometric point of view multisymplectic structure is a G-structure on M, where $G \subset Gl(n)$ is the isotropy group at the canonical 3-form. In the paper we use the classification of 3-forms in dimension 7 (see [W], [D]), and we describe the isotropy groups of the individual canonical forms. The study of related geometric structures will be postponed to subsequent papers.

1. ALGEBRAIC PROPERTIES OF 3-FORMS ON A REAL VECTOR SPACE

Let V be an n-dimensional vector space over the field $\mathbb{R}$. The general linear group $Gl(V)$ has natural action on V, and the induced natural actions on V^* and on the spaces $\Lambda^k V$, $\Lambda^k V^*$ for any k.

$$(\varphi\omega)(X) = \omega(\varphi X) \quad \text{for every} \quad \varphi \in Gl(V),\ \omega \in \Lambda^k V^*,\ X \in \Lambda^k V.$$

We shall say that two k-forms are of the same algebraic type if they lie in the same orbit under the action of $Gl(V)$. In every orbit we can choose a k-form which will be called canonical. Instead of the notation $\varphi\omega$ we shall use the more common notation $\varphi^*\omega$.

Let us consider now a 3-form $\omega \in \Lambda^3 V^*$. There is a set of invariants of the form ω under the induced action of $Gl(n)$ on $\Lambda^3 V^*$.

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1. Rank of ω, denoted by $\rho(\omega)$. It is defined as the minimal dimension of the subspaces $W \subset V^*$ such that $\omega \in \Lambda^3 W$.

2. Irreducible length of ω, denoted by $l(\omega)$. It is the minimal number of decomposable summands in all possible representations of ω (number of summands in the shortest representation of ω).

3. The numbers $m(\omega)$ and $r(\omega)$ defined in the following way. Let $0 \neq w \in V$ and let W be a complement of $[w]$ in V. Then for $\theta \in V^*$ satisfying $\theta(w) = 1, \theta(W) = 0$ we have a decomposition

$$\Lambda^3 V^* = (\theta \wedge \Lambda^2 W^*) \oplus \Lambda^3 W^*,$$

and for any $\omega \in \Lambda^3 V^*$ there are uniquely defined elements $\gamma_1 \in \Lambda^2 W^*, \gamma_2 \in \Lambda^3 W^*$ with

$$\omega = \theta \wedge \gamma_1 + \gamma_2.$$

Let us denote by $D(\omega)$ the set of all γ_1 arising in this way, and similarly by $E(\omega)$ the set of all γ_2. We define

$$m(\omega) = \min\{l(\gamma_1); \gamma_1 \in D(\omega)\},$$
$$r(\varphi) = \min\{l(\gamma_2); \gamma_2 \in E(\omega)\}.$$

The quadruple of numbers

$$p(\omega) = (\rho(\varphi), l(\omega), m(\omega), r(\omega))$$

enables to distinguish among the orbits of $Gl(V)$ in $\Lambda^3 V^*$. It is constant on each orbit, and to two different orbits correspond two different quadruples.

We are interested in 3-forms of maximal rank on V, i. e. in the 3-forms ω satisfying $\rho(\omega) = \dim V$. Such forms are usually called multisymplectic forms. It is well known that a form ω is multisymplectic if and only if the map

$$V \to \Lambda^2 V^*, \quad v \mapsto \iota(v)\omega = \omega(v, \cdot, \cdot)$$

is injective.

1. Remark. For each dimension $n \leq 8$ there is only a finite number of types, for each dimension $n \geq 9$ there is always an infinite number of types. The first interesting nontrivial case appears for $n = 6$, where the 3-forms of maximal rank under the action of $Gl(V)$ have three orbits. Two of them are open in $\Lambda^3 V^*$, the third one has codimension 1. Open orbits exist also for $n = 7$ and 8. This fenomenon cannot occur if= $n \geq 9$. This can be easily deduced by comparing dimensions n^2 of $Gl(V)$ and $\binom{n}{3}$ of $\Lambda^3 V^*$. We shall restrict now to the case of dimension 7.

2. Classification of 3-forms of maximal rank 7

We fix a basis $e_1, \ldots, e_7$ of V and we denote the dual basis of V^* by $\alpha_1, \ldots, \alpha_7$.

2. Lemma. *There is the following classification of 3-forms* ω *of maximal rank* $\rho(\omega) = 7$ *in the 7-dimensional space* V *with respect to the action of the group* $Gl(V)$.

Type 1. $p(\omega_1) = (7, 3, 1, 1)$. *Representative of the orbit is*

$$\omega_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6.$$

Type 2. $p(\omega_2) = (7, 3, 1, 2)$. *Representative of the orbit is*

$$\omega_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 - \alpha_2 \wedge \alpha_3 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7.$$

Type 3. $p(\omega_3) = (7, 3, 1, 0)$. *Representative of the orbit is*

$$\omega_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5).$$

Type 4. $p(\omega_4) = (7, 4, 1, 1)$. *Representative of the orbit is*

$$\omega_4 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) + \alpha_2 \wedge \alpha_4 \wedge \alpha_6.$$

Type 5. $p(\omega_5) = (7, 4, 2, 2)$. *Representative of the orbit is*

$$\omega_5 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7$$

$$+\alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

Type 6. $p(\omega_6) = (7, 4, 1, 2)$. *Representative of the orbit is*

$$\omega_6 = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6.$$

Type 7. $p(\omega_7) = (7, 4, 2, 3)$. *Representative of the orbit is*

$$\omega_7 = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 + \alpha_2 \wedge \alpha_3 \wedge \alpha_7 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5.$$

Type 8. $p(\omega_8) = (7, 5, 3, 3)$. *Representative of the orbit is*

$$\omega_8 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7$$
$$+\alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

3. Remark. Let V^C be the complexification of V, and let ω^C denote the complexification of ω. Then the following couples belong to the same orbit with respect to the action of the complex linear group $Gl(V^C)$:

$$\omega_1^C \text{ and } \omega_6^C, \quad \omega_2^C \text{ and } \omega_7^C, \quad \omega_5^C \text{ and } \omega_8^C.$$

3. Further invariants

We would like to find further invariants of the orbits of the forms $\omega_i, i = 1, \dots, 8$ with respect to the action of $Gl(V)$. With any 3-form ω we associate the subsets $\Delta^k(\omega) \subset V$, $k = 2, 3$

$$\Delta^k(\omega) = \{v \in V; (i(v)\omega)^{\wedge k} = 0\}.$$

Instead of $\Delta^k(\omega_j)$ we shall use the notation Δ_j^k. We introduce also a symmetric bilinear form on V with values in $\Lambda^7 V^*$

$$b(v, w) = \iota(v)\omega \wedge \iota(w)\omega \wedge \omega.$$

This means that b determines a conformal class of scalar bilinear forms, and simultaneously the corresponding conformal class of quadratic forms. A representative of the first class we shall denote by B, and the corresponding representative of the second class we denote by Q. As an invariant of a conformal class of bilinear forms we get the common kernel of its elements.

For a 3-form ω we introduce its group of automorphisms

$$O(\omega) = \{\varphi \in Aut(V); \varphi^*\omega = \omega\}.$$

It is obvious that $O(\omega)$ is the isotropy group of the action of $Gl(V)$ on $\Lambda^3 V^*$ at the point ω. Instead of $O(\omega_i)$ we write O_i. One of the main aims of this paper is the determination of these groups.

4. Further study of types.

Let us start to study further properties of the individual types from the above list of forms. If $v, w \in V$ we write $v = c_1 e_1 + \cdots + c_7 e_7$, $w = d_1 e_1 + \cdots + d_7 e_7$.

Type 1. The form ω_1.

We have

$$\Delta_1^2 = V_3^a \cup V_3^b, \quad \Delta_1^3 = V_6^a \cup V_6^b,$$

where

$$V_3^a = [e_3, e_4, e_7], \quad V_3^b = [e_5, e_6, e_7]$$

and

$$V_6^a = [e_1, e_3, e_4, e_5, e_6, e_7], \quad V_6^b = [e_2, e_3, e_4, e_5, e_6, e_7].$$

Further invariant subspaces of V are $V_1 = V_3^a \cap V_3^b$, $V_5 = V_6^a \cap V_6^b$. Moreover, we have the quotients

$$W_2 = V/V_5,\ W_4^a = V/V_3^a,\ W_4^b = V/V_3^b,\ W_6 = V/V_1,$$

and

$$Z_1^a = V_6^a/V_5,\ Z_1^b = V_6^b/V_5,\ Z_2^a = V_3^a/V_1,\ Z_2^b = V_3^b/V_1,$$
$$\tilde{Z}_2^a = V_5/V_3^b,\ \tilde{Z}_2^b = V_5/V_3^a,\ Z_4 = V_5/V_1.$$

We have obviously $W_2 = Z_1^a \oplus Z_1^b$ and $Z_4 = Z_2^a \oplus Z_2^b$. We get also

$$(\iota(v)\omega_1) \wedge (\iota(w)\omega_1) \wedge \omega_1 = 3(c_1 d_2 + c_2 d_1)\alpha_1 \wedge \cdots \wedge \alpha_7.$$

The scalar form $B(v,w) = c_1 d_2 + c_2 d_1$ induces a regular form of signature $(1,1)$ on W_2. We denote it by B_2. In fact, we obtain on W_2 a conformal structure of signature $(1,1)$.

Obviously, for every $\varphi \in O_1$ we have either $\varphi V_3^a = V_3^a$ and $\varphi V_3^b = V_3^b$ or $\varphi V_3^a = V_3^- b$ and $\varphi V_3^b = V_3^a$. We define a homomorphism

$$\mathrm{sg} : O_1 \to \mathbb{Z}_2$$

in the following way

$$\mathrm{sg}\,\varphi = 1 \quad \text{if } \varphi V_3^a = V_3^a \text{ and } \varphi V_3^b = V_3^b,$$

$$\mathrm{sg}\,\varphi = -1 \quad \text{if } \varphi V_3^a = V_3^b \text{ and } \varphi V_3^b = V_3^a.$$

We obtain easily a split short exact sequence

$$0 \to O_1^+ \to O_1 \xrightarrow{\mathrm{sg}} \mathbb{Z}_2 \to 0,$$

where $O_1^+ = \ker \mathrm{sg}$.

It is also obvious that every automorphism $\varphi \in O_1$ induces an automorphism $\tilde{\varphi} \in GL(Z_4)$, and that $\tilde{\varphi}(Z_2^a \oplus Z_2^b) = Z_2^a \oplus Z_2^b$. We can easily prove that

$$\tilde{\varphi} Z_2^a = Z_2^a \text{ and } \tilde{\varphi} Z_2^b = Z_2^b \text{ iff } \mathrm{sg}\,\varphi = 1,$$
$$\tilde{\varphi} Z_2^a = Z_2^b \text{ and } \tilde{\varphi} Z_2^b = Z_2^a \text{ iff } \mathrm{sg}\,\varphi = -1.$$

This means that we can define a homomorphism

$$\nu : O_1^+ \to GL(Z_2^a) \oplus GL(Z_2^b), \quad \nu\varphi = \tilde{\varphi}.$$

It can be proved that this homomorphism is an epimorphism, and we obtain a split short exact sequence

$$0 \to K \to O_1^+(\omega_1) \xrightarrow{\nu} GL(Z_2^a) \oplus GL(Z_2^b) \to 0,$$

where $K = \ker \nu$.

It is not difficult to see that the mapping $v \mapsto (\iota(v)\omega_1)|V_5$ induces a homomorphism $W_2 \to \Lambda^2 Z_4^*$. The image of this homomorphism is the subspace $\Lambda^2 Z_2^{a*} \oplus \Lambda^2 Z_2^{b*}$. Consequently, we get an isomorphism

$$\kappa : W_2 \to \Lambda^2 Z_2^{a*} \oplus \Lambda^2 Z_2^{b*}$$

with $\kappa Z_1^a = \Lambda^2 Z_2^{a*}$ and $\kappa Z_1^b = \Lambda^2 Z_2^{b*}$.

Similarly, the mapping $v \mapsto \iota(v)\omega_1$ induces an isomorphism

$$\lambda : V_1 \to \Lambda^2 W_2^*.$$

Let us consider again $\varphi \in O_1^+$. We denote $\hat{\varphi}$ the automorphism induced on W_2 by φ. An easy computation shows that

$$\kappa(\hat{\varphi}w) = \frac{1}{\det \bar{\varphi}} \kappa w, \quad \text{for every } w \in W_2.$$

This formula implies that

$$\hat{\varphi} Z_1^a = Z_1^a, \quad \hat{\varphi} Z_1^b = Z_1^b.$$

Similar result we get for the isomorphism λ.

$$\lambda(\varphi v) = \frac{1}{\det \hat{\varphi}} \lambda v \quad \text{for every } v \in V_1.$$

We are now going to investigate the subgroup K. It is obvious that an element $\varphi \in K$ has the form

$$\begin{aligned}
\varphi e_1 &= \varphi_{11}e_1 + \varphi_{12}e_2 + \varphi_{13}e_3 + \varphi_{14}e_4 + \varphi_{15}e_5 + \varphi_{16}e_6 + \varphi_{17}e_7 \\
\varphi e_2 &= \varphi_{21}e_1 + \varphi_{22}e_2 + \varphi_{23}e_3 + \varphi_{24}e_4 + \varphi_{25}e_5 + \varphi_{26}e_6 + \varphi_{27}e_7 \\
\varphi e_3 &= e_3 + \varphi_{37}e_7 \\
\varphi e_4 &= e_4 + \varphi_{47}e_7 \\
\varphi e_5 &= e_5 + \varphi_{57}e_7 \\
\varphi e_6 &= e_6 + \varphi_{67}e_7 \\
\varphi e_7 &= \varphi_{77}e_7
\end{aligned}$$

and the previous considerations show that

$$\varphi_{11} = 1, \ \varphi_{12} = 0, \ \varphi_{21} = 0, \ \varphi_{22} = 1, \ \varphi_{77} = 1.$$

Considering the equality $\varphi^*\omega_1 = \omega_1$, we find that an automorphism φ of the above form belongs to O_1 if and only if

$$\varphi_{37} = \varphi_{24}, \quad \varphi_{47} = -\varphi_{23}, \quad \varphi_{57} = -\varphi_{16}, \quad \varphi_{67} = \varphi_{15}.$$

Let us consider a mapping

$$\nu_1 : K \to \tilde{Z}_2^a, \quad \nu_1(\varphi) = [(\varphi - id)e_1],$$

where [ ] denotes the class of the corresponding element from V_5 in the quotient $\tilde{Z}_2^a$. Similarly we define a mapping

$$\nu_2 : K \to \tilde{Z}_2^b, \quad \nu_2(\varphi) = [(\varphi - id)e_2].$$

Considering now $\tilde{Z}_2^a$ and $\tilde{Z}_2^b$ as commutative groups with the addition inherited from the vector space structures, we find easily that

$$\nu_1 \oplus \nu_2 : K \to \tilde{Z}_2^a \oplus \tilde{Z}_2^b$$

is a surjective homomorphism, i. e. an epimorphism. We get again a split short exact sequence

$$0 \to L \to K \overset{\nu_1 \oplus \nu_2}{\to} \tilde{Z}_2^a \oplus \tilde{Z}_2^b \cong \mathbb{R}^4 \to 0,$$

where L denotes the kernel of $\nu_1 \oplus \nu_2$. Finally, it is not difficult to find that $L = H \oplus H$, where H is a Lie group diffeomorphic as a manifold with $\mathbb{R}^3$, and with a multiplication given by the formula

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 - x_1 y_2 + x_2 y_1).$$

In other words, H is the Heisenberg group.

Summarizing, we obtain the following

4. Proposition.

$$O_1 = [((H \oplus H) \ltimes \mathbb{R}^4) \ltimes (GL(W_2^a) \oplus GL(W_2^b))] \ltimes \mathbb{Z}_2$$

with $\dim O_1 = 18$.

In the above formula $\ltimes$ denotes various semidirect products with respect to the splittings in the above split short exact sequences. For the sake of brevity we have omitted their description.

Type 2. The form ω_2.

Here we have

$$\Delta_2^2 = \{v \in V; c_1 = c_2 = c_3 = c_4 = 0, c_5 c_6 + c_6 c_7 + c_7 c_5 = 0\},$$

$$\Delta_2^3 = \{v \in V; c_1 c_4 - c_2 c_3 = 0\},$$

and

$$(\iota(v)\omega_2) \wedge (\iota(v)\omega_2) \wedge \omega_2 = 6(c_1 d_4 - c_2 d_3 - c_3 d_2 + c_4 d_1)\alpha_1 \wedge \cdots \wedge \alpha_7.$$

This means that we get a subspace $V_3 = [e_5, e_6, e_7]$, and a quotient space $W_4 = V/V_3$. On the subspace V_3 we shall consider the quadratic form

$$Q_3(v) = c_5 c_6 + c_6 c_7 + c_7 c_5,$$

which is of course determined only up to a multiple. The corresponding bilinear form we denote by B_3. Q_3 has signature $(1, 2)$. On V we shall consider the quadratic form

$$Q(v) = 2(c_1 c_4 - c_2 c_3).$$

The corresponding bilinear form we denote by B. It is easy to see that the kernel of B is the subspace V_3. The bilinear form B induces on W_4 a regular bilinear form B_4. The corresponding quadratic form we denote by Q_4.

In V_3 we shall consider the orthonormal basis $f_5 = e_5 + e_6$, $f_6 = e_5 - e_=6$, $f_7 = e_7 - e_5 - e_6$ satisfying

$$B_3(f_5, f_5) = 1, \quad B_3(f_6, f_6) = -1, \quad B_3(f_7, f_7) = -1.$$

The mapping $v \mapsto \iota(v)\omega_2$ induces a monomorphism

$$\lambda : V_3 \to \Lambda^2 W_4^*.$$

We denote $\sigma_1 = \lambda(f_5)$, $\sigma_2 = \lambda(f_6)$, $\sigma_3 = \lambda(f_7)$. An easy computation shows that there are uniquely determined endomorphisms $E, F, G \in End(W_4)$ such that

$$\begin{aligned}\sigma_1(w_1, w_2) &= B_4(Ew_1, w_2),\\ \sigma_2(w_1, w_2) &= B_4(Fw_1, w_2),\\ \sigma_3(w_1, w_2) &= B_4(Gw_1, w_2).\end{aligned}$$

These endomorphisms satisfy the relations

$$E^2 = -I, \quad F^2 = I, \quad G^2 = I,$$

$$EF = -FE = G, \quad FG = -GF = -E, \quad GE = -EG = F,$$

which shows that the associative subalgebra of $End(W_4)$ generated by I, E, F, and G is isomorphic to the algebra $\tilde{\mathbb{H}}$ of pseudoquaternions. Obviously, W_4 is an 1-dimensional free $\tilde{\mathbb{H}}$-module.

We now start to investigate the group O_2 of automorphisms of the form ω_2. Any element $\varphi \in O_2$ preserves the subspace V_3, and preserves up to a positive multiple the quadratic form Q_3. Therefore we can define the restriction homomorphisms

$$\rho : O_2 \to CO(Q_3), \quad \rho\varphi = \bar{\varphi} = \varphi | V_3,$$

where

$$CO(Q_3) = \{\psi \in GL(V_3); \exists c > 0 \text{ such that } Q_3(\psi v) = cQ_3(v) \text{ for all } v \in V_3\}.$$

Similarly $\varphi \in O_2$ induces an automorphism $\tilde{\varphi}$ of W_4. It must preserve the form B_4 up to a non-zero multiple (we remind that B_4 has signature $(2, 2)$). Consequently, we can define a homomorphism

$$\mu : O_2 \to CO(Q_4),$$

where

$$CO(Q_4) = \{\chi \in GL(W_4); \exists d \neq 0 \text{ such that } Q_4(\chi w) = dQ_4(w) \text{ for all } w \in W_4\}.$$

Considering the homomorphism $\det \rho : O_2 \to \mathbb{R}^*$, we find that this homomorphism is an epimorphism. Consequently, we obtain a split short exact sequence

$$0 \to O_2^+ \to O_2 \stackrel{\det \rho}{\to} \mathbb{R}^* \to 0,$$

where $O_2^+ = \ker \det \rho$.

Having in mind the short exact sequence

$$0 \to SO(1,2) \to CO(1,2) \stackrel{\det}{\to} \mathbb{R}^* \to 0,$$

we find easily a split short exact sequence

$$0 \to K \to O_2^+ \stackrel{\rho}{\to} SO(Q_3) \to 0,$$

where we write simply ρ instead of $\rho|O_2^+$, and $K = \ker \rho$.

Next, we consider the restriction $\mu : K \to CO(Q_4)$ of the homomorphism μ to the subgroup K. It can be shown, that the image of μ consists precisely of automorphisms of the 1-dimensional $\tilde{\mathbb{H}}$-module W_4 preserving the bilinear form B_4. It is easy to see that this image can be identified with the group

$$\tilde{S}^3 = \{A \in \tilde{\mathbb{H}}; (A, A) = 1\},$$

where $(\cdot, \cdot)$ denotes the standard product of pseudoquaternions. In this way we get a homomorphism $\mu : K \to \tilde{S}^3$, and standard considerations show that this homomorphism is an epimorphism. Consequently, we obtain a split short exact sequence

$$0 \to L \to K \stackrel{\mu}{\to} \tilde{S}^3 \to 0.$$

We shall introduce a subspace $V_4 = [e_1, e_2, e_3, e_4]$. Obviously, there is a natural isomorphism of V_4 with W_4. Any element $\varphi \in L$ determines an endomorphism $D_\varphi : V \to V$ such that $D_\varphi V_4 \subset V_3$, $D_\varphi|V_3 = 0$, and

$$\varphi e_1 = e_1 + D_\varphi e_1, \quad \varphi e_2 = e_2 + D_\varphi e_2, \quad \varphi e_3 = e_3 + D_\varphi e_3, \quad \varphi e_4 = e_4 + D_\varphi e_4,$$

$$\varphi e_5 = e_5, \quad \varphi e_6 = e_6, \quad \varphi e_7 = e_7.$$

It is obvious that the group

$$\{id + D; D \in End(V) \text{ with } DV_4 \subset V_3 \text{ and } D|V_3 = 0\}$$

is commutative. This shows that also the group L is commutative. Considering an automorphism $\varphi = id + D$, we can easily see that

$$\omega_2(\varphi e_i, \varphi e_j, \varphi e_k) = \omega_2(e_i, e_j, e_k) \quad \text{if } \{i, j, k\} \cap \{5, 6, 7\} \neq \emptyset.$$

This means that $\varphi = id + D \in O_2$ if and only if

$$\omega_2(\varphi e_i, \varphi e_j, \varphi e_k) = \omega_2(e_i, e_j, e_k) = 0 \quad \text{for all } i, j, k \in \{1, 2, 3, 4\}.$$

We can find easily that $\varphi = id + D \in O_2$ if and only if the following four equations are satisfied.

$$\omega_2(De_1, e_2, e_3) + \omega_2(e_1, e_2, De_3) = 0$$
$$\omega_2(e_1, De_2, e_4) + \omega_2(e_1, e_2, De_4) = 0$$
$$\omega_2(De_1, e_3, e_4) + \omega_2(e_1, De_3, e_4) = 0$$
$$\omega_2(De_2, e_3, e_4) + \omega_2(e_2, e_3, De_4) = 0$$

Hence we can conclude that L is a Lie group isomorphic with the Lie group $\mathbb{R}^8$. It is well known that $\tilde{S}^3 \cong Spin(1,2)$. We thus obtain

5. Proposition.

$$O(\omega_2) = [(\mathbb{R}^8 \ltimes Spin(1,2)) \ltimes SO(1,2)] \ltimes \mathbb{R}^*$$

with $\dim O_2 = 15$.

Again, we have not specified the splittings hidden in the above formula.

Type 3. The form ω_3.

Here we have

$$\Delta_3^2 = \Delta_3^3 = [e_2, e_3, e_4, e_5, e_6, e_7] = V_6,$$
$$(\iota(v)\omega_3) \wedge (\iota(w)\omega_3) \wedge \omega_3 = 2c_1 d_1 \alpha_1 \wedge \cdots \wedge \alpha_7.$$

We denote also $W_1 = V/V_6$. Because $\omega_3|V_6 = 0$, we find easily that the correspondence $v \in V \mapsto (\iota(v)\omega_3)|V_6$ induces a monomorphism

$$\lambda : W_1 \to \Lambda^2 V_6^*.$$

We denote

$$\sigma = \lambda e_1 = \alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5.$$

This is obviously a symplectic form on the subspace V_6.

We start now to consider the group O_3 of automorphisms of the form ω_3. Every element $\varphi \in O_3$ preserves the subspace V_6, and we denote $\bar{\varphi} = \varphi|V_6$. It can be proved that $\bar{\varphi}$ belongs to the conformal symplectic group $CSp(\sigma) \cong CSp(3,\mathbb{R})$. This means that we can introduce a restriction homomorphism $\rho : O_3 \to CSp(3,\mathbb{R})$. This homomorphism is an epimorphism, and we obtain a split short exact sequence

$$0 \to K \to O_3 \xrightarrow{\rho} CSp(3,\mathbb{R}) \to 0,$$

where $K = \ker \rho$. It is easy to verify that for every $\varphi \in K$ we have

$$\varphi e_1 = e_1 + \varphi_{12} e_2 + \cdots + \varphi_{17} e_7$$
$$\varphi e_i = e_i \quad \text{for } 2 \le i \le 7.$$

On the other hand every element of this form belongs to K. Moreover, we can immediately see that K is isomorphic with $\mathbb{R}^6$ considered as a Lie group. Therefore we have

6. Proposition.

$$O_3 = \mathbb{R}^6 \ltimes CSp(3, \mathbb{R})$$

with $\dim O_3 = 28$.

Type 4. The form ω_4.

For technical reasons this time we shall renumber our basis in V. We set

$$f_1 = e_1, \quad f_2 = e_2, \quad f_3 = e_4, \quad f_4 = e_6, \quad f_5 = e_7, \quad f_6 = e_5, \quad f_7 = e_3.$$

We denote $\beta_1, \dots, \beta_7$ the dual basis to $f_1, \dots, f_7$. With respect to this basis we have

$$\omega_4 = \beta_1 \wedge (\beta_2 \wedge \beta_5 + \beta_3 \wedge \beta_6 + \beta_4 \wedge \beta_7) + \beta_2 \wedge \beta_3 \wedge \beta_4.$$

We find easily that

$$\Delta_4^2 = [f_5, f_6, f_7] = V_3, \quad \Delta_4^3 = [f_2, f_3, f_4, f_5, f_6, f_7] = V_6.$$

Further we define $W_1 = V/V_6$, $W_4 = V/V_3$, and $Z_3 = V_6/V_3$. Let us notice that the restriction of ω_4 onto V_6 is the form $\theta\beta_2 \wedge \beta_3 \wedge \beta_4$. This form obviously induces a 3-form on the quotient Z_3, which will be denoted again by θ.

We shall now investigate the group O_4. It is obvious that any automorphism $\varphi \in O_4$ preserves the subspaces V_3 and V_6. It is also clear that such φ induces an automorphism $\tilde{\varphi} \in GL(W_1)$, an automorphism $\hat{\varphi} \in GL(Z_3)$, and an automorphism $\bar{\varphi} \in GL(V_3)$. The automorphism $\tilde{\varphi}$ has the form $\tilde{\varphi} = c_\varphi \cdot id$, where $c_\varphi \in \mathbb{R}^*$.

We can define first a homomorphism

$$\mu : O_4 \to \mathbb{R}^*, \quad \mu\varphi = c_\varphi.$$

It is easy to see that this homomorphism is an epimorphism, and we get a split short exact sequence

$$0 \to O_4^+ \to O_4 \xrightarrow{\mu} \mathbb{R}^* \to 0,$$

where $O_4^+ = \ker \mu$.

It is also easy to see that if $\varphi \in O_4$, then $\hat{\varphi}^*\theta = \theta$, which means that $\hat{\varphi} \in SL(Z_3) \cong SL(3, \mathbb{R})$. Consequently, we can define a homomorphism

$$\nu : O_4 \to SL(Z_3), \quad \nu\varphi = \hat{\varphi}.$$

We shall use only the restriction of ν to the subgroup O_4^+, which we denote again by ν. This restriction is also an epimorphism, and we obtain a split short exact sequence

$$0 \to K \to O_4^+ \xrightarrow{\nu} SL(3, \mathbb{R}) \to 0,$$

where $K = \ker \nu$.

It can be shown that the subgroup K consists precisely of elements with the matrix expression

$$\varphi = \begin{pmatrix} 1 & \varphi_{37} - \varphi_{46} & \varphi_{45} - \varphi_{27} & \varphi_{26} - \varphi_{35} & \varphi_{15} & \varphi_{16} & \varphi_{17} \\ 0 & 1 & 0 & 0 & \varphi_{25} & \varphi_{26} & \varphi_{27} \\ 0 & 0 & 1 & 0 & \varphi_{35} & \varphi_{36} & \varphi_{37} \\ 0 & 0 & 0 & 1 & \varphi_{45} & \varphi_{46} & \varphi_{47} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We shall consider first a homomorphism

$$\tau_2 : K \to \mathbb{R}, \quad \tau_2\varphi = \varphi_{37} - \varphi_{46}.$$

This homomorphism is an epimorphism, and we get a split short exact sequence

$$0 \to L_2 \to K \xrightarrow{\tau_2} \mathbb{R} \to 0,$$

where $L_2 = \ker \tau_2$. Next, we define a homomorphism

$$\tau_3 : L_2 \to \mathbb{R}, \quad \tau_3\varphi = \varphi_{45} - \varphi_{27}.$$

Again this homomorphism is an epimorphism, and we have a split short exact sequence

$$0 \to L_3 \to L_2 \xrightarrow{\tau_3} \mathbb{R} \to 0,$$

where $L_3 = \ker \tau_3$. Finally, we define a homomorphism

$$\tau_4 : L_3 \to \mathbb{R}, \quad \tau_4\varphi = \varphi_{26} - \varphi_{35}.$$

This homomorphism is also an epimorphism, and we obtain a split short exact sequence

$$0 \to L_4 \to L_3 \xrightarrow{\tau_4} \mathbb{R} \to 0,$$

where $L_4 = \ker \tau_4$.

In order to describe the subgroup L_4, we consider the restriction homomorphism

$$\rho : L_4 \to GL(V_6), \quad \rho\varphi = \check{\varphi} = \varphi | V_6.$$

The image of this homomorphism is a commutative subgroup of $GL(V_6)$ which is isomorphic as a Lie group to $\mathbb{R}^6$. This time we get a split short exact sequence

$$0 \to M \to L_4 \xrightarrow{\rho} \mathbb{R}^6 \to 0,$$

where $M = \ker \rho$. It is not difficult to see that M is isomorphic as a Lie group to $\mathbb{R}^3$.

Summarizing we obtain

7. Proposition.

$$O_4 = [((([\mathbb{R}^3 \ltimes \mathbb{R}^6] \ltimes \mathbb{R}) \ltimes \mathbb{R}) \ltimes \mathbb{R}) \ltimes SL(3,\mathbb{R})] \ltimes \mathbb{R}^*$$

with $\dim O_4 = 21$.

Type 5. The form ω_5.

We have here

$$\Delta_5^2 = \{0\}, \quad \Delta_5^3 = \{v \in V; -c_1^2 - c_2^2 - c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2 = 0\},$$

$$(\iota(v)\omega_5) \wedge (\iota(w)\omega_5) \wedge \omega_5 =$$

$$= (-c_1d_1 - c_2d_2 - c_3d_3 + c_4d_4 + c_5d_5 + c_6d_6 + c_7d_7)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 \wedge \alpha_7.$$

The form ω_5 is well known. It arises in the following way. We recall that the algebra $\widetilde{\mathbb{C}a}$ of pseudoCayley numbers is defined as the double of the algebra of pseudoquaternions. Let $(\cdot,\cdot)$ denote the standard scalar product on the algebra of pseudoCayley numbers. We take $V = \operatorname{Im}\widetilde{\mathbb{C}a}$. Then for any $v_1, v_2, v_3 \in V$ we have

$$\omega_5(v_1, v_2, v_3) = (v_1v_2, v_3).$$

It is obvious that $Aut(\widetilde{\mathbb{C}a}) \subset O_5$, where $Aut(\widetilde{\mathbb{C}a})$ denotes the group of automorphisms of the algebra of pseudoCayley numbers. In fact, it can be proved that these two groups coincide. Moreover, the group $Aut(\widetilde{\mathbb{C}a})$ is isomorphic with the group $\tilde{G}_2$, the noncompact dual of the exceptional Lie group G_2. Thus, we have

8. Proposition. $O_5 = \tilde{G}_2$ *with* $\dim O_5 = 14$.

Type 6. The form ω_6.

Here we have

$$\Delta_6^2 = [e_7], \quad \Delta_6^3 = [e_3, e_4, e_5, e_6, e_7],$$

$$(\iota(v)\omega_6) \wedge (\iota(w)\omega_6) \wedge \omega_6 = 6(c_1d_1 + c_2d_2)\alpha_1 \wedge \cdots \wedge \alpha_7.$$

We denote $V_1 = \Delta_6^2$, $V_5 = \Delta_6^3$, $W_2 = V/V_5$, and $Z_4 = V_5/V_1$. We shall consider on V the quadratic form $Q(v) = c_1^2 + c_2^2$, and we denote the corresponding bilinear form by B. Q induces a definite form Q_2 on W_2. The corresponding bilinear form we denote by B_2.

The insertion $v \in V_1 \mapsto \iota(v)\omega_6$ induces an isomorphism

$$\kappa : V_1 \to \Lambda^2 W_2^*.$$

Furthermore, the insertion $v \mapsto (\iota(v)\omega_6)|V_5$ induces a monomorphism

$$\lambda : W_2 \to \Lambda^2 Z_4^*.$$

We denote $\lambda e_1 = \sigma_1$, $\lambda e_2 = \sigma_2$. Introducing on Z_4 an auxiliary positive definite bilinear form B_4 in such a way that with respect to it e_3, e_4, e_5, e_6 is an orthonormal basis, we can find uniquely determined endomorphisms E, F of Z_4 such that for every $w, w' \in Z_4$ we have

$$\sigma_1(w, w') = B_4(Ew, w'), \quad \sigma_2(w, w') = B_4(Fw, w').$$

An easy computation shows that $E^2 = -I$, $F^2 = -I$ and $EF = -FE$. Setting $G = EF$, we have the relations

$$E^2 = -I, \quad F^2 = -I, \quad G^2 = -I,$$

$$EF = -FE = G, \quad FG = -GF = E, \quad GE = -EG = F,$$

which show that the automorphisms I, E, F, G generate the algebra $\mathbb{H}$ of quaternions.

Any automorphism $\varphi \in O_6$ induces an automorphisms $\bar{\varphi}$ of Z_4. We define a homomorphism

$$\rho : O_6 \to GL(Z_4), \quad \rho\varphi = \bar{\varphi}.$$

Similarly any automorphism $\varphi \in O_6$ induces an automorphisms $\tilde{\varphi}$ of W_2. Obviously, $\tilde{\varphi} \in CO(Q_2)$. This means that we can define a homomorphism

$$\mu : O_6 \to CO(Q_2), \quad \mu\varphi = \tilde{\varphi}.$$

We shall start with the homomorphism

$$\det \mu : O_6 \to \mathbb{R}^*.$$

This homomorphism is an epimorphism, and we have a split short exact sequence

$$0 \to O_6^1 \to O_6 \overset{\det \mu}{\to} \mathbb{R}^* \to 0,$$

where $O_6^1 = \ker(\det \mu)$.

Next, we are going to investigate a restriction of the above homomorphism μ, namely the homomorphism

$$\mu : O_6^1 \to SO(Q_2).$$

It can be proved that this homomorphism is an epimorphism, and because $SO(Q_2) \equiv SO(2)$, we obtain a split short exact sequence

$$0 \to L \to O_6^1 \overset{\mu}{\to} SO(2) \to 0,$$

where $L = \ker \mu$.

Now, we can consider a restriction of the homomorphism ρ, namely the homomorphism $\rho : L \to GL(Z_4)$. Considering Z_4 as an 1-dimensional quaternionic vector space, we find that the image of the homomorphism ρ is isomorphic to the group $SL(2,\mathbb{C})$. We get then a split short exact sequence

$$0 \to M \to L \xrightarrow{\rho} SL(2,\mathbb{C}) \to 0,$$

where $M = \ker \rho$.

The matrix of an automorphism $\varphi \in M$ with respect to the basis $e_1, \dots, e_7$ has the form

$$\varphi = \begin{pmatrix} 1 & 0 & \varphi_{13} & \varphi_{14} & \varphi_{15} & \varphi_{16} & \varphi_{17} \\ 0 & 1 & \varphi_{23} & \varphi_{24} & \varphi_{25} & \varphi_{26} & \varphi_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & \varphi_{37} \\ 0 & 0 & 0 & 1 & 0 & 0 & \varphi_{47} \\ 0 & 0 & 0 & 0 & 1 & 0 & \varphi_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & \varphi_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

On the other hand, an automorphism of this form is not necessarily an element of M. It belongs to M if and only if it belongs to O_6. An easy computation shows that $\varphi \in M$ if and only if the following equations are satisfied

$$\begin{aligned} \varphi_{37} &= -\varphi_{15} - \varphi_{26}, \\ \varphi_{47} &= -\varphi_{16} + \varphi_{25}, \\ \varphi_{57} &= \varphi_{13} - \varphi_{24}, \\ \varphi_{67} &= \varphi_{14} + \varphi_{23}. \end{aligned}$$

We shall consider first the mapping

$$\nu_1 : M \to \mathbb{R}^2, \quad \nu_1\varphi = (\varphi_{37}, \varphi_{47}).$$

It is not difficult to prove that this mapping is an epimorphism of the group M onto the commutative group $\mathbb{R}^2$. In this way we obtain a split short exact sequence

$$0 \to N_1 \to M \xrightarrow{\nu_1} \mathbb{R}^2 \to 0,$$

where $N_1 = \ker \nu_1$. Next we define a mapping

$$\nu_2 : N_1 \to \mathbb{R}^2, \quad \nu_2\varphi = (\varphi_{57}, \varphi_{67}).$$

It is again not difficult to prove that this mapping is an epimorphism of the group N_1 onto the commutative group $\mathbb{R}^2$. We obtain a split short exact sequence

$$0 \to N_2 \to N_1 \xrightarrow{\nu_2} \mathbb{R}^2 \to 0,$$

where $N_2 = \ker \nu_2$. Moreover, one can easily see that N_2 is a commutative Lie group isomorphic with $\mathbb{R}^6$.

Finally, we obtain

9. Proposition.

$$O_6 = [(((\mathbb{R}^6 \ltimes \mathbb{R}^2) \ltimes \mathbb{R}^2) \ltimes SL(2,\mathbb{C})) \ltimes SO(2)] \ltimes \mathbb{R}^*,$$

with $\dim O_6 = 18$.

Type 7. The form ω_7.

Here we have

$$\Delta_7^2 = \{0\}, \quad \Delta_7^3 = [e_5, e_6, e_7],$$

and we denote $V_3 = [e_5, e_6, e_7]$, and $W_4 = V/V_3$. Moreover,

$$(\iota(v)\omega_7) \wedge (\iota(w)\omega_7) \wedge \omega_7 = (c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4)\alpha_1 \wedge \cdots \wedge \alpha_7.$$

Consequently, we consider on V the bilinear form

$$B(v,w) = c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4.$$

The kernel of B is the subspace V_3, and therefore there is an induced positive definite bilinear form B_4 on W_4. The corresponding quadratic form we denote by Q_4.

It is easy to see that the mapping $v \in V_3 \mapsto \iota(v)\omega_7$ induces a monomorphism

$$\lambda : V_3 \to \Lambda^2 W_4^*.$$

We denote

$$\lambda(e_5) = \sigma_1, \quad \lambda(e_6) = \sigma_2, \quad \lambda(e_7) = \sigma_3.$$

There are uniquely defined automorphisms E, F, G of W_4 such that

$$\sigma_1(w,w') = B_4(Ew,w'), \quad \sigma_2(w,w') = B_4(Fw,w'), \quad \sigma_3(w,w') = B_4(Gw,w'),$$

for every $w, w' \in W_4$. It is easy to verify that

$$E^2 = F^2 = G^2 = -I$$

$$EF = -FE = -G, \quad FG = -GF = -E \quad GE = -EG = -F,$$

which shows that the associative subalgebra of $End(W_4)$ generated by I, E, F, and G is isomorphic with the algebra $\mathbb{H}$ of quaternions. This means that W_4 can be considered as an 1-dimensional quaternionic vector space.

The scalar product B_4 induces a scalar product on W_4^* and this in turn induces a scalar product on $\Lambda^2 W_4^*$. We shall denote these products again by B_4. We find easily that the elements $\sigma_1, \sigma_2, \sigma_3$ are orthonormal with

$$B_4(\sigma_1,\sigma_1) = 1, \quad B_4(\sigma_2,\sigma_2) = 1, \quad B_4(\sigma_3,\sigma_3) = 1.$$

Using the monomorphism λ, we transfer the scalar product B_4 from $\Lambda^2 W_4^*$ to V_3. This product on V_3 we denote by B_3. With respect to B_3 the basis e_5, e_6, e_7 is an orthonormal basis with

$$B_3(e_1, e_1) = 1, \quad B_3(e_2, e_2) = 1, \quad B_3(e_3, e_3) = 1.$$

The corresponding quadratic form we shall denote by Q_3. Now, we can see that $\lambda : V_3 \to \Lambda^2 W_4^*$ is an isometry into.

We can start to investigate the group O_7 of automorphisms of the form ω_7. Any element $\varphi \in O_7$ preserves the subspace V_3, and induces an automorphism $\tilde{\varphi}$ of W_4. Obviously, $\tilde{\varphi}$ preserves up to a positive multiple the bilinear form B_4. Consequently, we can define a homomorphism

$$\mu : O_7 \to CO(Q_4), \quad \mu\varphi = \tilde{\varphi},$$

where

$$CO(Q_4) = \{\chi \in GL(W_4); \exists d > 0 \text{ such that } Q_4(\chi w) = dQ_4(w) \text{ for all } w \in W_4\}.$$

For every $\varphi \in O_7$ we find easily the formula

$$\lambda\varphi = \Lambda^2(\varphi^{-1})\lambda.$$

This means that for every $v \in V_3$ we have

$$B_3(\varphi v, \varphi v) = B_4(\lambda\varphi v, \lambda\varphi v) = B_4(\Lambda^2(\varphi^{-1})\lambda v, \Lambda^2(\varphi^{-1})\lambda v) =$$

$$= dB_4(\lambda v, \lambda v) = dB_3(v, v),$$

which means that the restriction $\bar{\varphi} = \varphi | V_3$ preserves the bilinear form B_3 up to a positive multiple. Therefore we can define the restriction homomorphisms

$$\rho : O_7 \to CO(Q_3), \quad \rho\varphi = \bar{\varphi} = \varphi | V_3,$$

where

$$CO(Q_3) = \{\psi \in GL(V_3); \exists c > 0 \text{ such that } Q_3(\psi v) = cQ_3(v) \text{ for all } v \in V_3\}.$$

It is possible to prove that the image of the homomorphism ρ is the subgroup $CSO(Q_3)$. Consequently, we obtain a split short exact sequence

$$0 \to K \to O(\omega_7) \xrightarrow{\rho} CSO(Q_3) \to 0,$$

where $K = \ker\rho$. We shall now consider the restriction $\mu : K \to CO(Q_4)$ of the homomorphism μ. It can be proved that an element $\psi \in CO(Q_4)$ belongs to the image of μ if and only if ψ is an automorphism of the 1-dimensional quaternionic vector space W_4 preserving the bilinear form B_4. Consequently, the image of μ can be identified with the group

$$S^3 = \{A \in \mathbb{H}; (A, A) = 1\},$$

where $(\cdot, \cdot)$ denotes the standard scalar product on $\mathbb{H}$. In this way we get a homomorphism $\mu : K \to S^3$, and our standard considerations show that this homomorphism is an epimorphism. Thus, we obtain a split short exact sequence

$$0 \to L \to K \xrightarrow{\mu} S^3 \to 0,$$

where $L = \ker \mu$.

Any element $\varphi \in L$ determines an endomorphism $D_\varphi : V \to V$ such that $D_\varphi V_4 \subset V_3$, $D_\varphi | V_3 = 0$, and

$$\varphi e_1 = e_1 + D_\varphi e_1, \quad \varphi e_2 = e_2 + D_\varphi e_2, \quad \varphi e_3 = e_3 + D_\varphi e_3, \quad \varphi e_4 = e_4 + D_\varphi e_4,$$

$$\varphi e_5 = e_5, \quad \varphi e_6 = e_6, \quad \varphi e_7 = e_7.$$

It is obvious that the group

$$\{id + D; D \in End(V) \text{ with } DV_4 \subset V_3 \text{ and } D|V_3 = 0\}$$

is commutative. This shows that also the group L is commutative. Considering an automorphism $\varphi = id + D$, we can easily see that

$$\omega_7(\varphi e_i, \varphi e_j, \varphi e_k) = \omega_7(e_i, e_j, e_k) \quad \text{if } \{i, j, k\} \cap \{5, 6, 7\} \neq \emptyset.$$

This means that $\varphi = id + D \in O_7$ if and only if

$$\omega_7(\varphi e_i, \varphi e_j, \varphi e_k) = \omega_7(e_i, e_j, e_k) = 0 \quad \text{for all } i, j, k \in \{1, 2, 3, 4\}.$$

We can find easily that $\varphi = id + D \in O_7$ if and only if the following four equations are satisfied:

$$\begin{aligned}
\omega_7(De_1, e_2, e_3) + \omega_7(e_1, De_2, e_3) + \omega_7(e_1, e_2, De_3) &= 0, \\
\omega_7(De_1, e_2, e_4) + \omega_7(e_1, De_2, e_4) + \omega_7(e_1, e_2, De_4) &= 0, \\
\omega_7(De_1, e_3, e_4) + \omega_7(e_1, De_3, e_4) + \omega_7(e_1, e_3, De_4) &= 0, \\
\omega_7(De_2, e_3, e_4) + \omega_7(e_2, De_3, e_4) + \omega_7(e_2, e_3, De_4) &= 0.
\end{aligned}$$

Hence we can conclude that L is a Lie group isomorphic with the Lie group $\mathbb{R}^8$. Finally, we obtain

10. Proposition.

$$O_7 = (\mathbb{R}^8 \ltimes S^3) \ltimes CSO(3).$$

with $\dim O(\omega_7) = 15$.

Type 8. The form ω_8.

We have here

$$\Delta_8^2 = \{0\}, \quad \Delta_8^3 = \{0\},$$

$$(\iota(v)\omega_8) \wedge (\iota(w)\omega_8) \wedge \omega_8 =$$

$$= (c_1 d_1 + c_2 d_2 + c_3 d_3 + c_4 d_4 + c_5 d_5 + c_6 d_6 + c_7 d_7)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 \wedge \alpha_7.$$

The form ω_8 is the best known one. It arises in the following way. We recall that the algebra $\mathbb{C}a$ of Cayley numbers is defined as the double of the algebra of quaternions. Let $(\cdot,\cdot)$ denote the standard scalar product on the algebra of Cayley numbers. We take $V = \operatorname{Im}\mathbb{C}a$. Then for any $v_1, v_2, v_3 \in V$ we have

$$\omega_8(v_1, v_2, v_3) = (v_1 v_2, v_3).$$

It is obvious that $Aut(\mathbb{C}a) \subset O_8$, where $Aut(\mathbb{C}a)$ denotes the group of automorphisms of the algebra of Cayley numbers. In fact, it can be proved that these two groups coincide. Moreover, the group $Aut(\mathbb{C}a)$ is isomorphic with the exceptional Lie group G_2. Thus, we have

11. Proposition. $O_8 = G_2$ *with* $\dim O_8 = 14$.

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