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DISTINGUISHED CURVES ON 6-DIMENSIONAL CR-MANIFOLDS OF CODIMENSION 2

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ABSTRACT. 6-dimensional CR-manifolds of codimension 2 represent a very special and a very distinguished class of geometric structures among CR-structures of higher codimension. General properties of distinguished curves are studied for hyperbolic and elliptic cases in the framework of parabolic geometries.

The notion of distinguished curves is developed for any Cartan geometry, see e.g. [4] or [7]. These curves generalize geodesics in affine geometries so they are called generalized geodesics too. Similarly to affine geodesics, the general properties of such curves are visible already in the homogeneous model of the Cartan geometry. If the Cartan geometry is parabolic we can say much more, especially with regard to the order of jet which determines a unique distinguished curve in any single point.

The general approach of [1] is applied here to find out the essential properties of generalized geodesics in CR-geometries of codimension 2 on 6-dimensional manifolds. These are very distinguished cases of CR-structures of higher codimension since there is only a finite number of nonisomorphic homogeneous models in these dimensions. More precisely, there are three such models where two of them carry the structure of a parabolic Cartan geometry. The two cases are called hyperbolic and elliptic, respectively, and the third one presents some singular transition. This was discovered and further developed in [6]. In nearly all of other cases, there is either a continuum of nonisomorphic models or the classification is discrete but none of the models admits a structure of parabolic geometry.

In sections 3 and 4 we present all types of generalized geodesics and their properties for hyperbolic and elliptic CR-geometries. In particular, there is an important class of curves generalizing chains, the invariant set of curves well known from CR-geometries of the hypersurface type. Compared to these geometries, the discussion will be a bit richer.

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1. Introduction

The modelling CR-manifolds are the so called CR-quadrics which appear as follows. Let M be a real submanifold in $\mathbb{C}^N$ such that the dimension of the maximal complex subspace $T_x^{CR}M \subset T_xM$ does not depend on the base point $x \in M$. Let us denote the complex dimension by n. If the dimension of M is $2n+k$ then M is called an embedded CR-manifold of codimension k. Locally, any such CR-manifold can be viewed as an embedded CR-manifold in $\mathbb{C}^{n+k}$ and expressed as a graph $\mathrm{Im}(w_j) = f_j(z, \bar{z}, \mathrm{Re}(w))$ for $j = 1, \dots, k$ where $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_k)$ are coordinates in $\mathbb{C}^{n+k}$ and f_j are real functions. Let us assume f is chosen in such a way that $f(0) = 0$ and $df(0) = 0$, i.e. M passes through the origin and $T_0M = \{\mathrm{Im}(w) = 0\}$. Moreover, we may assume that $T_0^{CR}M = \{w = 0\}$. After some manipulation, the above equations can be written as

$$\mathrm{Im}(w) = h(z, \bar{z}) + o(3) \tag{1}$$

where h is a Hermitian form with values in $\mathbb{C}^k$ determined by the second derivatives of f_j with respect to z and $\bar{z}$ in 0, i.e. the Levi form of M in the origin. All details can be found e.g. in [5].

The geometrical meaning of the last equation is that M is osculated by the quadric $Q = \{\mathrm{Im}(w) = h(z, \bar{z})\}$ up to second order in any fixed point. If the codimension k equals to 1 then there is a finite list of osculating quadrics which depends only on the signature of the Levi form. For general k, the list of classifying quadrics has mostly the cardinality of continuum. One of the few exceptions are 6-dimensional embedded CR-manifolds of codimension 2. In order to get the quadrics in that case it remains to classify all nondegenerated Hermitian forms on $\mathbb{C}^2$ with values in $\mathbb{C}^2$. According to [5], any of the mentioned forms can be expressed in suitable coordinates in one of the following forms

$$h(z, \bar{z}) = (|z_1|^2, |z_2|^2)\,, \tag{2}$$

$$h(z, \bar{z}) = (|z_1|^2, \mathrm{Re}(z_1\bar{z}_2))\,, \tag{3}$$

$$h(z, \bar{z}) = (\mathrm{Re}(z_1\bar{z}_2), \mathrm{Im}(z_1\bar{z}_2)) \tag{4}$$

and these three cases are called hyperbolic, parabolic, and elliptic, respectively.

Any embedded nondegenerated CR-manifold of codimension 2 in $\mathbb{C}^4$ is divided into disjunct sets of points with respect to the type of osculating quadrics. For any hyperbolic and elliptic point there is a neighbourhood of points of the same type while the parabolic points have not got this property. So both hyperbolic and elliptic points form open areas on M while parabolic points form some closed transitions.

The last point to remind is that any embedded CR-manifold whose all points are either hyperbolic or elliptic carry the structure of a $|2|$-graded parabolic geometry, which was found in [6]. This is due to the fact that in both cases the automorphism group of the corresponding quadric is a semisimple Lie group and the stabilizer of some its point is a parabolic subgroup.

This section introduces necessary notions and definitions. Further, we present some technical results of [1] which will provide a recipe for applications.

2.1. **Definitions.** Let $p : \mathcal{G} \to M$ be the principal bundle of a Cartan geometry of type (G, P) and let $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ be the Cartan connection. P is a closed subgroup in a Lie group G, $\mathfrak{p}$ and $\mathfrak{g}$ are corresponding Lie algebras. The absolute parallelism ω defines the so called constant vector fields $\omega^{-1}(X)$ on $\mathcal{G}$, for any $X \in \mathfrak{g}$, determined by the condition $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in \mathcal{G}$. Moreover, the same property provides a natural identification $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ where the action of P on $\mathfrak{g}/\mathfrak{p}$ is induced by the restricted adjoint action $\mathrm{Ad} : G \to GL(\mathfrak{g})$. The identification is given by the assignment $\{u, X + \mathfrak{p}\} \mapsto Tp \cdot \omega^{-1}(X)(u)$.

Let us suppose the subalgebra $\mathfrak{p}$ has got a complementary subalgebra $\mathfrak{n}$ in $\mathfrak{g}$. Then $\mathfrak{n}$ is identified with $\mathfrak{g}/\mathfrak{p}$ and via this identification the subalgebra $\mathfrak{n}$ comes to be a P-module; the "truncated" adjoint action is denoted by $\underline{\mathrm{Ad}}$. Any complementary subalgebra $\mathfrak{n}$ gives rise to a general connection on $\mathcal{G}$ which is principal if and only if $\mathfrak{n}$ is invariant with respect to the restricted Ad-action of P on $\mathfrak{g}$, in that case the Cartan geometry is called reductive.

Generalized geodesics on M are defined as projections of flow lines of horizontal constant vector fields, i.e. as curves of the shape $c^{u,X}(t) = p(\mathrm{Fl}_t^{\omega^{-1}(X)}(u))$ for any $u \in \mathcal{G}$ and $X \in \mathfrak{n}$. Obviously, the tangent vector of curve $c^{u,X}$ in $c(0) = p(u)$ is $\{u, X\}$ with respect to the above identification. Another representative of that vector defines another curve in general if the geometry is not reductive. The essential question appears in this context: What are all generalized geodesics with the common tangent vector in some point? The answer may depend on the type of the tangent vector if there are distinguished ones (as nullvectors in conformal geometries are). Distinguished directions of the same type in any tangent space are contained in a common orbit of action of the structure group P. They correspond to P-invariant subsets in $\mathfrak{n}$ so the classification of generalized geodesics always depends on the classification of P-invariant subsets in $\mathfrak{n}$.

For any fixed subset $A \subset \mathfrak{n}$ all curves $c^{u,X}$ with $X \in A$ form a subset among all generalized geodesics which is denoted by the symbol $\mathcal{C}_A$. Generalized geodesics of type $\mathcal{C}_A$ emanate in those tangent directions which correspond to the P-orbit of the subset A in $\mathfrak{n}$.

The homogeneous model of a Cartan geometry of type (G, P) consists of the principal bundle $G \to G/P$ with the Maurer–Cartan form playing the role of the Cartan connection. Constant vector fields on G are the left invariant ones and their flows are left shifts of 1-parametric subgroups. So generalized geodesics of type $\mathcal{C}_A$ on the homogeneous space G/P take the form $c^{g,X}(t) = g\exp(tX)\cdot P$ for any $g \in G$ and $X \in A \subseteq \mathfrak{n}$.

2.2. **Developments.** The essential tool for our purposes is the notion of development of curves via the construction of the Cartan's space $SM = \mathcal{G} \times_P G/P$ (with the obvious action of P on the homogeneous space G/P). The Cartan connection on $\mathcal{G}$ induces a general connection on SM which allows to define for any curve c on the base manifold M and any fixed point $x \in c$ its development into the fibre over x which is identified with G/P. This construction is often used to distinguish curves on M (or generally, on all manifolds endowed with a Cartan geometry of type (G, P)) by means of distinguished curves in the homogeneous model. The following statement holds, see [7].

Lemma. *Let $p : \mathcal{G} \to M$ be a Cartan geometry of type (G, P) with the Lie algebra decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ and let $c^{u,X}$ be a generalized geodesic for $u \in \mathcal{G}$ and $X \in \mathfrak{n}$. Then the development of $c^{u,X}$ in $x = p(u)$ is the curve $\{u, \exp(tX) \cdot P\} \subset S_x M$.*

Hereby defined developments generalize the classical concept of the development of curves on manifolds with affine connection. In that case the homogeneous space $G/P = \mathbb{R}^m$ is globally identified with $\mathfrak{n}$ so that the two actions of the structure group $P = GL(m, \mathbb{R})$ coincide. Then the Cartan's space SM equals to the tangent bundle TM and a curve is an affine geodesic if and only if it develops into a straight line within the tangent space of any single point.

2.3. **Technicalities.** Due to the above correspondence of generalized geodesics on M and their developments in G/P we may focus only on generalized geodesics in the homogeneous space G/P going through the origin $e \cdot P$. In other words, we are interested in curves of the form $c^{b,X}(t) = b \exp(tX) \cdot P = \exp(t \operatorname{Ad}_b X) \cdot P$ with $b \in P$ and $X \in A \subseteq \mathfrak{n}$. Further, let us assume the geometry in question is parabolic.

Let $G \to G/P$ be a homogeneous model of a parabolic geometry, i.e. G is a semisimple Lie group and P its parabolic subgroup. On the infinitesimal level, the subalgebra $\mathfrak{p} \subset \mathfrak{g}$ defines a grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. The subalgebra $\mathfrak{n} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is a canonical complement to $\mathfrak{p}$, usually denoted by $\mathfrak{g}_-$. The subalgebra $\mathfrak{g}_0$ is reductive and $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k = \mathfrak{p}_+$ is nilpotent. Altogether, P is the semidirect product $G_0 \rtimes \exp \mathfrak{p}_+$ where G_0 is the subgroup of P (with the Lie algebra $\mathfrak{g}_0$) whose all elements respect the gradation of $\mathfrak{g}$.

Now any element $b \in P$ is uniquely written as $b = b_0 \exp Z$, with $b_0 \in G_0$ and $Z \in \mathfrak{p}_+$, but it can be rewritten $b = \exp(\operatorname{Ad}_{b_0} Z) \cdot b_0$ as well. Hence, generalized geodesics $c^{b_0 \exp Z, X}$ and $c^{\exp(\operatorname{Ad}_{b_0} Z), \operatorname{Ad}_{b_0} X}$ coincide by the definition, so we will assume only curves of the form $c^{\exp Z, X}$ hereafter. If we restrict to the curves of type $\mathcal{C}_A$, the subset $A \subset \mathfrak{g}_-$ must be G_0-invariant for the above elimination to be valid. This convention will be kept in the rest of paper. Moreover, two curves c^{b_1, X_1} and c^{b_2, X_2} coincide if and only if curves c^{e, X_1} and $c^{b_1^{-1} b_2, X_2}$ do, so we will further suppose $b_1 = e$.

Let us compare generalized geodesics $c^{e,X}$ and $c^{\exp Z, Y}$. The equation

$$\exp(tX) = \exp(t \operatorname{Ad}_{\exp Z} Y) \cdot u(t) \tag{1}$$

defines a curve $u : \mathbb{R} \to G$, at least locally, which is analytic for arbitrary entries X, Y, and Z. The two curves coincide if and only if u takes values in P which is equivalent to all derivatives $u^{(i)}(0)$ are tangent to P. Similarly, the two curves have got a common r-jet in 0 if and only if the previous condition holds for all $i \leq r$.

By technical reasons we prefer to differentiate the map δu instead of u, where $\delta u : T\mathbb{R} \to \mathfrak{g}$ is the so called left logarithmic derivative of u. The map δu is defined by $\delta u = u^* \omega$ where ω is the Maurer–Cartan form on G. By definition, δu is determined by Tu so the following lemma certainly holds.

Lemma. *Curves $c^{e,X}$ and $c^{\exp Z,Y}$ determine a common r-jet in 0 if and only if the derivatives $(\delta u)^{(i)}(0)$ belong to $\mathfrak{p}$ for all $i \leq r-1$, where u is defined by (1).*

Now we go quickly through the technical results of [1] which will be essential for other computations. The Leibniz rule and other basic properties of the left logarithmic derivative lead to the lemma:

Lemma. *In the above setting, following equalities hold for all $i \geq 1$,*

$$\delta u(t) = X - \operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{\exp Z} Y\,, \tag{2}$$

$$(\delta u)^{(i)}(t) = (-\operatorname{ad}_X)^i \delta u(t)\,. \tag{3}$$

In particular, the condition $\delta u(0) \in \mathfrak{p}$ is satisfied if and only if $X = \underline{\operatorname{Ad}}_{\exp Z} Y$, i.e. the curves have got the same tangent vector in the origin. Let us suppose k is the length of grading of $\mathfrak{g}$. An easy consequence of (3) is that the assumption $(\delta u)^{(i)}(0) \in \mathfrak{p}$ for all $i \leq k+1$ implies $(\delta u)^{(i)}(0) = 0$ for all $i \geq k+1$. So any two generalized geodesics share the same $(k+2)$-jet in 0 if and only if they coincide. The number $k+2$ only proves the finiteness of the order but the estimate is not sharp at all, as one can see in next sections.

2.4. **Reparametrizations.** The previous paragraph will serve us to find all generalized geodesics of a certain type C_A which share a common tangent vector. However, some of those curves may parametrize the same unparametrized geodesic. In this paragraph we solve the question when two generalized geodesics coincide up to some reparametrization? In this context we have to rewrite the equation (1) as follows:

$$\exp(\varphi(t)X) = \exp(t \operatorname{Ad}_{\exp Z} Y) \cdot u(t)\,, \tag{4}$$

where φ is a reparametrization, i.e. $\varphi'(t) \neq 0$, and we further assume $\varphi(0) = 0$ for simplicity. The general formula for $(\delta u)^{(i)}(t)$ can be found in [1] but it is rather complicated. For our purposes expressions with $i \leq 2$ will suffice.

Lemma. *With the above notation, following equalities hold,*

$$\delta u(t) = \varphi'(t)X - \operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{\exp Z} Y\,, \tag{5}$$

$$(\delta u)'(t) = \varphi''(t)X - \varphi'(t)[X, \delta u(t)]\,, \tag{6}$$

$$(\delta u)''(t) = \varphi'''(t)X - \varphi''(t)[X, \delta u(t)] + \varphi'(t)^2[X,[X,\delta u(t)]]\,. \tag{7}$$

By (5), the condition $\delta u(0) \in \mathfrak{p}$ is satisfied if and only if tangent vectors of the two curves in the origin differ by the multiple $\varphi'(0)$. The other conditions define relations among derivatives of φ in 0 which gives a hint to guess an appropriate reparametrization satisfying (4).

In the case of $|1|$-graded parabolic geometries one can directly compute that if two generalized geodesics coincide up to parametrizations then the reparametrization must be projective, i.e. $\varphi(t) = \frac{At+B}{Ct+D}$ where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. In particular, all such functions solve the Schwartzian differential equation $\varphi''' = \frac{3}{2}\frac{\varphi''^2}{\varphi'}$. We will see later that geometries studied here admit the projective reparametrizations too besides the affine ones which are settled by the condition $\varphi''(t) = 0$.

2.5. **Recipe.** (1) Given the homogeneous model G/P of a parabolic geometry, we start with the description of distinguished types of tangent vectors. According to 2.1, they correspond to P-invariant subsets in $\mathfrak{g}_-$ with respect to the truncated adjoint action $\underline{\mathrm{Ad}}$. To any such subset we will look for its G_0-invariant subsets which define generalized geodesics emanating in the actual directions. Sometimes two such distinct subsets define the same class of curves so one of them is omitted in further discussions. For G_0-invariant subsets $A, B \subset \mathfrak{g}_-$ the classes of curves $\mathcal{C}_A$ and $\mathcal{C}_B$ obviously coincide if for any $X \in A$ there is an element $b \in P$ such that $\mathrm{Ad}_b(X) \in B$, and conversely, as happens in the elliptic case.

(2) Let A be a G_0-invariant subset in $\mathfrak{g}_-$ and $c^{e,X}$ be a generalized geodesic of type $\mathcal{C}_A$. Step by step, we search $Z \in \mathfrak{p}_+$ and $Y \in A$ such that the curves $c^{e,X}$ and $c^{\exp Z,Y}$ coincide. At the same time we get the order of jet which decides the two curves are equal.

The first condition $\delta u(0) \in \mathfrak{p}$ restricts $Z \in \mathfrak{p}_+$ and $Y \in A$ to fulfil the equality $Y = \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X)$. For any $|2|$-graded parabolic geometry the latter condition means $Y = X - [Z_1, X_2]$, where Z_1 is the $\mathfrak{g}_1$-part of Z and X_2 the $\mathfrak{g}_{-2}$-part of X. The other conditions $(\delta u)^{(i)}(0) \in \mathfrak{p}$ further reduce possible $Z \in \mathfrak{p}_+$ for the two curves share a common $(i+1)$-jet. All such elements form a subset in $\mathfrak{p}_+$ denoted by the symbol B^{i+1}. More precisely, for any $r \geq 1$ we put $B^r = \{Z \in \mathfrak{p}_+ : j^r_0 c^{e,X} = j^r_0 c^{\exp Z,Y}\}$ where $Y = \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X)$. In particular, $B^1 = \{Z \in \mathfrak{p}_+ : \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X) \in A\}$. Whenever the condition $(\delta u)^{(r)}(0) \in \mathfrak{p}$ implies $(\delta u)^{(r+1)}(0) = 0$ then generalized geodesics of a given type are uniquely determined by a jet of order $r+1$. This is equivalent to the condition $B^{r+1} = B^s$ to be true for all $s > r+1$.

(3) Now we are interested in the dimension of the set of parametrized generalized geodesics of type $\mathcal{C}_A$ sharing the same tangent vector ξ. This set is denoted by $\mathcal{C}^\xi_A$ and its dimension does not depend on the vector of the type in question. In view of the above arguments, let $\xi = \{e, X\}$ be the fixed vector and further let r be an order of jet which determines generalized geodesics with the given tangent vector uniquely. Obviously, for any $Z \in B^r$ the curves $c^{e,X}$ and $c^{\exp Z, \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X)}$ coincide. If another representative of the vector $\{e, X\}$ is chosen then the analogously defined subsets $B^i \subseteq \mathfrak{p}_+$ are naturally identified with the initial ones. So the set $\mathcal{C}^\xi_A$ is parametrized by the quotient B^1/B^r which is easy to describe.

(4) Finally, we are interested in distinguished parametrizations of generalized geodesics of a given type. Lemma 2.4 allows us to find all reparametrizations which appear if two generalized geodesics parametrize the same curve. The function φ is guessed according to its behaviour in 0, so we must always check the guess is right, i.e. the equation (4) in 2.4 holds true for all t. Reparametrizations which appear in this way will be either projective or affine.

On the other hand, for any generalized geodesic $c^{g,X}$ and any reparametrization φ of the admissible type the curve $c^{g,X} \circ \varphi$ is a generalized geodesic too. This is trivially satisfied if φ is affine. Otherwise, the statement is true if and only if $\varphi''(0)$ (expressed during the computation in variables X and Z) can get all values. This can be easily checked in all cases discussed below.

Remark. Most of necessary computations was done with the help of the computational system Maple, so computational details in classifications below are available in the format of Maple worksheets in [8].

3. Hyperbolic structures

The section starts with a complete classification of generalized geodesics on an embedded CR-quadric of codimension 1 in $\mathbb{C}^2$ which is viewed as the homogeneous model of 3-dimensional CR-structures of the hypersurface type. Let us remind that the homogeneous model is the only one in this case since the whole classification depends on the signature of the Levi form on the CR-subspace whose complex dimension is 1. The main purpose of that example is to simplify the discussion on the hyperbolic quadric of codimension 2 in $\mathbb{C}^4$ which is a product of two above mentioned quadrics.

3.1. **Opening example.** Let Q be a real hypersurface in $\mathbb{C}^2$ defined by the equation

$$\mathrm{Im}(w) = |z|^2$$

in coordinates (z, w). Usually, it is called the 3-dimensional CR-sphere although the standard sphere is given by $|z|^2 + |w|^2 = 1$. However, these two surfaces are locally equivalent and, moreover, there is a global biholomorphism between them if one point of the standard sphere is removed (or mapped to ∞), see [2].

On the quadric Q the semisimple Lie group $G = SU(2, 1)$ acts as follows. The group $SU(2, 1)$ consists of all linear automorphisms of $\mathbb{C}^3$ which preserve a Hermitian inner product of the signature $(2, 1)$. Let us suppose the Hermitian form given by the matrix $\begin{pmatrix} 0 & 0 & -i/2 \\ 0 & 1 & 0 \\ i/2 & 0 & 0 \end{pmatrix}$. The set of all nonzero nullvectors forms a cone $\frac{i}{2}(z_1\bar{z}_3 - z_3\bar{z}_1) + z_2\bar{z}_2 = 0$ invariant with respect to the action of G. The complex projectivization of the cone is naturally identified with Q via the inclusion $\mathbb{C}^2 \to \mathbb{C}^3$ defined by $(z, w) \mapsto (1, z, w)$. The action of G on the cone factors to a transitive action on the CR-sphere Q where any element of G acts by a CR-automorphism. Conversely, the group of CR-automorphisms of Q is just the quotient of G by a noneffective kernel which is isomorphic to $\mathbb{Z}_3$. The stabilizer P of the origin is the Borel subgroup in the semisimple G. Altogether, Q can be identified with the quotient space G/P and the principal bundle $G \to G/P$ represents a homogeneous model of 3-dimensional CR-geometries of the hypersurface type.

In the above described representation of the principal group $G = SU(2, 1)$, the Lie algebra $\mathfrak{g} = \mathfrak{su}(2, 1)$ looks like $\left\{ \begin{pmatrix} y & -2iz & b \\ x & -2i\,\mathrm{Im}\,y & -\bar{z} \\ a & -2ix & -\bar{y} \end{pmatrix} : x, y, z \in \mathbb{C}, a, b \in \mathbb{R} \right\}$. The parabolic subalgebra $\mathfrak{p}$ consists of upper triangular matrices and the gradation $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is given by the five diagonals. With respect to the identification $TQ = G \times_P \mathfrak{g}_-$ from 2.1, the CR-subbundle is associated as $T^{CR}Q = G \times_P \mathfrak{g}_{-1}$. Now we can apply the process suggested in 2.5 to describe generalized geodesics in this case.

First of all, there are two distinguished kinds of vectors in TQ. The first ones belong to the CR-subspace $T^{CR}Q$, the second ones are transverse. The two classes of vectors correspond to the complementary P-invariant subsets $\mathfrak{g}_{-1}$ and $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$ in $\mathfrak{g}_-$. There are no other distinguished directions in TQ, i.e. no other nontrivial P-invariant subsets in $\mathfrak{g}_-$. The previous two subsets are clearly G_0-invariant and there is another G_0-invariant subset in $\mathfrak{g}_-$, the last component $\mathfrak{g}_{-2}$ in the gradation of $\mathfrak{g}_-$. Its complement in $\mathfrak{g}_-\setminus\mathfrak{g}_{-1}$ is also G_0-invariant and represents the generic case. Curves determined by the subset $\mathfrak{g}_{-2}$ are the well known Chern–Moser chains which form a CR-invariant class of curves with particularly nice properties. Especially, the P-orbit of $\mathfrak{g}_{-2}$ is the entire $\mathfrak{g}_-\setminus\mathfrak{g}_{-1}$ so the chains emanate in all directions transversal to the CR-subspace. Let us discuss all mentioned types of generalized geodesics:

(1) $A=\mathfrak{g}_{-1}$. For an arbitrary $Z\in\mathfrak{p}_+$, the curves $c^{e,X}$, $c^{\exp Z,Y}$ share the same tangent vector in origin if and only if $Y=X$. The curves have got a common 2-jet if and only if Z belongs to $\mathfrak{g}_2$. The condition on 3-jet, i.e. $(\delta u)''(0)\in\mathfrak{p}$, is satisfied if and only if $Z=0$ which implies $(\delta u)''(0)=0$. Altogether, curves of this type are determined by a 3-jet.

Parametrized curves $c^{\exp Z,X}(t)$, for any $Z\in\mathfrak{p}_+$, are different from each other and keep the common tangent vector $\xi=\{e,X\}$ hence the set $\mathcal{C}^{\xi}_{\mathfrak{g}_{-1}}$ is parametrized by elements of $\mathfrak{p}_+$, the dimension of which is 3. With the notation of 2.5, $B^1=\mathfrak{p}_+$, $B^2=\mathfrak{g}_2$, and $B^3=B^4=\cdots=0$.

Let $X=\begin{pmatrix}0&0&0\\x&0&0\\0&-2i\bar{x}&0\end{pmatrix}\in\mathfrak{g}_{-1}$. Looking for the admissible reparametrizations one can find that all curves from $\mathcal{C}^{\xi}_{\mathfrak{g}_{-1}}$ which coincide up to parametrization with $c^{e,X}$ are of the form $c^{\exp(sZ),X}$ where $s\in\mathbb{R}$ and $Z=\begin{pmatrix}0&2\bar{x}&0\\0&0&ix\\0&0&0\end{pmatrix}$. Reparametrizations which appear in this way are just the projective ones and $\varphi''(0)=-2s\varphi'(0)^2|x|^2$ really takes all values.

(2) $A=\mathfrak{g}_{-2}$. The curves $c^{e,X}$, $c^{\exp Z,Y}$ have got the same tangent vector in the origin if and only if $Z\in\mathfrak{g}_2$ and $Y=X$. They share a common 2-jet if and only if $Z=0$. Then $(\delta u)'(0)=0$ so the chains are determined by a 2-jet and the set $\mathcal{C}^{\xi}_{\mathfrak{g}_{-2}}$ is parametrized by elements of $\mathfrak{g}_2$ whose dimension is 1. Moreover, all curves from $\mathcal{C}^{\xi}_{\mathfrak{g}_{-2}}$ parametrize the same unparametrized chain and the admissible reparametrizations are projective. For $X=\begin{pmatrix}0&0&0\\0&0&0\\a&0&0\end{pmatrix}$ fixed and $Z=\begin{pmatrix}0&0&s\\0&0&0\\0&0&0\end{pmatrix}\in\mathfrak{g}_2$ arbitrary, the value of $\varphi''(0)$ is $-2s\varphi'(0)^2a$.

Altogether, in any direction which does not belong to the CR-subspace of the tangent space there is a unique unparametrized chain endowed with a canonical projective structure. This is a very classical result which can be also found in [2] or [3].

(3) $A=\mathfrak{g}_-\setminus(\mathfrak{g}_{-1}\cup\mathfrak{g}_{-2})$. The curves $c^{e,X}$, $c^{\exp Z,Y}$ share the same tangent vector if and only if $Y=X-[Z_1,X_2]$ and $Z=Z_1+Z_2\in\mathfrak{p}_+$ is arbitrary. They have got the same 2-jet if and only if $Z=0$. Then $Y=X$ and $(\delta u)'(0)=0$ so the curves of this type are determined by a 2-jet and the set $\mathcal{C}^{\xi}_{A}$ is 3-dimensional, parametrized by all elements of $\mathfrak{p}_+$.

The distinct difference against the above two cases is that there are no two curves in $\mathcal{C}^{\xi}_{A}$ which would coincide up to some reparametrization. It turns out the class of admissible reparametrizations on any curve of this type is only affine. For any tangent vector transversal to the CR-subspace there is, besides the chains, the 3-dimensional family of uniquely parametrized generalized geodesics of the generic type.

3.2. **Hyperbolic quadric.** The hyperbolic quadric H is given by the equality (2) in the first section, i.e. it is expressed as

$$\operatorname{Im}(w_1)=|z_1|^2\quad\operatorname{Im}(w_2)=|z_2|^2 \tag{1}$$

with respect to the coordinates (z_1, z_2, w_1, w_2) of $\mathbb{C}^4$. Obviously, the hyperbolic quadric is the direct product of two CR-spheres discussed in 3.1 which lie in the subspaces (z_1, w_1) and (z_2, w_2), respectively. The action of the group $G = SU(2,1) \times SU(2,1)$ on H is given by the product of the two actions of $SU(2,1)$ on each CR-sphere. The isotropy subgroup P of the origin is the product of two copies of the parabolic subgroup in the CR-sphere case. Obviously, G is semisimple and P parabolic. Any element of G acts on H by a CR-automorphism. Conversely, the group of automorphisms of H is isomorphic to the semidirect product

$$(SU(2,1)/\mathbb{Z}_3 \times SU(2,1)/\mathbb{Z}_3) \rtimes \mathbb{Z}_2 \tag{2}$$

where the group $\mathbb{Z}_2$ consists of the identity and an involutive automorphism which interchanges the two spheres in the product.

The discussion on generalized geodesics is based only on results of 3.1 due to the product structure on H. Especially, the tangent bundle TH is a direct product of $T^L H$ and $T^R H$, the left and the right subbundle, which correspond to the vanishing right and left part of $\mathfrak{g}_- = \mathfrak{g}_-^L \times \mathfrak{g}_-^R$, respectively. Vectors belonging to one of these subspaces are called singular.

Let us suppose the principal group G of the geometry is the group (2). Then $P = (B/\mathbb{Z}_3 \times B/\mathbb{Z}_3) \rtimes \mathbb{Z}_2$ where B is the Borel subgroup of $SU(2,1)$, i.e. the parabolic subgroup from the CR-sphere case. All P-invariant and G_0-invariant subsets in $\mathfrak{g}_- = \mathfrak{g}_-^L \times \mathfrak{g}_-^R$ are obtained as products of P and G_0-invariant subsets in each slot up to their interchanging. Now the singular directions correspond to the P-invariant subset $\mathfrak{g}_-^L \cup \mathfrak{g}_-^R$ which is just the P-orbit of $\mathfrak{g}_-^L$. Similarly, all P and G_0-invariant subsets in $\mathfrak{g}_-$ discussed below are written in the brief form, i.e. $A \times B \subset \mathfrak{g}_-^L \times \mathfrak{g}_-^R$ means its P and G_0-orbit $A \times B \cup B \times A$, respectively.

Altogether, there are 5 kinds of tangent vectors in TH and 9 types of generalized geodesics on H. The corresponding G_0-invariant subsets in $\mathfrak{g}_-$ are

$$\begin{aligned}
A_1 &= \{0\} \times \mathfrak{g}_{-1}^R\,, & A_4 &= \mathfrak{g}_{-1}^L \times \mathfrak{g}_{-1}^R \setminus S\,, & A_7 &= \mathfrak{g}_{-2}^L \times \mathfrak{g}_{-2}^R \setminus S\,,\\
A_2 &= \{0\} \times \mathfrak{g}_{-2}^R\,, & A_5 &= \mathfrak{g}_{-1}^L \times \mathfrak{g}_{-2}^R \setminus S\,, & A_8 &= \mathfrak{g}_{-2}^L \times \mathfrak{g}_{--}^R \setminus S\,,\\
A_3 &= \{0\} \times \mathfrak{g}_{--}^R\,, & A_6 &= \mathfrak{g}_{-1}^L \times \mathfrak{g}_{--}^R \setminus S\,, & A_9 &= \mathfrak{g}_{--}^L \times \mathfrak{g}_{--}^R \setminus S\,.
\end{aligned}$$

The symbol $\mathfrak{g}_{--}^L$ denotes the set of generic vectors $\mathfrak{g}_-^L \setminus (\mathfrak{g}_{-1}^L \cup \mathfrak{g}_{-2}^L)$ in $\mathfrak{g}_-^L$, alike for $\mathfrak{g}_{--}^R$ and $\mathfrak{g}_{--}$, and S represents singular vectors, i.e. $S = A_1 \cup A_2 \cup A_3$. Subsets A_1 and A_4 are P-invariant so they define distinguished directions in TH, namely they are singular and nonsingular vectors in $T^{CR}H$, respectively. Further, the P-orbit of A_2, denoted by $P(A_2)$, coincide with $P(A_3) = \{0\} \times (\mathfrak{g}_-^R \setminus \mathfrak{g}_{-1}^R)$ and this set describes all singular directions which do not belong to $T^{CR}H$. Next type of tangent vectors is given by $P(A_7) = P(A_8) = P(A_9) = \mathfrak{g}_- \setminus \mathfrak{g}_{-1}$ which corresponds to nonsingular directions not belonging to $T^{CR}H$. The last type of distinguished directions correspond to the orbit $P(A_5) = P(A_6) = \mathfrak{g}_{-1}^L \times (\mathfrak{g}_-^R \setminus \mathfrak{g}_{-1}^R)$.

Curves which emanate in singular directions are fully classified in 3.1 so there are 6 types of generalized geodesics emanating in nonsingular directions of TH left to be discussed. The product structure of H with the isolated action of the structure group on each slot leads directly to the following results compiled only from 3.1:

For A_4, A_5, and A_6, one side in the product is isomorphic to $\mathfrak{g}^L_{-1}$ and so the corresponding generalized geodesics are determined by a 3-jet, otherwise a 2-jet is enough. The dimensions of families of generalized geodesics with the common tangent vector are obtained by summing the numbers from 3.1, i.e. $\dim \mathcal{C}^\xi_{A_4} = 6$, $\dim \mathcal{C}^\xi_{A_5} = 4$, $\dim \mathcal{C}^\xi_{A_6} = 6$, $\dim \mathcal{C}^\xi_{A_7} = 2$, $\dim \mathcal{C}^\xi_{A_8} = 4$, and $\dim \mathcal{C}^\xi_{A_9} = 6$.

The only difficulties are to express explicitly all generalized geodesics with a common tangent vector parametrizing the same curve if the admissible reparametrizations are projective. In general, if at least one slot of a G_0-invariant subset is $\mathfrak{g}^L_{--}$ then the admissible class of reparametrizations is affine. This is the case of A_6, A_8, and A_9. Otherwise the reparametrizations are projective. All details can be found in [8].

Among all nonsingular types of generalized geodesics there are three of them of a particular interest. The first one is given by the choice $A_4 = \mathfrak{g}_{-1}$ where corresponding generalized geodesics emanate in generic directions of the CR-distribution $T^{CR}H$. Curves of this type are determined by a 3-jet in one point and they admit the projective class of reparametrizations. The second distinguished choice is $A_7 = \mathfrak{g}_{-2}$, which defines chains in nonsingular directions as discussed in [6], and the last choice of $A_9 = \mathfrak{g}_{--}$ defines curves of the generic type. For any nonsingular direction which does not belong to $T^{CR}H$ there is a 1-dimensional family of unparametrized chains with the projective class of parametrizations and 6-dimensional family of uniquely parametrized curves of type $\mathcal{C}_{A_9}$.

4. Elliptic structures

The elliptic quadric E is expressed in section 1 by the equality (4). In coordinates (z_1, z_2, w_1, w_2) of $\mathbb{C}^4$ it is the graph of

$$\mathrm{Im}(w_1) = \mathrm{Re}(z_1\bar{z}_2) \quad \mathrm{Im}(w_2) = \mathrm{Im}(z_1\bar{z}_2)\,. \tag{1}$$

The group of automorphisms of the CR-structure on E is less visible than in the hyperbolic case. However, the group is isomorphic to

$$G = SL(3,\mathbb{C})/\mathbb{Z}_3 \rtimes \mathbb{Z}_2 \tag{2}$$

as shown in [6]. Similarly to the hyperbolic case, $\mathbb{Z}_3$ represents a noneffective kernel in $SL(3,\mathbb{C})$ and, on the infinitesimal level, $\mathbb{Z}_2$ provides the interchanging of the two components of $T^{CR}E$, see below. Isotropy parabolic subgroup P of the origin is the semidirect product $B/\mathbb{Z}_3 \rtimes \mathbb{Z}_2$ where B is the Borel subgroup of $SL(3,\mathbb{C})$ consisting of upper triangular matrices. The Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, viewed as a real Lie algebra, and the corresponding parabolic subalgebra $\mathfrak{p}$ defines the gradation of $\mathfrak{g}$ according to the five diagonals.

In addition, the subspace $\mathfrak{g}_{-1} = \left\{ \left(\begin{smallmatrix} 0&0&0\\ x&0&0\\ 0&y&0 \end{smallmatrix}\right) : x, y \in \mathbb{C} \right\} \subset \mathfrak{g}_-$ defining the CR-distribution decomposes into $\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-1}$ which induces a product structure on $T^{CR}E$. At the same time, subspaces $\mathfrak{g}^L_{-1}$ and $\mathfrak{g}^R_{-1}$ distinguish tangent vectors in the CR-subspace. More precisely, the minimal P-invariant subset in $\mathfrak{g}_{-1}$ containing $\mathfrak{g}^L_{-1}$ is $\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}$. Its complement in $\mathfrak{g}_{-1}$ is P-invariant too and the complement of $\mathfrak{g}_{-1}$ in $\mathfrak{g}_-$ is the last P-invariant subset in $\mathfrak{g}_-$ defining directions transversal to the CR-distribution. There are 4 types of generalized geodesics on the elliptic quadric which correspond to the following G_0-invariant subsets in $\mathfrak{g}_-$:

$$\begin{aligned} A_1 &= \mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}, \\ A_2 &= \mathfrak{g}_{-1} \setminus (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}), \\ A_3 &= \mathfrak{g}_{-2}, \\ A_4 &= \mathfrak{g}_- \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2} \times (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})). \end{aligned}$$

The natural candidate on the post of A_4 is the complement of $\mathfrak{g}_{-2}$ in $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$, i.e. the subset $\mathfrak{g}_- \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2})$. However, the actual A_4 arises by excluding the G_0-invariant subset $A = \mathfrak{g}_{-2} \times (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})$ from the latter set. The only reason of its excluding is that the class of geodesics of type $\mathcal{C}_A$ coincide with the class of chains determined by $A_3 = \mathfrak{g}_{-2}$; this is shown in [8]. Moreover, the P-orbit of both A_3 and A_4 is $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$, hence for any tangent vector transversal to $T^{CR}E$ there are generalized geodesics of these two types.

(1) $A_1 = \mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}$. Two curves $c^{e,X}$ and $c^{\exp Z,Y}$ share the same tangent vector in the origin if and only if $Y = X$ and $Z \in \mathfrak{p}_+$ is arbitrary. Let be $X = \begin{pmatrix} 0&0&0 \\ x&0&0 \\ 0&0&0 \end{pmatrix} \in \mathfrak{g}^L_{-1}$. The curves have got a common 2-jet if and only if Z belongs to $B^2 = \left\{ \begin{pmatrix} 0&0&z \\ 0&0&w \\ 0&0&0 \end{pmatrix} : w, z \in \mathbb{C} \right\}$. Now already $(\delta u)'(0) = 0$ so the curves of this type are determined by a 2-jet. The set $\mathcal{C}^\xi_{A_1}$ is parametrized by all elements of $\mathfrak{p}_+/B^2 = \left\{ \begin{pmatrix} 0&v&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix} : v \in \mathbb{C} \right\}$ so its dimension is 2. All curves from $\mathcal{C}^\xi_{A_1}$ which parametrize the same curve as $c^{e,X}$ look like $c^{\exp(sZ),X}$ for any $s \in \mathbb{R}$ and $Z = \begin{pmatrix} 0&\bar{x}&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$. Admissible reparametrizations are projective and $\varphi''(0) = -2s\varphi'(0)^2|x|^2$ can get all values.

(2) $A_2 = \mathfrak{g}_{-1} \setminus (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})$. The curves $c^{e,X}$ and $c^{\exp Z,Y}$ have got the same 1-jet if and only if $Y = X$ and Z is arbitrary, the same 2-jet if and only if $Z \in \mathfrak{g}_2$, and the same 3-jet if and only if $Z = 0$. Then $(\delta u)''(0) = 0$ so generic generalized geodesics tangent to CR-subspaces are determined by a 3-jet. The set $\mathcal{C}^\xi_{A_2}$ is parametrized by $\mathfrak{p}_+$ whose dimension is 6. Curves from $\mathcal{C}^\xi_{A_2}$ parametrizing the same curve as $c^{e,X}$, $X = \begin{pmatrix} 0&0&0 \\ x&0&0 \\ 0&y&0 \end{pmatrix}$, are of the form $c^{\exp(sZ),X}$ where $s \in \mathbb{R}$ and $Z = \begin{pmatrix} 0&x^{-1}&0 \\ 0&0&y^{-1} \\ 0&0&0 \end{pmatrix}$. Admissible reparametrizations are the projective ones and $\varphi''(0) = -s\varphi'(0)$ takes all values.

(3) $A_3 = \mathfrak{g}_{-2}$. The curves $c^{e,X}$ and $c^{\exp Z,Y}$ share the same tangent vector if and only if $Y = X$ and $Z \in \mathfrak{g}_2$. The condition on 2-jet implies $Z = 0$ and $(\delta u)'(0) = 0$. Hence chains are determined by a 2-jet and the set $\mathcal{C}^\xi_{\mathfrak{g}_{-2}}$ is parametrized by $\mathfrak{g}_2$ whose dimension is 2. All curves which coincide up to parametrization with $c^{e,X}$, $X = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ x&0&0 \end{pmatrix}$, take the form $c^{\exp(sZ),X}$ where $s \in \mathbb{R}$ and $Z = \begin{pmatrix} 0&0&\bar{x} \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$. Then $\varphi''(0) = -2s\varphi'(0)^2|x|^2$ and the admissible reparametrizations are projective.

(4) $A_4 = \mathfrak{g}_- \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2} \times (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}))$. The curves $c^{e,X}$ and $c^{\exp Z,Y}$ share the same 1-jet if and only if $Y = X - [Z_1 X_2]$ for an arbitrary $Z = Z_1 + Z_2 \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Their 2-jets coincide if and only if $Z = 0$ which implies $Y = X$ and $(\delta u)'(0) = 0$. Hence curves of this type are determined by a 2-jet and the set $\mathcal{C}^\xi_{A_4}$ is parametrized by whole $\mathfrak{p}_+$ so its

dimension is 6. There are no two distinct geodesics in $\mathcal{C}^{\xi}_{A_4}$ which would parametrize the same curve, so the admissible reparametrizations are affine.

There are some technical problems in the computation of properties of curves of the last type. The system of equations which appears there is too large so that Maple is not able to solve it. For the order of jet which determines curves of type $\mathcal{C}_{A_4}$ uniquely it is sufficient to work over complex numbers instead of reals since all matrices are complex.

The same approach applied to the reparametrizations gives results which look like the above ones with the following one and only difference. Originally, the parameter s describing all curves from $\mathcal{C}^{\xi}_{A}$ (in any particular case) which parametrize the same curve as $c^{e,X}$ is real but now it is complex, so it is not possible to deduce the previous results in this way. Nevertheless, if there is no freedom in the latter approach then there is no freedom in the former one, so the only admissible reparametrizations are affine. This just happens in the case of curves of the generic type. All details can be found in Maple worksheets [8].

Remark. The above classification is very similar to that in 3.1 for the CR-sphere. The only exceptions are that chains here behave like chains in the hyperbolic case and there are no distinguished directions in the CR-distribution of the CR-sphere. However, for CR-spheres of higher CR-dimension and indefinite signature there are distinguished ones of zero length. The corresponding generalized geodesics are called null-chains and their properties are the same as for curves of type $\mathcal{C}_{A_1}$ here, see [3].

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