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PRODUCTS AND MEASURABLE CARDINALS

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Following van Douwen [1], let us call a space X to be <u>extremally disconnected at</u> a point p, $p \in X$, if for every two disjoint open sets U and V in X, $p \notin \mathrm{cl}U \cap \mathrm{cl}V$. This is a local version of the well known concept of extremal disconnectedness: a space is <u>extremally disconnected</u> if every its two disjoint open subsets have disjoint closures, or equivalently, if the closure of any its open subset is again open. Results of van Douwen's paper [1] decide generally the question whether a space is extremally disconnected at some point for spaces being the Čech-Stone compactification of another one.

We aim to decide the same question but for spaces being (topological) products of two or more factors. Our main results are:

there exists a measurable cardinal iff there are two Hausdorff spaces X and Y and their non-isolated points $x \in X$ and $y \in Y$ such that the space $X \times Y$ is extremally disconnected at the point (x,y);

Con(ZFC + "there exists a measurable cardinal") $\longrightarrow$ Con(ZFC + + "there exist two completely regular Hausdorff dense-in-itself spaces whose product is extremally disconnected" + "for every three Hausdorff spaces X, Y and Z and for every their non-isolated points $x \in X$, $y \in Y$ and $z \in Z$, the space $X \times Y \times Z$ is not extremally disconnected at the point (x,y,z)").

The motivation for considering extremal disconnectedness for topological products comes also from Boolean algebras (BA for shortening). It is well known that the free product of two infinite BA's cannot be complete BA. The question arises, "how much" is it incomplete? To give an answer, consider the following local version of completeness of BA's. Call two ideals I, J of a BA B <u>complementary</u> if $I \wedge J = \{0\}$ (see [2]). We say that the BA B is <u>complete at</u> the ultrafilter $U \subset B$ if for any two complementary ideals $I, J \subset B$ there is a u, $u \in U$, such that either $u \wedge i = 0$ for every $i \in I$, or $u \wedge j = 0$ for every $j \in J$. One can verify that a BA is complete iff it is complete at each its ultrafilter. Now, some of our results can be stated as follows (see Corollary 1.3 and Corollry 2.6): under V = L, the free product of two infinite BA's is complete at no ultrafilter, but assuming the existence of a measurable cardinal, there are two infinite complete BA's whose free product is complete at "many" ultrafilters.

We have used the topological language instead of Boolean algebraic one, as it allows to get results in much more general form. We refer to Engelking's book [3] and Jech's book [4] for undefined topological and set-theoretical notions.

1. Cellular families. A family $\mathcal{R}$ of pairwise disjoint open subsets of a space X is a cellular family for p, $p \in X$, if $p \notin \mathrm{cl}U$ for $U \in \mathcal{R}$ and $p \in \mathrm{cl}\bigcup\mathcal{R}$. If p is a non-isolated point of a Hausdorff space X, then there exists a cellular family for p of cardinality not greater than the character of p, and there exists a cellular family for p of cardinality not greater than the cellularity of X.

For a non-isolated point p of a Hausdorff space X let us set $c(p,X) = \inf\{|\mathcal{R}| : \mathcal{R}$ is a cellular family for p$\}$. Then c(p,X) is a regular infinite cardinal.

If κ is an infinite cardinal, then $P^*(\kappa)$-point of a space X is a point that lies in the interior of every intersection of κ regular open sets containing the point.

Theorem 1.1. If p is a non-isolated point of a Hausdorff space X, then $c(p,X) = \inf\{\kappa : p$ is not a $P^*(\kappa)$-point of $X\}$.

Proof. Let $\lambda = \inf\{\kappa : p$ is not a $P^*(\kappa)$-point of $X\}$. It is obvious that if p is a $P^*(\kappa)$-point of X, then $c(p,X) > \kappa$. Hence $c(p,X) \geq \lambda$. To prove the converse inequality, let $\{U_\alpha : \alpha < \lambda\}$ be a family of regular open neighborhoods of p such that p does not lie in the interior of the intersection of that family. Set $V_0 = X - \mathrm{cl}U_0$ and $V_\beta = \mathrm{int}\bigcap\{U_\alpha : \alpha < \beta\} - \mathrm{cl}U_\beta$, for $\beta < \lambda$. The family $\mathcal{R} = \{V_\alpha : \alpha < \lambda\}$ is a cellular family for p. To see this we verify the condition $p \in \mathrm{cl}\bigcup\mathcal{R}$, only; remaining conditions can be easily checked. Let G be arbitrary open neighborhood of p. Let β, $\beta < \lambda$, be the least ordinal such that $G - U_\beta \neq \emptyset$. Since U_β is regular open $G - \mathrm{cl}U_\beta \neq \emptyset$. Hence $G \cap V_\beta \neq \emptyset$ □

An uncountable cardinl κ is said to be measurable if there is a non-principal κ-complete ultrafilter over κ, i.e., if there is a family ξ of subsets of κ such that:

(1) $\{\alpha\} \notin \xi$ for every $\alpha \in \kappa$,

(2) if $A \in \xi$ and $A \subset B \subset \kappa$, then $B \in \xi$,

(3) for every $A \subset \varkappa$, either $A \in \xi$ or $\varkappa - A \in \xi$,

(4) if $S \subset \xi$ and $|S| < \varkappa$, then $\bigcap S \in \xi$.

Theorem 1.2. If p is a non-isolated point of a Hausdorff space X and the space X is extremally disconnected at p, then either $c(p,X) = \aleph_0$ or $c(p,X)$ is a measurable cardinal.

Proof. Let us assume that $\varkappa = c(p,X) > \aleph_0$. Let $\mathcal{R}$ be a cellular family for p such that $|\mathcal{R}| = \varkappa$, say $\mathcal{R} = \{U_\alpha : \alpha < \varkappa\}$. Consider a set ξ defined in the following way: $\xi = \{A \subset \varkappa : p \in \mathrm{cl} \bigcup \{U_\alpha : \alpha \in A\}\}$. It can be easily verified that ξ satisfies conditions (1)-(3) for $\varkappa$ to be measurable. We verify the condition (4). Let $S \subset \xi$ and let $|S| < \varkappa$. For $A \in S$ we set $V_A = X - \mathrm{cl} \bigcup \{U_\alpha : \alpha \in \varkappa - A\}$. Then $\{V_A : A \in S\}$ is a family of less than $\varkappa$ regular open sets containing p. By Theorem 1.1, there is an open neighborhood G of p such that $G \subset \bigcap \{V_A : A \in S\}$. Hence $p \notin \mathrm{cl} \bigcup \{U_\alpha : \alpha \in \bigcup \{\varkappa - A : A \in S\}\}$ and therefore $p \in \mathrm{cl} \bigcup \{U_\alpha : \alpha \in \bigcap \{A : A \in S\}\}$. This means $\bigcap S \in \xi$ ▫

Theorem 1.3. Let X_s, $s \in S$, be Hausdorff spaces such that $|X_s| > 1$ for every $s \in S$ and let $|S| \geq \aleph_0$. If X is the product of all spaces X_s, then $c(p,X) = \aleph_0$ for every $p \in X$.

Proof. Let $s_n \in S$, $n \in \omega$, and $s_n \neq s_m$ if $n \neq m$. For a given point $p \in X$ let U_n be a regular open neighborhood of the point $p(s_n)$ such that $U_n \neq X_{s_n}$ (such U_n exists because $|X_{s_n}| > 1$ and X_{s_n} is Hausdorff). We set $V_n = \{x \in X : x(s_0) \in U_0 \wedge \ldots \wedge x(s_n) \in U_n\}$. Then each V_n, $n \in \omega$, is a regular open neighborhood of the point p and p does not lie in the interior of their intersection. Hence p is not a $P^*(\omega)$-point of X. By Theorem 1.1, $c(p,X) = \aleph_0$, p being non-isolated point of the Hausdorff space X ▫

2. Are products extremally disconnected at any point? If x is an isolated point of a space X and a space Y is extremally disconnected at a point y, the answer is yes: the product $X \times Y$ is extremally disconnected at the point (x,y). However if we restrict ourself to the non-trivial case, i.e., both x and y are non-isolated, then the answer is much more complicated. It turns out, we have to deal with measurable cardinals.

Theorem 2.1. Suppose X and Y are Hausdorff spaces and let $x \in X$ and $y \in Y$ be non-isolated points. If $c(x,X) = c(y,Y)$, then the space $X \times Y$ is not extremally disconnected at the point (x,y).

Proof. One can choose cellular families $\mathcal{R}$ for x and $\mathcal{S}$ for y such that $|\mathcal{R}| = |\mathcal{S}| = \varkappa$, where $\varkappa = c(x,X) = c(y,Y)$. Enumerate $\mathcal{R}$ as $\{U_\alpha : \alpha < \varkappa\}$ and $\mathcal{S}$ as $\{V_\alpha : \alpha < \varkappa\}$. Let us put $G_\beta = U_\beta \times (Y - \mathrm{cl} \bigcup \{V_\alpha : \alpha < \beta\})$ and $H_\beta = (Y - \mathrm{cl} \bigcup \{U_\alpha : \alpha < \beta\}) \times V_\beta$ for $\beta < \varkappa$. If we set G =

$= \bigcup\{G_\beta : \beta<\kappa\}$ and $H = \bigcup\{H_\beta : \beta<\kappa\}$, then G and H are disjoint open subsets of the space $X\times Y$. It remains to show that $(x,y)\in \mathrm{cl}G \cap \mathrm{cl}H$. Let $A\times B$ be a basic neighborhood of the point (x,y). There is a γ, $\gamma<\kappa$, such that $A\cap U_\gamma \neq \emptyset$. Since $y \notin \mathrm{cl}\bigcup\{V_\alpha : \alpha<\gamma\}$, there is a β, $\gamma\leqslant\beta<\kappa$, such that $B\cap V_\beta \neq \emptyset$. Hence $\emptyset \neq (A\cap U_\gamma)\times(B\cap V_\beta) \subset \subset (A\times B)\cap G_\gamma \subset (A\times B)\cap G$. This shows that $p\in \mathrm{cl}G$. Similar proof works for H ◻

From this theorem one can immediately deduce that the space $X\times X$ is extremally disconnected at no point of the form (x,x), whenever X is a Hausdorff space and x is a non-isolated point of X. Also, the space $X\times Y$ is not extremally disconnected at some point, whenever X and Y are infinite countably compact Hausdorff spaces (compare to Corollary 2.6).

<u>Theorem 2.2.</u> Suppose X and Y are Hausdorff spaces and let $x\in X$ and $y\in Y$ be non-isolated points. If the space $X\times Y$ is extremally disconnected at the point (x,y), then one of cardinals $c(x,X)$ or $c(y,Y)$ is measurable.

Proof. One can observe that the space X is extremally disconnected at the point x and the space Y is extremally disconnected at the point y. By Theorem 2.1, $c(x,X) \neq c(y,Y)$. Hence one of cardinals $c(x,X)$ or $c(y,Y)$ is uncountable and therefore, by Theorem 1.2, it is measurable ◻

If the axiom of constructibility, $V = L$, holds, then by Scott's theorem there is no measurable cardinal. Hence

<u>Corollary 2.3.</u> ($V = L$). If X and Y are Hausdorff spaces and $x\in X$ and $y\in Y$ are non-isolated points, then the space $X\times Y$ is not extremally disconnected at the point (x,y) ◻

The next theorem shows that the non-existence of measurable cardinals is essential to get the above corollary.

<u>Theorem 2.4.</u> Suppose that λ is a measurable cardinal. Then there exists a completely regular Hausdorff extremally disconnected dense-in-itself space X such that the product $X\times Y$ is an extremally disconnected space, whenever Y is an extremally disconnected space and $|Y|<\lambda$.

Proof. We shall begin with a general construction of topologies induced by ultrafilters. Before some notations.

Let Z be a set and let SeqZ be the set of all finite sequences in Z (i.e., $s\in \mathrm{Seq}Z$ iff $\mathrm{dom}s\in\omega$ and $\mathrm{rng}s \subset Z$). If $s\in \mathrm{Seq}Z$ is a sequence of lenght n (i.e., $\mathrm{dom}s = n$) and $z\in Z$, then $s^\frown z$ denotes the sequence of lenght n+1 that extends s and whose n-th term is z.

Let κ be cardinal and let ξ be an ultrafilter over κ. We are ready to define the promised topology, denoted from now by T_ξ.

The underlying set is $\mathrm{Seq}\kappa$; $U \in T_\xi$ iff $\forall s(s \in U \longrightarrow \{\alpha \in \kappa : s^\frown\alpha \in U\} \in \xi)$. It can be easily verified that T_ξ is a topology on $\mathrm{Seq}\kappa$ (even in the case when ξ is a filter over κ). Properties of the topology T_ξ depend on the ultrafilter ξ we have taken.

(i) if ξ is a non-principal ultrafilter over κ, then the space $(\mathrm{Seq}\kappa, T_\xi)$ is completely regular Hausdorff extremally disconnected and dense-in-itself.

We first prove that the space $(\mathrm{Seq}\kappa, T_\xi)$ is extremally disconnected. So let $s \in \mathrm{cl}U$, where $U \in T_\xi$.

<u>Claim.</u> The set $\{\alpha \in \kappa : s^\frown\alpha \in \mathrm{cl}U\}$ is in ξ.

Otherwise, the set $A = \{\alpha \in \kappa : s^\frown\alpha \notin \mathrm{cl}U\}$ would be in ξ. For $\alpha \in A$ let $U_\alpha \in T_\xi$ be an open neighborhood of $s^\frown\alpha$ that is disjoint with U. Then the set $V = \{s\} \cup \{s^\frown\alpha : \alpha \in A\} \cup \bigcup\{U_\alpha : \alpha \in A\}$ would be an open neighborhood of s disjoint with U, which is impossible; the claim is proved.

Now, we define inductively sets $V_n \subset \mathrm{Seq}\kappa$, $n \in \omega$, in the following way: $V_0 = \{s\}$, $V_{n+1} = \{s^\frown\alpha : s \in V_n \text{ and } s^\frown\alpha \in \mathrm{cl}U\}$. Each of the sets V_n is contained in clU and, by the claim, the set $V = \bigcup\{V_n : n \in \omega\}$ is an open neighborhood of s. This shows that the closure of any open set is open, i.e., the space $(\mathrm{Seq}\kappa, T_\xi)$ is extremally disconnected.

Undoubtly it can be verified that the space $(\mathrm{Seq}\kappa, T_\xi)$ is Hausdorff and dense-in-itself. So we pass to the proof that it is completely regular. We shall do this by showing that the topology T_ξ has a base consisting of closed-open sets.

If $s \in U \in T_\xi$, then let $V_0 = \{s\}$ and $V_{n+1} = \{t \in U :$ there are $\alpha \in \kappa$ and $p \in V_n$ such that $t = p^\frown\alpha\}$ and finally $V = \bigcup\{V_n : n \in \omega\}$. The set V is open. To prove that it is also closed, take r outside V. There are possible two cases: r extends s or not. If r extends s, then take W to be the set of all possible extensions of r. We claim that this set, being an open neighborhood of r, is disjoint with V. If not, then let t be an element of $W \cap V$ having the shortest lenght, say n. Then t must be a proper extension of the sequence r. Hence $t|n-1$ yet extends r, so it is in W. On the other hand, t belongs to V_m for some $m \in \omega$. Hence $t = z^\frown\alpha$ for some $z \in V_{m-1} \subset U$. In consequence, $t|n-1 = z$ and therefore $t|n-1$ is in $W \cap V$ which contradicts the choice of t.

Now assume that r does not extend s. In such a case we take W to be the set of all possible extensions of r that are not extensions of s. Because V consists of extensions of the sequence s only, the sets W and V are disjoint. It remains to show that W is an open neighborhood of r. Let $t \in W$. If dom$t \geqslant$ dom s, then any extension of t is not extension of s and therefore $\{\alpha \in \kappa\colon t^\frown\alpha \in W\} = \kappa \in \xi$. If dom$t <$ dom$s = n$, then $\{\alpha \in \kappa\colon t^\frown\alpha \in W\} \supset \kappa - \{s(n-1)\}$ which is in ξ because ξ is a non-principal ultrafilter over κ. The proof of (i) is finished.

The next property we shall establish is

(ii) if the intersection of less than γ sets from ξ is again in ξ (i.e., ξ is γ-complete), then the intersection of less than γ sets from T_ξ is again in T_ξ.

Indeed, take $\mathcal{R} \subset T_\xi$ with $|\mathcal{R}| < \gamma$. Let $s \in \bigcap\mathcal{R}$. For any $U \in \mathcal{R}$ the set $A_U = \{\alpha \in \kappa\colon s^\frown\alpha \in U\}$ is in ξ. By γ-completeness of ξ, $A = \bigcap\{A_U\colon U \in \mathcal{R}\}$ is in ξ. Hence $\{\alpha \in \kappa\colon s^\frown\alpha \in \bigcap\mathcal{R}\} = A \in \xi$.

Having (i) and (ii) we are ready to define our space X which satisfies the hypothesis of the theorem. It is just the space $(\mathrm{Seq}\,\lambda, T_\xi)$, where ξ is a non-principal λ-complete ultrafilter over λ. By (i) such a space is completely regular Hausdorff extremally disconnected and dense-in-itself. Now, let Y be an extremally disconnected space such that $|Y| < \lambda$. To prove that $X \times Y$ is extremally disconnected take an open set $U \subset X \times Y$ and $(p,q) \in \mathrm{cl}U$. For any y, $y \in Y$, let $U_y = \{x \in X\colon (x,y) \in U\}$. Then each of the sets U_y is an open subset of X. Since X is extremally disconnected, $\mathrm{cl}U_y$ is closed-open in X for every $y \in Y$. Since $|Y| < \lambda$, the condition (ii) guarantees that the set $G = \bigcap\{X - \mathrm{cl}U_y\colon p \notin \mathrm{cl}U_y\} \cap \bigcap\{\mathrm{cl}U_y\colon p \in \mathrm{cl}U_y\}$ is an open (in fact, closed-open) neighborhood of p. Since $G \times Y$ is an open neighborhood of the point (p,q), the set $V = U \cap (G \times Y)$ is open and $(p,q) \in \mathrm{cl}V$. In consequence, the projection of V into the Y-axis, say H, is open in Y and contains the point q in its closure. Since Y is extremally disconnected, clH is an open (in fact, closed-open) neighborhood of q. It remains to show that $G \times \mathrm{cl}H \subset \mathrm{cl}U$. Suppose on the contrary that $(x,y) \notin \mathrm{cl}U$ for some $(x,y) \in G \times \mathrm{cl}H$. There are open neighborhoods A of x and B of y, such that $(A \times B) \cap \mathrm{cl}U = \emptyset$. Since $y \in \mathrm{cl}H$, $B \cap H \neq \emptyset$ and let $z \in B \cap H$. Since $z \in H$, $p \in \mathrm{cl}U_z$ and therefore $G \subset \mathrm{cl}U_z$. But $(x,z) \in \mathrm{cl}U_z \times \{z\} \subset (X \times \{z\}) \cap \mathrm{cl}U \subset \mathrm{cl}U$ from which it follows that $(x,z) \in \mathrm{cl}U \cap (A \times B)$; a contradiction □

If, in the above theorem, we take Y to be the absolute over the Cantor set $2^{\aleph_0}$, then, since Y is compact Hausdorff extremally disconnected dense-in-itself and $|Y| = 2^{2^{\aleph_0}} < \lambda$ (λ being strongly inaccessible cardinal), we get

Corollary 2.5. If there exists a measurable cardinal, then there is a completely regular Hausdorff extremally disconnected dense-in-itself space X and there is a compact Hausdorff extremally disconnected dense-in-itself space Y such that the space $X \times Y$ is extremally disconnected ▫

Corollary 2.6. If there exists a measurable cardinal, then there are two compact Hausdorff extremally disconnected dense-in-itself spaces W and Y such that the space $W \times Y$ is extremally disconnected at a dense set of points.

Proof. As W we take the Čech-Stone compactification of the space X from Theorem 2.4 and as Y we take the absolute over the Cantor set. By Corollary 2.5, the space $W \times Y$ is extremally disconnected at each point of the subspace $X \times Y$ which is dense in $W \times Y$ ▫

Corollary 2.7. There exists a measurable cardinal if and only if there are two Hausdorff dense-in-itself spaces such that their product is extremally disconnected at some point.

Proof. The "if" part follows from Theorem 1.2 and the "only if" part is in Corollary 2.5

Knowing that there are at least two Hausdorff dense-in-itself spaces whose product is extremally disconnected, one can ask whether the same can be said for products of three or more factors. We shall give some results showing that it is not the case. But first discard the infinite case.

Theorem 2.8. Let X_s, $s \in S$, be Hausdorff spaces such that $|X_s| > 1$ for every $s \in S$, and let $|S| \geq \aleph_0$. If X is the product of all spaces X_s, then X extremally disconnected at no point.

Proof. Let us decompose S into two disjoint infinite parts S_1, S_2. If Y is the product of all spaces X_s where $s \in S_1$, and Z is the product of all spaces X_s where $s \in S_2$, then $X = Y \times Z$. In virtue of Theorem 1.3, $c(y,Y) = \aleph_0 = c(z,Z)$ for every $y \in Y$ and $z \in Z$. Hence, by Theorem 1.1, the space $Y \times Z$ is extremally disconnected at no point ▫

For $n \in \omega$, $n \geq 1$, let Q_n, P_n, M_n denote the following statements:

Q_n: there are Hausdorff spaces $X_0, \dots, X_n$ and non-isolated points $x_0 \in X_0, \dots, x_n \in X_n$ such that the space $X_0 \times \dots \times X_n$ is extremally disconnected at the point $(x_0, \dots, x_n)$;

P_n: there are completely regular Hausdorff extremally disconnected dense-in-itself spaces $X_0, \dots, X_n$ such that $X_0 \times \dots \times X_n$ is an extremally disconnected space;

M_n: there are (at least) n different measurable cardinals.

Theorem 2.9. The statements Q_n, P_n and M_n are equivalent each other.

Proof. First we show $M_n \longrightarrow P_n$. Let $\varkappa_1 < \varkappa_2 < \dots < \varkappa_n$ be measurable cardinals and let ξ_j be a $\varkappa_j$-complete non-principal ultrafilter over $\varkappa_j$ for $1 \leq j \leq n$. The space $X_j = (\mathrm{Seq}\,\varkappa_j, T_{\xi_j})$, $1 \leq j \leq n$, and X_0, the absolute over the Cantor set, satisfy the requirements of P_n. To see this it is enough to show that the space $X_0 \times \dots \times X_n$ is extremally disconnected. We do this by induction. For $n = 1$ it was done in Theorem 2.4. So assuming that the space $X_0 \times \dots \times X_{n-1}$ is extremally disconnected we apply again Theorem 2.4 and we get that the space $(X_0 \times \dots \times X_{n-1}) \times X_n$ is extremally disconnected, because $|X_0 \times \dots \times X_{n-1}| = \varkappa_{n-1} < \varkappa_n$.

The implication $P_n \longrightarrow Q_n$ is obvious.

To get $Q_n \longrightarrow M_n$ we proceed as in Theorem 2.2, which is in fact the case $n = 1$ □

Corollary 2.10. $\mathrm{Con}(ZFC + M_n) \longrightarrow \mathrm{Con}(ZFC + P_n + \neg Q_{n+1})$ for $n \geq 1$.

Proof. From the consistency of $ZFC + M_n$ one can easily derive the consistency of $ZFC + M_n + \neg M_{n+1}$ which, in virtue of Theorem 2.9, is the same as the consistency of $ZFC + P_n + \neg Q_{n+1}$ □

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