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ON EXTENSIONS OF MEASURES WHICH ARE MAXIMAL WITH RESPECT TO A CHAIN

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Let (X,A,μ) be a fixed probability measure space and let Z be a chain of subsets of X. We consider the problem of extending μ to a measure ν defined on the σ-algebra $\sigma(A \cup Z)$, generated by A and Z.

We say that ν is a strongly Z-maximal extension of μ if $\nu(Z) = \mu^*(Z)$ for $Z \in Z$ (compare with the notion of Z-maximality, [2], Definition 2.7). This concept was investigated by Lipecki in the context of finitely additive set functions with values in an order complete Abelian lattice group ([4], [5]). In particular, he observed that a strongly Z-maximal extension is unique if it exists ([5], Proposition 1).

The following theorem is a particular case of Weber'a Satz 3 ([7])

THEOREM 1. If Z is well-ordered by inclusion then there exists a strongly Z-maximal extension of μ.

In general, the assumption of well-ordering cannot be replaced by linear ordering of Z ([4]).

Let $D(Z)$ denote the closure of the set $\{\mu^*(Z) : Z \in Z\}$ in the unit interval. Note that $D(Z)$ is countable in case Z is well-ordered. In particular $|D(Z)| = 0$, where $|\cdot|$ stands for the Lebesque measure on $[0,1]$. Moreover, μ^* is continuous from above on Z i.e

$$\mu^*(\cap Z') = \inf\{\mu^*(Z) : Z \in Z'\}$$

for every countable $Z' \subset Z$. We will prove the following generalization of Theorem 1

This paper is in finel form and no version of it will be submitted for publication elsewhere.

THEOREM 2. Assume that $|D(\mathcal{Z})| = 0$ and μ^* is continuous from above on $\mathcal{Z}$. Then there exists a strongly $\mathcal{Z}$-maximal extension of μ.

We will use the result stated below as Theorem 3. It was proved in [6] (Théorème 27). This principle for extending a measure was applied by Ascherl and Lehn to obtain a generalization of the theorem of Bierlein ([1]).

THEOREM 3. Let $\mathcal{G}$ be a σ-ideal of subsets of X such that $\mu_*(G) = 0$ for $G \in \mathcal{G}$. Then μ can be extended to a measure on $\sigma(\mathcal{A} \cup \mathcal{G})$ vanishing on every element of $\mathcal{G}$.

We also need the following simple lemma.

LEMMA. Let $\{Y_i : i \le k\}$ be a chain, and $\{A_i : i \le k\} \subset \mathcal{A}$. Then

$$\mu_*(\bigcup_i (A_i - Y_i)) \le \sum_i \mu_*(A_i - Y_i).$$

PROOF. We may assume that $Y_1 \supset Y_2 \supset \dots \supset Y_k$.

$$\bigcup_{i\le k} (A_i - Y_i) = \bigcup_{i<k} (A_i - Y_i) \cup (A_k - Y_k) = (\bigcup_{i<k} A_i - Y_i) - A_k) \cup (A_k - Y_k).$$

Hence $\mu_*(\bigcup_{i\le k} (A_i - Y_i)) \le \mu_*(\bigcup_{i<k} (A_i - Y_i)) + \mu_*(A_k - Y_k)$. We see that Lemma follows by induction.

PROOF OF THEOREM 2. For every $Z \in \mathcal{Z}$ choose a measurable cover of Z, say $H(Z)$, such that $\mu(H(Z)) = \mu^*(Z)$. Let $\mathcal{G} = \{H(Z) - Z : Z \in \mathcal{Z}$ We will prove that $\mu_*(G) = 0$ for each set of the form

$$G = \bigcup_n (H(Z_n) - Z_n)$$

where $Z_n \in \mathcal{Z}$. Fix $\varepsilon > 0$. There exists a finite collection of closed intervals $\{I_i : i \le k\}$ such that

$$D(\mathcal{Z}) \subset \bigcup_i I_i \quad \text{and} \quad \sum_i |I_i| < \varepsilon,$$

since $D(\mathcal{Z})$ is a compact set of Lebesgue measure zero.
Let $A_i = \bigcup\{H(Z_n) : \mu^*(Z_n) \in I_i\}$, $Y_i = \bigcap\{Z_n : \mu^*(Z_n) \in I_i\}$.

Observe that $\mu(A_i) \in I_i$ and $\mu^*(Y_i) \in I_i$ by the continuity of μ^*. Therefore $\mu_*(A_i - Y_i) < |I_i|$.
The sets Y_i form a chain and $G \subset \bigcup_i (A_i - Y_i)$. Using Lemma we have

$$\mu_*(G) \le \mu_*(\bigcup_i (A_i - Y_i)) \le \sum_i \mu_*(A_i - Y_i) \le \sum_i |I_i| < \varepsilon$$

Hence $\mu_*(G) = 0$ and the rest follows from Theorem 3.

The assumption of the continuity of μ^* in Theorem 2 is evidently necessary but it is not sufficient in itself, as the following example shows.

EXAMPLE. Let $I = [0,1]$, $X = \{(x,y) \in I^2 : y > x\}$,
$\mathcal{A} = \{(A \times I) \cap X : A \in \mathcal{B}\}$ where $\mathcal{B}$ is the σ-field of Borel sets.
Define $\mu((A \times I) \cap X) = |A|$ and consider the chain $\mathcal{Z} = \{Z_t : t \in I\}$,
$Z_t = (I \times [0,t]) \cap X$.
Then $\mu^*(Z_t) = t$, so μ^* is continuous on $\mathcal{Z}$.
Suppose that ν is a maximal extension of μ. Then $\nu(W_t) = 0$,
where $W_t = ([0,t] \times [t,1]) \cap X$.
Since $X = \bigcup\{W_t : t \in Q\}$, ν cannot be σ-additive.

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