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ON A CLASS OF QUEUE DISCIPLINES

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1. It is a rather well known fact that FIFO and LIFO can be considered extremal cases of queue disciplines. It was proved in [2] and [3] that, in a very general class of queue disciplines, the steady state waiting time variance is minimal with the FIFO and maximal with the LIFO discipline.

In [4] a general pattern of random selection of customers from the queue[1]) was introduced and analogous inequalities for the waiting time variance in an $M/M/n$ queueing system in equilibrium were derived.

In this paper we shall investigate a rather special one-parametric class of queue disciplines of the general type described in [4]. As usual, we shall be interested in the corresponding waiting time distribution as well as in the outtaking of customers during the waiting time (see [5], [7]). Like the so-called "mixed" discipline (see [5], [6], [7]), the disciplines belonging to our class can be regarded as mixtures of the FIFO and the LIFO disciplines. Although the mixing operation seems to be different, we shall see that the final results are essentially the same for our class and for the mixed discipline (see also [8]). Anyway, the useful random walk methods we apply here again are interesting since they allow a unified approach to the problems.

2. The queue disciplines dealt with in the present paper can be described as follows:

Customers waiting for service form a simple queue. Newly arriving customers join the queue at its end; there is no balking nor reneging. Whenever a server becomes free and there are some customers waiting, either the *first* customer in the queue or the *last* one is chosen to be served as the next. The choice is made *at random*, with fixed probabilities δ for the first and $1 - \delta$ for the last customer: δ is a given real number, $0 \leqq \delta \leqq 1$.[2]) The successive choices are independent of each other.

When $\delta = 1$ we get the FIFO discipline; when $\delta = 0$ we get the LIFO discipline. The other cases, $0 < \delta < 1$, form a continuous transition between these two extremes.

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*) The paper resumes the main results established in a thesis written by the first author under the supervision of the other at Charles University, Prague, in 1973—74.

[1]) It embraces the FIFO and the LIFO as well as the classical random queueing disciplines.

[2]) If there is only one customer waiting, the choice is made as well, but the result is of course always the same: the only waiting customer is taken to be served.

3. We shall start by considering a queueing system $M/M/1$ with an input rate λ, $0 < \lambda < \infty$, and a mean service time μ^{-1}, $0 < \mu < \infty$. Since we are interested only in the steady state behaviour of the system, we shall always suppose that $\lambda < \mu$, so that $\varrho = \lambda/\mu < 1$.

The probabilities p_k ($k = 0, 1, 2, \ldots$) of finding exactly k customers in the system (in equilibrium) are well known to be (see e.g. [1])

$$p_k = (1 - \varrho)\,\varrho^k\,, \quad k = 0, 1, 2, \ldots \tag{1}$$

This also means that an arriving customer begins his waiting at the k-th place in the queue with the probability p_k; the place zero is to be interpreted as the situation of a customer admitted to service.

While the probabilities p_k are not affected by the queue discipline adopted in our system, the waiting time distribution clearly depends on δ. Our first aim will be to find this distribution.

4. We shall proceed by the method we have already found useful in the case of the mixed queue discipline (see [5],]6], [7]): we shall track the successive positions in the queue of a waiting customer $\mathscr{C}$. The position of the customer will be characterized this time by a pair of non-negative integers, say $[k, m]$, indicating that our customer $\mathscr{C}$ is just at the k-th place in the queue when counted from the top of the queue and, simultaneously, at the m-th place when counted from the end of the queue. There are then just $k + m - 1$ customers waiting.[3])

While waiting in the queue, our customer $\mathscr{C}$ changes his position from time to time. When a new customer arrives and joins the queue, the position of $\mathscr{C}$ changes from $[k, m]$ to $[k, m + 1]$. When the server becomes free and a customer leaves the queue in order to be served, the position of $\mathscr{C}$ can change from $[k, m]$ either to $[k - 1, m]$ – if the first customer in the queue is chosen – or to $[k, m - 1]$ – if the last customer is taken. If, moreover, $k = 1$ in the former case or $m = 1$ in the latter, then it is just our customer $\mathscr{C}$ who happens to be selected to be served as the next; his waiting time stops at this moment and no further changes of position can occur. Formally we shall write this as a (final) step from $[1, m]$ to $[0, m]$, or from $[k, 1]$ to $[k, 0]$. Thus the positions $[k, m]$ with $km = 0$[4]) will be interpreted as describing the situation of a customer that has already reached the service.

The time intervals separating the successive moments at which such changes of position occur are independent and exponentially distributed random variables with the mean value $(\mu + \lambda)^{-1}$. Moreover, $\lambda(\mu + \lambda)^{-1}$ and $\mu(\mu + \lambda)^{-1}$ are the probabilities that the next change will be caused by an arrival or by a departure, respectively. It is then easy to see that the process of successive changes of position of our customer $\mathscr{C}$ is a two-dimensional Markov process with possible states $[k, m]$,

[3]) Since our customer $\mathscr{C}$ is really waiting, we can never have $k + m = 0$.

[4]) As we have already seen, the pair [0, 0] will never appear as an actual position of a customer.

$k = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots; k + m > 0$. The states with $km = 0$ are absorbing. The transition intensities are

$$\begin{array}{llll} \lambda & \text{from } [k, m] & \text{to} & [k, m+1], \\ \delta\mu & \text{from } [k, m] & \text{to} & [k-1, m], \\ (1-\delta)\mu & \text{from } [k, m] & \text{to} & [k, m-1], \end{array}$$

(provided the corresponding transition leads to a possible state).

5. Let $\varphi_{km}(s)$ be the characteristic function of the remaining waiting time of a customer whose actual position is $[k, m]$, $k \geqq 0$, $m \geqq 0$, $k + m > 0$. Since every customer starts in a position $[k, 1]$ with the probability p_k, $k = 0, 1, 2, \ldots$, the characteristic function $\varphi_W(s)$ of the total waiting time W of an arbitrary customer ("chosen at random") will be

$$\varphi_W(s) = \sum_{k=0}^{\infty} p_k\, \varphi_{k1}(s) = (1 - \varrho) \sum_{k=0}^{\infty} \varrho^k\, \varphi_{k1}(s)\,. \tag{2}$$

We have of course $\varphi_{0m}(s) \equiv 1$, $\varphi_{k0}(s) \equiv 1$ for all $m \geqq 1$, $k \geqq 1$; the interpretation is clear: the waiting time of a customer that has already reached the service is almost surely zero.

Given the Markov character of the process, the functions $\varphi_{km}(s)$ satisfy the following system of linear equations (see also (5.22) in [5] or (3.1) in [6])

$$\varphi_{km}(s) = \frac{\mu + \lambda}{\mu + \lambda - is} \left\{ \frac{\lambda}{\mu + \lambda}\, \varphi_{k\,m+1}(s) + \frac{\delta\mu}{\mu + \lambda}\, \varphi_{k-1\,m}(s) + \frac{(1-\delta)\,\mu}{\mu + \lambda}\, \varphi_{k\,m-1}(s) \right\},$$

or

$$(\mu + \lambda - is)\, \varphi_{km}(s) = \lambda\, \varphi_{k\,m+1}(s) + \delta\mu\, \varphi_{k-1\,m}(s) + (1 - \delta)\, \mu\, \varphi_{k\,m-1}(s)\,, \tag{3}$$

for all $k \geqq 1$, $m \geqq 1$.

In order to solve the system (3) we shall use the generating functions

$$F_m(w, s) = \sum_{k=1}^{\infty} w^k\, \varphi_{km}(s)\,, \quad m = 0, 1, 2, \ldots$$

and

$$H(z, w, s) = \sum_{m=1}^{\infty} z^m\, F_m(w, s) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} z^m w^k\, \varphi_{km}(s)\,.$$

Since $|\varphi_{km}(s)| \leqq 1$ for all $k \geqq 0$, $m \geqq 0$, $k + m > 0$, and all real s, the functions $F_m(w, s)$, $m = 0, 1, 2, \ldots$ are analytic in the domain $|w| < 1$; also $H(z, w, s)$ is analytic for $|w| < 1$, $|z| < 1$.

From (3) we obtain, after a tedious but straightforward calculation, the equation

$$H(z, w, s) = \frac{\lambda z\, F_1(w, s) - \delta\mu w z^2 (1 - z)^{-1} - (1 - \delta)\, \mu w z^2 (1 - w)^{-1}}{(1 - \delta)\, \mu z^2 - (\mu + \lambda - \delta\mu w - is)\, z + \lambda}\,.$$

Since $\varrho < 1$, we may put $w = \varrho$; this value is important because of (2). Hence

$$H(z, \varrho, s) = \frac{\lambda z\, F_1(\varrho, s) - \delta\lambda z^2(1 - z)^{-1} - (1 - \delta)\,\lambda z^2(1 - \varrho)^{-1}}{(1 - \delta)\,\mu z^2 - (\mu + \lambda - \delta\lambda - is)\,z + \lambda}. \tag{4}$$

The denominator of (4) has always one root, say α, in the domain $|z| < 1$. For $\delta = 1$ this root is $\alpha = \lambda(\mu - is)^{-1}$, for $\delta \neq 1$ it is

$$\alpha = \frac{\mu + \lambda - \delta\lambda - is - \{(\mu + \lambda - \delta\lambda - is)^2 - 4\lambda\,\mu(1 - \delta)\}^{1/2}}{2\mu(1 - \delta)}. \tag{5}$$

Since $H(z, \varrho, s)$ is analytic in $|z| < 1$, α must be a zero of the numerator of (4), too. We have therefore

$$F_1(\varrho, s) = \frac{\delta\alpha}{1 - \alpha} + \frac{(1 - \delta)\,\alpha}{1 - \varrho}$$

or

$$F_1(\varrho, s) = \frac{\varrho}{1 - \varrho} + \frac{is\varrho\alpha}{\lambda(1 - \varrho)\,(1 - \alpha)}.$$

Now (2) shows that

$$\varphi_W(s) = 1 - \varrho + (1 - \varrho)\,F_1(\varrho, s)\,,$$

i.e.

$$\varphi_W(s) = 1 - \varrho + \varrho\left[1 + \frac{is\alpha}{\lambda(1 - \alpha)}\right]; \tag{6}$$

this is *the characteristic function of the waiting time distribution* desired.

If a customer arrives when the system is empty, he is served at once, without waiting. This event occurs with the probability

$$\mathbf{P}\{W = 0\} = p_0 = 1 - \varrho\,.$$

We see then from (6) that

$$\psi(s) = 1 + \frac{is\alpha}{\lambda(1 - \alpha)} \tag{7}$$

is the characteristic function of the conditional distribution of W, provided $W > 0$.

6. Starting from (6) we can easily obtain expressions for the first two moments of the waiting time W. First, we find that

$$\mathbf{E}[W] = -i\,\varphi'_W(0) = \varrho(\mu - \lambda)^{-1}\,,$$

which is not very interesting, because it is well known that the mean waiting time does not depend on the queue discipline. The other result

$$\mathbf{E}[W^2] = -\varphi_W''(0) = \frac{2\lambda}{(\mu - \lambda)^2 (\mu - \lambda + \delta\lambda)}$$

shows that the second moment of W, and therefore also its variance

$$\mathbf{D}^2[W] = \frac{2\lambda}{(\mu - \lambda)^2 (\mu - \lambda + \delta\lambda)} - \frac{\varrho^2}{(\mu - \lambda)^2}, \tag{8}$$

are decreasing functions of the parameter δ. The extreme values of the variance are $\varrho(2 - \varrho)(\mu - \lambda)^{-2}$ for $\delta = 1$ (FIFO) and $\lambda(\varrho^2 - \varrho + 2)(\mu - \lambda)^{-3}$ for $\delta = 0$ (LIFO).

7. Hitherto we have assumed that there is only one server in the system. In order to extend our results to the more general case of systems $M/M/n$ some slight alterations are necessary in our formulae.

The random walk of the waiting customers on different positions in the queue, and hence also the equations (3), are affected only formally: we just write $n\mu$ instead of μ everywhere. The expression for $F_1(\varrho, s)$ remains formally unchanged, but of course ϱ is now $\lambda/n\mu$ (we still suppose $\varrho < 1$). Instead of (5) we have now

$$\alpha = \frac{n\mu + \lambda - \delta\lambda - is - \{(n\mu + \lambda - \delta\lambda - is)^2 - 4n\mu\lambda(1 - \delta)\}^{1/2}}{2n\mu(1 - \delta)}.$$

However, the presence of n servers in the system affects the probabilities p_k as well. Instead of the simple geometric distribution (1) we have now (see e.g. [1])

$$p_k = p_0 \frac{\lambda^k}{k!\, \mu^k} \quad \text{for} \quad 0 \leqq k \leqq n,$$

$$p_k = p_n \varrho^{k-n} \quad \text{for} \quad n \leqq k,$$

with

$$p_0^{-1} = \sum_{k=0}^{n-1} \frac{\lambda^k}{k!\, \mu^k} + \frac{n^n}{n!} \sum_{k=n}^{\infty} \varrho^k.$$

The probability of immediate service without waiting is now

$$\mathbf{P}\{W = 0\} = \sum_{k=0}^{n-1} p_k = 1 - \sum_{k=n}^{\infty} p_k = 1 - p_n(1 - \varrho)^{-1}.$$

Instead of (2) we get for the characteristic function $\varphi_W(s)$ of the waiting time distribution

$$\varphi_W(s) = 1 - \frac{p_n}{1 - \varrho} + \frac{p_n}{1 - \varrho}\left[1 + \frac{is\alpha}{\lambda(1 - \alpha)}\right];$$

we see that the function $\psi(s)$ of (7) is formally the same as with $n = 1$; only α has changed.

By differentiation we obtain again

$$\mathbf{E}[W] = -i\,\varphi_W'(0) = \frac{p_n}{1-\varrho}\,\frac{1}{n\mu-\lambda}$$

and

$$\mathbf{D}^2[W] = \frac{p_n}{1-\varrho}\,\frac{2(n\mu-\lambda) - n\mu p_n - \lambda(1-\delta)(1-\varrho)}{(n\mu-\lambda)^2\,(n\mu-\lambda+\delta\lambda)(1-\varrho)}\,.$$

8. As we have just seen, the extension from $M/M/1$ to $M/M/n$ is rather straightforward. We now come back to the simpler case of systems with one server; also in the sequel we shall deal with the system $M/M/1$ only.

With our queue discipline a waiting customer can leave the queue in two different ways: either from the first place or from the last place in the queue, i.e. either from a position $[1, m]$, with $m \geqq 1$, or from a position $[k, 1]$, with $k \geqq 1$. There is perhaps some ambiguity in the situation of a customer that leaves the queue from position $[1, 1]$ – he is both the first and the last customer in the queue. For the sake of simplicity, it is assumed that even in this case the customer to be served as the next is chosen at random[5]): with the probability δ the leaving customer is considered the first and with the probability $1 - \delta$ the last customer in the queue.

We shall now be interested in the probability f that an arbitrary customer ("chosen at random") will leave the queue *from the first place*. We remind that with probability $1 - \varrho$ an arriving customer finds the system empty and is therefore served immediately without waiting. If he joins the queue – this occurs with the probability ϱ – he starts in a position $[k, 1]$, $k = 1, 2, 3, \ldots$, the probability of his doing so is $p_k = (1-\varrho)\,\varrho^k$.

Let us denote by f_{km} the probability with which a customer, now waiting in a position $[k, m]$, $k \geqq 1$, $m \geqq 1$, will finally leave the queue from the first place. We have then

$$f = \sum_{k=1}^{\infty} = (1-\varrho)\sum_{k=1}^{\infty} \varrho^k f_{k1}\,. \tag{8}$$

On the other hand, taking into consideration the random walk of the customers on the positions in the queue, we can see that the probabilities f_{km} fulfil

$$(\mu+\lambda)\,f_{km} = \lambda f_{k\,m+1} + \delta\mu f_{k-1\,m} + (1-\delta)\,\mu f_{k\,m-1} \tag{10}$$

for $k \geqq 1$, $m \geqq 1$, with boundary conditions

$$f_{0m} = 1 \quad \text{for all} \quad m \geqq 1\,,$$

$$f_{k0} = 0 \quad \text{for all} \quad k \geqq 0\,.$$

[5]) See footnote[2]).

Here again we shall use the standard tool of generating functions. We put

$$F_m(w) = \sum_{k=1}^{\infty} w^k f_{km}, \quad m = 1, 2, 3, \ldots$$

for $|w| < 1$ and

$$H(z, w) = \sum_{m=1}^{\infty} z^m F_m(w)$$

for $|w| < 1$, $|z| < 1$; (10) then implies

$$H(z, w) = \frac{\lambda z F_1(w) - \delta\mu w z^2 (1 - z)^{-1}}{\mu(1 - \delta) z^2 - (\mu + \lambda - \delta\mu w) z + \lambda}.$$

The denominator vanishes for $w = \varrho$, $z = \varrho$; since $\varrho < 1$, the numerator must vanish as well. Thus we have

$$F_1(\varrho) = \delta\varrho(1 - \varrho)^{-1}.$$

On the other hand, (9) shows that $f = (1 - \varrho) F_1(\varrho)$ so that finally

$$f = \delta\varrho. \tag{11}$$

Analogously we could find that the probability with which an arbitrary customer will wait and finally leave the queue from the last place is $(1 - \delta)\varrho$.

9. When investigating the mixed queue discipline in [5] and [7] we were also interested in the *outtaking of customers* during the waiting time.[6] We shall now study the same problems for our class of queue disciplines.

Given an arbitrary customer $\mathscr{C}$ let us denote by X the number of customers that had entered in the system earlier than $\mathscr{C}$ (we shall call them older than $\mathscr{C}$) but are still waiting when $\mathscr{C}$ leaves the queue; by Y we denote the number of customers that arrived later than $\mathscr{C}$ (we shall call them younger than $\mathscr{C}$) but left the queue before $\mathscr{C}$. As $\mathscr{C}$ is arbitrary (chosen at random) X and Y are two random variables; we may ask for their probability distributions.[7]

We shall start with the variable X, this case being a bit easier to deal with (see also [7]).

Once more we shall take advantage of the random walk of waiting customers on the different positions in the queue. Let us suppose that the actual position of the customer $\mathscr{C}$ is $[k, m]$, with some $k \geqq 1$, $m \geqq 1$. This means that there are precisely $k - 1$ older customers waiting before $\mathscr{C}$ (and $m - 1$ younger customers behind $\mathscr{C}$).

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[6]) We remind that a customer $\mathscr{C}_1$ outtakes a customer $\mathscr{C}_2$ if $\mathscr{C}_1$ arrives in the system later than $\mathscr{C}_2$ but leaves the queue earlier than $\mathscr{C}_2$.

[7]) In [5], X was called the number of active outtakings and Y the number of passive outtakings of customer $\mathscr{C}$; in [7] they were called respectively active and passive inversions.

Since no more older customer can arrive, the conditional distribution of X, given the actual position $[k, m]$ of $\mathscr{C}$, is restricted to the values $0, 1, 2, \ldots, k-1$. Let this distribution be

$$P_{km}(x) = \mathbf{P}\{X = x \mid [k, m]\}\,, \quad x \geqq 0 \quad \text{integer}\,;$$

we see that

$$P_{km}(x) = 0 \quad \text{for} \quad x \geqq k\,.$$

If we add, by convention, the formal probabilities

$$P_{0m}(0) = 1\,, \quad P_{0m}(x) = 0 \quad \text{for all} \quad m \geqq 1, \quad x > 0\,,$$

and

$$\begin{aligned} P_{k0}(x) &= 1 \quad \text{for} \quad x = k-1\,, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

then taking into consideration the usual random walk of $\mathscr{C}$ in the queue we see that the probabilities $P_{km}(x)$ satisfy the equations

$$(12) \qquad (\mu + \lambda)\, P_{km}(x) = \lambda\, P_{k\,m+1}(x) + \delta\mu\, P_{k-1\,m}(x) + (1-\delta)\,\mu\, P_{k\,m-1}(x)\,.$$

As usual, we write

$$F_m(x, w) = \sum_{k=1}^{\infty} w^k\, P_{km}(x)\,, \quad m = 0, 1, 2, \ldots$$

for $|w| < 1$; the boundary conditions give

$$F_0(x, w) = w^{x+1}\,.$$

Putting then

$$H(x, z, w) = \sum_{m=1}^{\infty} z^m\, F_m(x, w)$$

for $|z| < 1$, $|w| < 1$, we obtain from (12)

$$(13) \qquad H(x, z, w) = \frac{\lambda z\, F_1(x, w) - (1-\delta)\,\mu z^2 w^{x+1} - \delta\mu w z\, G_0(x, z)}{(1-\delta)\,\mu z^2 - (\mu + \lambda - \delta\mu w)\, z + \lambda}\,,$$

where

$$G_0(x, z) = \sum_{m=1}^{\infty} z^m\, P_{0m}(x)\,.$$

The denominator in (13) vanishes for $w = \varrho$, $z = \varrho$, and the usual argument yields

$$F_1(x, \varrho) = (1-\delta)\,\varrho^{x+1} + \delta\, G_0(x, \varrho)\,.$$

This implies that the unconditional probabilities

$$P_{\text{act}}(x) = \mathbf{P}\{X = x\}$$

are on the one hand (for $x = 0$)

$$(14)\qquad P_{\text{act}}(0) = 1 - \varrho + \sum_{k=1}^{\infty} p_k P_{k1}(0) = 1 - \varrho + (1 - \varrho) F_1(0, \varrho) =$$
$$= 1 - \varrho + \delta\varrho + (1 - \delta)\varrho(1 - \varrho) = 1 - \varrho^2(1 - \delta),$$

and on the other hand (for $x > 0$)

$$(15)\qquad P_{\text{act}}(x) = \sum_{k=1}^{\infty} p_k P_{k1}(x) = (1 - \varrho) F_1(x, \varrho) = (1 - \delta)(1 - \varrho)\varrho^{x+1}.$$

The equality (14) can be interpreted as follows: In order that an (arbitrary) customer $\mathscr{C}$ may not outtake any older customer, he must either arrive when the system is empty (probability $1 - \varrho$) or wait and then leave the queue from the first place (the probability is $\delta\varrho$ as we have seen in the preceding section), or else wait and leave the queue as the last customer, provided he is then alone in the queue (he leaves from the position $[1, 1]$) – the corresponding probability is $(1 - \varrho)\varrho(1 - \delta)$.

10. We shall now examine the probability distribution of Y. Let $R_{km}(y)$, $k \geqq 1$, $m \geqq 1$, $y \geqq 0$, integer, denote the probability of the following event: there are exactly y customers younger than $\mathscr{C}$ that leave the queue before $\mathscr{C}$, the actual position of $\mathscr{C}$ being $[k, m]$.[8])

The natural boundary values of $R_{km}(y)$ are

$$(16)\qquad R_{0m}(0) = 1, \quad R_{0m}(y) = 0 \quad \text{for all} \quad m \geqq 1, \quad y > 0,$$
$$R_{k0}(0) = 1, \quad R_{k0}(y) = 0 \quad \text{for all} \quad k \geqq 1, \quad y > 0.$$

We shall first consider the particular case of $y = 0$. The usual random walk argument leads to the equations

$$(17)\qquad (\mu + \lambda) R_{k1}(0) = \lambda R_{k2}(0) + \delta\mu R_{k-1\,1}(0) + (1 - \delta)\mu$$

and

$$(18)\qquad (\mu + \lambda) R_{km}(0) = \lambda R_{k\,m+1}(0) + \delta\mu R_{k-1\,m}(0), \quad m > 1.$$

Then for

$$F_m^0(w) = \sum_{k=1}^{\infty} w^k R_{km}(0), \quad m = 1, 2, 3, \dots; |w| < 1,$$

(16) and (17) yield

$$(19)\quad (\mu + \lambda) F_1^0(w) = \lambda F_2^0(w) + \delta\mu w F_1^0(w) + \delta\mu w + (1 - \delta)\mu w(1 - w)^{-1},$$

[8]) It is not supposed that all the younger customers are already in the system; they may arrive while $\mathscr{C}$ is still waiting. On the other hand, younger customers that had already left the queue are not counted here.

while (18) implies

(20) $$(\mu + \lambda)\, F_m^0(w) = \lambda\, F_{m+1}^0(w) + \delta\mu w\, F_m^0(w) + \delta\mu w\,, \quad m > 1\,.$$

Putting as usual

$$H^0(z, w) = \sum_{m=1}^{\infty} z^m\, F_m^0(w)$$

for $|z| < 1$, $|w| < 1$, we obtain from (19) and (20)

(21) $$H^0(z, w) = \frac{\lambda z\, F_1^0(w) - \delta\mu w z^2 (1 - z)^{-1} - (1 - \delta)\, \mu w z^2 (1 - w)^{-1}}{\lambda - z(\mu + \lambda - \delta\mu w)}\,.$$

Since $\lambda < |\mu + \lambda - \delta\mu w|$ for all $|w| < 1$, $0 \leqq \delta \leqq 1$, we conclude further by the usual argument

(22) $$F_1^0(w) = \frac{\delta w}{1 - \delta w} + \frac{(1 - \delta)\, w}{(1 - w)\,(1 + \varrho - \delta w)}\,,$$

whence

(23) $$H^0(z, w) = \frac{\delta w z}{(1 - \delta w)\,(1 - z)} + \frac{(1 - \delta)\, w z}{(1 - w)\,(1 + \varrho - \delta w)}\,.$$

For the particular value $w = \varrho$ (23) gives

(24) $$H^0(z, \varrho) = \frac{\delta \varrho z}{(1 - \delta \varrho)\,(1 - z)} + \frac{(1 - \delta)\, \varrho z}{(1 - \varrho)\,(1 + \varrho - \delta \varrho)}\,;$$

we shall make use of it later.

The probability $P_{\text{pas}}(0) = \mathbf{P}\{Y = 0\}$ clearly equals

$$1 - \varrho + \sum_{k=1}^{\infty} p_k\, R_{k1}(0) = 1 - \varrho + (1 - \varrho)\, F_1^0(\varrho)\,,$$

i.e.

(25) $$P_{\text{pas}}(0) = 1 - \varrho + \frac{(1 - \varrho)\, \delta\varrho}{1 - \delta\varrho} + \frac{(1 - \delta)\, \varrho}{1 + \varrho - \delta\varrho} =$$

$$= 1 - \varrho + \varrho\, \frac{1 - \delta\varrho - \delta\varrho^2 + \delta^2\varrho^2}{(1 - \delta\varrho)\,(1 + \varrho - \delta\varrho)} = 1 - \frac{(1 - \delta)\, \varrho^2}{(1 - \delta\varrho)\,(1 + \varrho - \delta\varrho)}\,.$$

Now the case of $y > 0$. The equations for $R_{km}(y)$ are

(26) $$(\mu + \lambda)\, R_{11}(y) = \lambda\, R_{12}(y)\,,$$

$$(\mu + \lambda)\, R_{k1}(y) = \lambda\, R_{k2}(y) + \delta\mu\, R_{k-1\,1}(y)\,, \quad k > 1\,,$$

and for $m > 1$

(27) $$(\mu + \lambda) R_{1m}(y) = \lambda R_{1\,m+1}(y) + (1 - \delta) \mu R_{1\,m-1}(y - 1),$$

$$(\mu + \lambda) R_{km}(y) = \lambda R_{k\,m+1}(y) + \delta\mu R_{k-1\,m}(y) + (1 - \delta) \mu R_{k\,m-1}(y - 1), \quad k > 1.$$

If we write

$$F_m^y(w) = \sum_{k=1}^{\infty} w^k R_{km}(y)$$

for $|w| < 1$, (26) gives

(28) $$(\mu + \lambda) F_1^y(w) = \lambda F_2^y(w) + \delta\mu w F_1^y(w),$$

and (27) implies

(29) $$(\mu + \lambda) F_m^y(w) = \lambda F_{m+1}^y(w) + \delta\mu w F_m^y(w) + (1 - \delta) \mu F_{m-1}^{y-1}(w), \quad m > 1.$$

For

$$H^y(z, w) = \sum_{m=1}^{\infty} z^m F_m^y(w), \quad |z| < 1,$$

it follows from (28) and (29) that

(30) $$H^y(z, w) = \frac{\lambda z F_1^y(w) - (1 - \delta) \mu z^2 H^{y-1}(z, w)}{\lambda - z(\mu + \lambda - \delta\mu w)}.$$

Since we know already $H^0(z, w)$ from (23), we can find successively all the functions $H^y(z, w)$, $y = 1, 2, 3, \ldots$, in the usual way: the denominator in (30) vanishes for $z = \lambda(\mu + \lambda - \delta\mu w)^{-1}$, etc. Moreover, we can restrict ourselves to $w = \varrho$, because we are mainly interested in the probabilities

$$P_{\text{pas}}(y) = \mathbf{P}(Y = y) = \sum_{k=1}^{\infty} p_k R_{k1}(y) = (1 - \varrho) F_1^y(\varrho), \quad y = 1, 2, 3, \ldots$$

We take then

(31) $$H^y(z, \varrho) = \frac{\lambda z F_1^y(\varrho) - (1 - \delta) \mu z^2 H^{y-1}(z, \varrho)}{\lambda - (\mu + \lambda - \delta z) z}$$

instead of (30) and (24) instead of (23). We can further write

$$G(v, z) = \sum_{y=1}^{\infty} v^y H^y(z, \varrho)$$

and

$$K(v) = \sum_{y=1}^{\infty} v^y F_1^y(\varrho),$$

then it follows from (31) that

$$G(v, z) = \frac{\lambda z\, K(v) - (1 - \delta)\, \mu v z^2\, H^0(z, \varrho)}{(1 - \delta)\, \mu v z^2 - (\mu + \lambda - \delta\lambda)\, z + \lambda}. \tag{32}$$

The denominator of (32) has one root, say α, in the domain $|z| < 1$; it is

$$\alpha = \alpha(v) = \frac{\mu + \lambda - \delta\lambda - \{(\mu + \lambda - \delta\lambda)^2 - 4\lambda\mu v(1 - \delta)\}^{1/2}}{2\mu v(1 - \delta)}.$$

Hence

$$K(v) = (1 - \delta)\, v\alpha^2 \left[\frac{\delta}{(1 - \delta\varrho)(1 - \alpha)} + \frac{1 - \delta}{(1 - \varrho)(1 + \varrho - \delta\varrho)}\right], \tag{33}$$

and we get $F_1^y(\varrho)$ for $y = 1, 2, 3, \ldots$ by differentiating $K(v)$ (with respect to v) y-times and then putting $v = 0$.

Anyway, we shall not do that explicitly. We only notice that $\alpha(1) = \varrho$ and

$$\mathbf{P}\{Y > 0\} = (1 - \varrho)\, K(1) = \frac{(1 - \delta)\, \varrho^2}{(1 - \delta\varrho)(1 + \varrho - \delta\varrho)}; \tag{34}$$

this together with (25) makes 1 as it ought to.

11. For the reasons we have already given in [5], p. 363, the variables X and Y must have the *same mean values*. For X we find from (15)

$$\mathbf{E}[X] = (1 - \delta)(1 - \varrho)\, \varrho^2 \sum_{x=1}^{\infty} x \varrho^{x-1} = (1 - \delta)\, \varrho^2 (1 - \varrho)^{-1}.$$

For Y we get the same result from (33); since

$$\alpha(1) = \varrho, \quad \alpha'(1) = \frac{(1 - \delta)\, \varrho^2}{1 - \varrho + \delta\varrho}$$

we have indeed

$$\mathbf{E}[Y] = (1 - \varrho)\, K'(1) = (1 - \delta)\, \varrho^2 (1 - \varrho)^{-1}.$$

12. As we have already announced in the introduction, the main results we have derived here for our class of queue disciplines are essentially the same as the results established for the mixed queue discipline. In fact, it suffices only to write $1 - \vartheta$ instead of δ in our formulae to obtain the corresponding formulae of [5] or [6].

Let us consider e.g. the outtaking probabilities: changing suitably the notation in our formulae (14) and (15) we get the expressions for $P_{\text{akt}}(k)$ and $P_{\text{akt}}(0)$ given on p. 379 in [5]; analogously, our formula (25) becomes formula (6.19) of [5]. It would be perhaps a bit tedious to verify that (33) is the generating function for the probabilities (6.18) of [5], but it is easy to see that our expression for $\mathbf{E}[Y]$ is in accordance with the results quoted on p. 380 in [5], and in the special case of $\vartheta = 1$ ($\delta = 0$) – i.e. for the LIFO discipline – we have

$$K(v) = v\,\alpha^2(v)\,(1 - \varrho^2)^{-1} \tag{35}$$

with

$$\alpha(v) = (1/2v)\,\{1 + \varrho - [(1 + \varrho)^2 - 4\varrho v]^{1/2}\}\,,$$

which corresponds to (6.4) of [5].

A similar equality holds for the waiting time distribution: our formulae (6) and (5) become – again with $1 - \vartheta$ instead of δ – the corresponding formulae given on pp. 199–200 in [6] (for $r = 1$, of course).

These coincidences may look rather surprising so far as we have not found any natural explication based on an intuitive interpretation of the mixing operations involved in the disciplines of the two classes. However, we shall come back to this problem elsewhere (see also [8]).

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Souhrn

O JEDNÉ TŘÍDĚ FRONTOVÝCH REŽIMŮ

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V článku se podrobně vyšetřují systémy hromadné obsluhy typu $M/M/n$, v nichž se při uvolnění obsluhové linky vybere do obsluhy s danou pravděpodobností první, resp. poslední zákazník z fronty. S využitím metody vytvořujících funkcí se odvozuje stacionární rozložení doby čekání, pravděpodobnosti opuštění fronty z prvního, resp. posledního místa a rozložení počtu aktivních a pasivních předstihů zákazníků ve stabilisovaném systému. Ukazuje se, že některé výsledky se shodují s výsledky odvozenými již dříve pro tzv. smíšený frontový režim, v němž si zákazníci analogickým způsobem vybírají místo při zařazování do fronty.

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