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A FINITE ELEMENT ANALYSIS FOR THE SIGNORINI PROBLEM IN PLANE ELASTOSTATICS

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INTRODUCTION

If an elastic body rests upon a rigid frictionless support, the equilibrium can be formulated by means of the Signorini unilateral problem (cf. [1]). A systematic mathematical analysis of the problem was given by Fichera ([2], [3]).

For numerical solution of the Signorini problem, one can employ the finite element technique – see [8], [9]. It is the aim of our paper to present some a priori asymptotic error estimates for the finite element procedure, provided the solution is sufficiently regular. We also prove the convergence without any regularity assumptions.

1. FORMULATION OF THE SIGNORINI PROBLEM

In this section we shall introduce a variational formulation of the Signorini problem within the range of plane elastostatics, involving linear stress-strain relations and small deformations of a non-homogeneous, anisotropic body. Let $\Omega \subset R^2$ be a bounded plane domain with Lipschitz boundary[1]), occupied by an elastic body and let $\boldsymbol{x} = (x_1, x_2)$ be a Cartesian coordinate system. Let $\boldsymbol{n} = (n_1, n_2)$ denote the unit outward normal to the boundary Γ. We shall use the Sobolev spaces $H^k(\Omega)$, $k = 1, 2, \ldots$ of functions, the generalized derivatives of which up to the order k exist and are square-integrable in Ω. The usual norm of u in $H^k(\Omega)$ will be denoted by $\|u\|_k$, $H^0(\Omega) = L_2(\Omega)$,[2])

$$(f, g)_0 = \int_\Omega fg \, dx .$$

[1]) See [5] for the definition of Lipschitz boundary.

[2]) The same notation $\|\boldsymbol{u}\|_k$ will be used for vector-functions in $[H^k(\Omega)]^2$ and the corresponding euclidean norms.

Let the displacement vector $\boldsymbol{u} = (u_1, u_2) \in [H^1(\Omega)]^2$. The strain-displacement relations are

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad (i, j = 1, 2) \tag{1.1}$$

and the stress-strain relations

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl} \quad (i, j = 1, 2) \tag{1.2}$$

hold, where the coefficients satisfy:

$$c_{ijkl} \in L_\infty(\Omega), \quad c_{ijkl} = c_{klij} = c_{jikl}, \tag{1.3}$$

$$\exists c_0 = \text{const} > 0, \quad c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqq c_0\varepsilon_{ij}\varepsilon_{ij} \quad \forall \varepsilon_{ij} = \varepsilon_{ji}. \tag{1.4}$$

A repeated index implies summation over the range 1, 2.

The stress field satisfies the following equilibrium equations

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad (i = 1, 2), \tag{1.5}$$

where $\boldsymbol{F}$ denotes the vector of body forces.

The traction-vector on the boundary

$$T_i = \tau_{ij}n_j$$

can be decomposed into the normal component

$$T_n = T_i n_i = \tau_{ij}n_i n_j$$

and the tangential component

$$T_t = T_i t_i = \tau_{ij}t_i n_j,$$

where $\mathbf{t} = (t_1, t_2) = (-n_2, n_1)$ is the unit tangential vector.

The displacement vector can be decomposed similarly:

$$u_n = u_i n_i, \quad u_t = u_i t_i.$$

Suppose that the boundary Γ consists of three mutually disjoint parts

$$\Gamma = \bar{\Gamma}_u \cup \bar{\Gamma}_\tau \cup \bar{\Gamma}_a$$

and on Γ_u and Γ_τ the displacements or tractions are prescribed, i.e.

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_u, \tag{1.6}$$

$$\boldsymbol{T} = \bar{\boldsymbol{T}} \quad \text{on} \quad \Gamma_\tau, \tag{1.7}$$

whereas on Γ_a the Signorini's conditions

$$(1.8) \qquad u_n \leqq 0\,, \quad T_n \leqq 0\,, \quad u_n T_n = 0\,, \quad T_t = 0$$

hold.

Assume that $\mathbf{F} \in [L_2(\Omega)]^2$, $\bar{\mathbf{T}} \in [L_2(\Gamma_\tau)]^2$ are given and that *both* Γ_u *and* Γ_a *contain sets open in* Γ.

Introducing

$$A(\mathbf{u}, \mathbf{v}) = \int_\Omega c_{ijkm}\, \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{km}(\mathbf{v})\, \mathrm{d}x\,,$$

$$L(\mathbf{v}) \quad = \int_\Omega F_i v_i\, \mathrm{d}x + \int_{\Gamma_\tau} \bar{T}_i v_i\, \mathrm{d}s\,,$$

the functional of potential energy can be defined as

$$\mathscr{L}(\mathbf{v}) = \tfrac{1}{2} A(\mathbf{v}, \mathbf{v}) - L(\mathbf{v})\,.$$

Let

$$V = \{\mathbf{v} \mid \mathbf{v} \in [H^1(\Omega)]^2\,, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_u\}\,,$$

and

$$K = \{\mathbf{v} \mid \mathbf{v} \in V,\ v_n \leqq 0 \text{ on } \Gamma_a\}$$

be the subspace of virtual displacements and the convex cone of admissible virtual displacements, respectively.

The problem (1.1), (1.2), (1.5)–(1.8) can be formulated as follows: to find $\mathbf{u} \in K$ such that

$$(1.9) \qquad \mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K\,.$$

There is a close relation between the "classical" solution of (1.1), (1.2), (1.5)–(1.8) and a solution of (1.9). In fact, if holds

Lemma 1.1. *Any "classical" solution of* (1.1), (1.2), (1.5)–(1.8) *satisfies* (1.9). *On the contrary, if a solution of* (1.9) *is sufficiently regular, then it is a "classical" solution, as well.*

Proof. First we recall that $\mathbf{u} \in K$ is a solution of (1.9) if and only if

$$(1.10) \qquad A(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geqq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K\,.$$

Let $\mathbf{u}$ be a "classical" solution. Multiplying (1.5) by a vector $\mathbf{w} \in V$, integrating by parts and using (1.6), (1.7), (1.8), we obtain

$$(1.11) \qquad 0 = \int_\Omega \left(-\tau_{ij}(\mathbf{u}) \frac{\partial w_i}{\partial x_j} + F_i w_i \right) \mathrm{d}x + \int_{\Gamma_\tau \cup \Gamma_a} \tau_{ij}(\mathbf{u})\, n_j w_i\, \mathrm{d}s =$$

$$= -A(\mathbf{u}, \mathbf{w}) + L(\mathbf{w}) + \int_{\Gamma_a} T_n(\mathbf{u})\, w_n\, \mathrm{d}s\,.$$

Choose an arbitrary $\mathbf{v} \in K$ and set $\mathbf{w} = \mathbf{v} - \mathbf{u}$. If $u_n(x) < 0$, then $T_n(\mathbf{u})(x) = 0$. If $u_n(x) = 0$, then $w_n(x) = v_n(x) \leqq 0$ and $T_n(\mathbf{u})(x) \leqq 0$. Altogether, the last integral is non-negative and we have

$$A(\mathbf{u}, \mathbf{w}) - L(\mathbf{w}) \geqq 0 ,$$

i.e., (1.10).

On the contrary, let $\mathbf{u} \in K$ be a sufficiently regular solution of (1.10). Denoting $\mathbf{v} - \mathbf{u} = \mathbf{w} \in V$ and integrating by parts, we obtain

$$A(\mathbf{u}, \mathbf{w}) - L(\mathbf{w}) = - \int_\Omega \left(\frac{\partial \tau_{ij}(\mathbf{u})}{\partial x_j} + F_i \right) w_i \, dx + \int_\Gamma \tau_{ij}(\mathbf{u}) \, n_j w_i \, ds - \int_{\Gamma_\tau} \bar{T}_i w_i \, ds . \tag{1.12}$$

Choosing $\mathbf{w} = \pm \varphi \in [\mathscr{D}(\Omega)]^2$ (where $\mathscr{D}(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω), we obtain the equilibrium equations (1.5). Consequently, from (1.12) and (1.10) it follows that

$$0 \leqq \int_{\Gamma_\tau} (\tau_{ij}(\mathbf{u}) \, n_j - \bar{T}_i) \, w_i \, ds + \int_{\Gamma_a} (T_n(\mathbf{u}) \, w_n + T_t(\mathbf{u}) \, w_t) \, ds . \tag{1.13}$$

The choice of $\mathbf{w} \in V$ such that the traces of w_i vanish on Γ_a leads to the boundary conditions (1.7) on Γ_τ. Thus in (1.13) only the last integral remains. Next choosing $\mathbf{w}$ on Γ_a such that $w_n = 0$, $w_t = \pm \psi$, we obtain $T_t(\mathbf{u}) = 0$ on Γ_a.

Let $w_n \leqq 0$ be arbitrary. Then from (1.13) $T_n(\mathbf{u}) \leqq 0$ on Γ_a follows.

Finally,

$$A(\mathbf{u}, \mathbf{u}) - L(\mathbf{u}) = 0 \tag{1.14}$$

can be deduced from (1.10), inserting $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}$.

Consequently, repeating the above procedure for $\mathbf{w} = \mathbf{u}$, we obtain

$$0 = \int_{\Gamma_a} T_n(\mathbf{u}) \, u_n \, ds .$$

As the product $T_n(\mathbf{u}) \, u_n$ is non-negative, it must vanish on Γ_a.

Proposition 1.1. *There exists a unique solution of the problem* (1.9).

Proof. The set K, being closed and convex in $[H^1(\Omega)]^2$, is weakly closed. For the second Gâteaux differential of $\mathscr{L}$ we may write

$$D^2 \mathscr{L}(\mathbf{u}; \mathbf{v}, \mathbf{v}) = A(\mathbf{v}, \mathbf{v}) \geqq c_0 \int_\Omega \varepsilon_{ij}(\mathbf{v}) \, \varepsilon_{ij}(\mathbf{v}) \, dx \geqq c_1 \|\mathbf{v}\|_1^2 \tag{1.15}$$

$$\forall \mathbf{u} \in [H^1(\Omega)]^2 , \quad \mathbf{v} \in V ,$$

(where the Korn's inequality has been used in the last step). Therefore $\mathscr{L}$ is weakly lower semi-continuous and coercive and the existence and uniqueness of a solution follows.

2. FINITE ELEMENT APPROXIMATIONS

For simplicity, we restrict ourselves to polygonal domains. (For domains with smooth boundary, we refer the reader to the paper by Scarpini and Vivaldi [6], whose technique could be extended to the above problem).

Let $\Omega \subset R^2$ be a polygonal bounded domain (multiply connected, in general). We carve it into triangles T generating a triangulation $\mathscr{T}_h$. Denote h the maximal side of all triangles in $\mathscr{T}_h$. Let V_h be the space of continuous piecewise linear functions on the tringulation $\mathscr{T}_h$, vanishing on Γ_u.[1])

We say that a family of triangulations $\{\mathscr{T}_h\}$, $0 < h \leqq 1$, is $\alpha - \beta$-regular, if there exist positive α and β, independent of h and such that (i) the minimal angle of all triangles in $\mathscr{T}_h$ is not less than α and (ii) the ratio between any two sides of $\mathscr{T}_h$ is less than β.

For any $h \in (0, 1\rangle$ we define

$$K_h = \{\mathbf{v} \mid \mathbf{v} \in [V_h]^2,\ v_n \leqq 0 \text{ on } \Gamma_a\}\ .$$

Obviously, $K_h \subset K\ \forall h \in (0, 1\rangle$. We say that $\mathbf{u}_h \in K_h$ is a *finite element approximation to the problem* (1.9) if

$$\mathscr{L}(\mathbf{u}_h) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K_h\ . \tag{2.1}$$

It is readily seen that there exists a unique finite element approximation. This assertion can be verified by following the proof of Proposition 1.1.

We focus our attention to the estimate of the error $\mathbf{u} - \mathbf{u}_h$ between the solutions of the problem (1.9) and (2.1), respectively. To this end we shall use the idea proposed by Mosco and Strang [10], like in [4] – Sect. 2. Let us recall the

Lemma 2.1. *Let $\mathscr{J}$ be the functional defined on a closed convex subset K of a Banach reflexive space B. Assume that $\mathscr{J}$ is twice differentiable in B and the second differential satisfies the following inequalities*

$$\alpha_0 \|z\|^2 \leqq D^2 \mathscr{J}(u;\, z, z) \leqq c\|z\|^2 \quad \forall u \in K\,, \quad z \in B\,. \tag{2.2}$$

Let $K_h \subset K$ be a closed convex set. Denote the minimizing element of $\mathscr{J}$ over K and K_h by u and u_h, respectively. Assume that a $w_h \in K_h$ exists such that $2u - w_h \in K$. Then it holds

$$\|u - u_h\| \leqq (c/\alpha_0)^{1/2}\, \|u - w_h\|\ . \tag{2.3}$$

For the proof – see [4] – Lemma 2.1.

[1]) The end-points of Γ_u coincide with the vertices of $\mathscr{T}_h$.

Hence the problem is to find a $w_h \in K_h$ sufficiently close to u and such that $2u - w_h \in K$. We can prove the following

Theorem 2.1. *Assume that* $\mathbf{u} \in [H^2(\Omega)]^2$ *and* $u_n \in H^2(\Gamma_a \cap \Gamma_m)$, *where* Γ_m, $m = 1, 2, \ldots, G$, *denotes any side of the polygonal boundary.*

Then there exists a $\mathbf{w}_h \in [V_h]^2$ *such that*

$$0 \geqq w_{hn} \geqq u_n \quad on \quad \Gamma_a\,, \tag{2.4}$$

and, if the triangulations are $\alpha - \beta$*-regular, it achieves the optimal order of approximation, i.e.*

$$\|\mathbf{u} - \mathbf{w}_h\|_1 \leqq Ch\{\|\mathbf{u}\|_2 + \max_{m=1,\ldots,G} \|u_n\|_{H^2(\Gamma_a \cap \Gamma_m)}\} \tag{2.5}$$

(*with* C *independent of* h *and* $\mathbf{u}$).

Proof is based on two lemmas.

Lemma 2.2. (*One-sided approximation of* u_n *on the boundary*). *Let* $u_n \in H^2(\Gamma_a \cap \Gamma_m)$, $m = 1, \ldots, G$. *Then there exist linear spline functions* $\psi_h^{(m)} \in C(\Gamma_m)$, (*with nodes determined by the vertices of the triangulation* $\mathcal{T}_h$), *such that*

$$0 \geqq \psi_h^{(m)} \geqq u_n \quad \text{on} \quad \Gamma_m \cap \Gamma_a\,, \tag{2.6}$$

$$\|u_{nI} - \psi_h^{(m)}\|^2_{\Gamma_m} \leqq h^3 \int_{\Gamma_m \cap \Gamma_a} [\mathrm{d}^2 u_n/\mathrm{d}s^2]^2 \,\mathrm{d}s \tag{2.7}$$

holds for any $m = 1, 2, \ldots, G$, *where* u_{nI} *is the linear Lagrange interpolate of* u_n *on* $\bar{\Gamma}_m$ (*with the same nodes*) *and*

$$\|\varphi\|_{\Gamma_m} = \sup_{s \in \Gamma_m} |\varphi(s)|\,.$$

Proof is parallel to that of Lemma 2.2 in [4], where Γ is replaced by $\bar{\Gamma}_m \cap \bar{\Gamma}_a$. We set $\psi_h^{(m)} = 0$ on $\bar{\Gamma}_u$ and $\psi_h^{(m)} = u_{nI}$ at the vertices of $\bar{\Gamma}_\tau$ except the points $\bar{\Gamma}_a \cap \bar{\Gamma}_\tau$, where the one-sided approximation on $\bar{\Gamma}_a$ is defined.

Lemma 2.3. *Let* $\varphi_m \in C(\bar{\Gamma}_m)$, $m = 1, 2, \ldots, G$ *be linear spline-functions with the nodes determined by a* $\alpha - \beta$*-regular triangulation* $\mathcal{T}_h$, $\varphi_m = 0$ *on* $\bar{\Gamma}_u$.

Then there exists a $\mathbf{v}_h \in [V_h]^2$ *such that* $v_{hn} = \varphi_m$ *on* Γ_m *for* $m = 1, \ldots, G$ *and*

$$\|\mathbf{v}_h\|_1 \leqq Ch^{-1/2} \max_{m=1,\ldots,G} \|\varphi_m\|_{\Gamma_m}\,. \tag{2.8}$$

Proof is analogous to that of Lemma 3.2 in [4].

The proof of Theorem 2.1. Let $\psi_h^{(m)}$ be the one-sided approximations of u_n defined in Lemma 2.2. Introducing

$$\varphi_m = u_{nI} - \psi_h^{(m)}\,, \quad m = 1, \ldots, G\,, \tag{2.9}$$

we construct the vector-function $\mathbf{v}_h \in [V_h]^2$ according to Lemma 2.3. Then the function

$$\mathbf{w}_h = \mathbf{u}_I - \mathbf{v}_h\,,$$

(where $\mathbf{u}_I = (u_{1I}, u_{2I})$ denotes the Lagrange linear interpolate of $\mathbf{u}$ over the triangulation $\mathcal{T}_h$) satisfies (2.4), (2.5). In fact, on every Γ_m it holds

$$w_{hn} = u_{In} - v_{hn} = u_{nI} - \varphi_m = \psi_h^{(m)}$$

and (2.6) implies (2.4).

Furthermore, it is well-known that

$$\|\mathbf{u} - \mathbf{u}_I\|_1 \leqq Ch\|\mathbf{u}\|_2\,. \tag{2.10}$$

Then

$$\|\mathbf{u} - \mathbf{w}_h\|_1 \leqq \|\mathbf{u} - \mathbf{u}_I\|_1 + \|\mathbf{u}_I - \mathbf{w}_h\|_1 \leqq \|\mathbf{u} - \mathbf{u}_I\|_1 + \|\mathbf{v}_h\|_1$$

holds and from (2.8), (2.9), (2.7) it follows

$$\|\mathbf{v}_h\|_1 \leqq Ch \max_{m=1,\ldots,G} \|u_n\|_{H^2(\Gamma_a \cap \Gamma_m)}\,. \tag{2.11}$$

Hence (2.10) and (2.11) yield the estimate (2.5).

Corollary 2.1. *Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of* (1.9) *and* (2.1), *respectively. If the assumptions of Theorem* 2.1 *are satisfied, then*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 = O(h)\,. \tag{2.12}$$

Proof. With regard to Lemma 2.1 and (2.5), it suffices to verify that (i) the functional $\mathcal{L}$ satisfies (2.2) and (ii) $\mathbf{w}_h \in K_h$, $2\mathbf{u} - \mathbf{w}_h \in K$. The positive-definiteness of (2.2) is an immediate consequence of the Korn's inequality (1.15), (for $B \equiv V$). From (2.4) it follows that $\mathbf{w}_h \in K_h$. Obviously, $2\mathbf{u} - \mathbf{w}_h \in V$ and on Γ_a we have

$$2u_n - w_{hn} \leqq u_n - w_{hn} \leqq 0$$

as a consequence of (2.4). Hence $2\mathbf{u} - \mathbf{w}_h \in K$ follows and the proof is complete.

3. CONVERGENCE WITHOUT ANY REGULARITY ASSUMPTIONS

From the results developed by Fichera (cf. [3]) one concludes that the regularity of the solution $\mathbf{u}$, assumed in Theorem 2.1, cannot be expected, in general. Therefore we study the convergence of the finite element approximations without any regularity assumptions. To this end, we employ the following abstract

Theorem 3.1. (cf. [7] – chpt. 4). Let V be a Hilbert space with the norm $\|.\|$, $K \subset V$ a convex closed subset, $h \in (0, 1\rangle$ a real parameter, $K_h \subset K$ convex closed sets for any h.

Let a differentiable functional $\mathscr{J}$ on V be given, the second differential of which exists and satisfies the inequalities (2.2) for any $u \in K$ and $z \in V$.

Denote u and u_h the minimizing elements of $\mathscr{J}$ over the sets K and K_h, respectively. Assume that $v_h \in K_h$ exist such that

$$\lim_{h \to 0} \|u - v_h\| = 0 . \tag{3.1}$$

Then it holds

$$\lim_{h \to 0} \|u - u_h\| = 0 . \tag{3.2}$$

Proof. From (2.2) the existence and uniqueness of u and u_h follows. Let $v_h \in K_h$ satisfy (3.1). Using the Taylor's theorem we may write

$$\mathscr{J}(v_h) = \mathscr{J}(u) + D\mathscr{J}(u, v_h - u) + \tfrac{1}{2}D^2 \mathscr{J}(u + \vartheta_h(v_h - u); v_h - u, v_h - u).$$

Consequently, by virtue of (2.2), we conclude

$$\lim \mathscr{J}(v_h) = \mathscr{J}(u) . \tag{3.3}$$

From the definition of u_h it follows

$$\mathscr{J}(u_h) \leqq \mathscr{J}(v_h) , \tag{3.4}$$

consequently,

$$\mathscr{J}(u_h) \leqq c < +\infty \quad \forall h .$$

Since $\mathscr{J}$ is coercive, it holds

$$\|u_h\| \leqq c_1 < +\infty \quad \forall h .$$

Thus we can choose a subsequence (denoted again by $\{u_h\}$) such that $u_h \in K_h$, u_h tends to u^* weakly. As K is weakly closed, $u^* \in K$. We have

$$\mathscr{J}(u^*) \leqq \lim \mathscr{J}(u_h) = \mathscr{J}(u) ,$$

consequently $u^* = u$.

There exist $\lambda_h \in (0, 1)$ such that

$$\mathscr{J}(u_h) = \mathscr{J}(u) + D\mathscr{J}(u, u_h - u) + \tfrac{1}{2}D^2 \mathscr{J}(u + \lambda_h(u_h - u); u_h - u, u_h - u)$$

and by virtue of (2.2)

$$\mathscr{J}(u_h) - \mathscr{J}(u) - D\mathscr{J}(u, u_h - u) \geqq \tfrac{1}{2}\alpha_0 \|u_h - u\|^2 .$$

From (3.3), (3.4) and the weak convergence $u_h \to u$ (3.2) follows for the subsequence. Since u is a unique solution, the whole sequence converges to u.

Theorem 3.2. *Assume that there is only a finite number of "end-points"* $\bar{\Gamma}_a \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_u \cap \bar{\Gamma}_a$. *Then the set*

$$K \cap [C^\infty(\bar{\Omega})]^2$$

is dense in K.

Proof. Let $\boldsymbol{u} \in K$ be a fixed vector-function. Consider a system of open domains $\{B_i\}$, $i = 0, 1, \ldots, r$, which cover $\bar{\Omega}$ and denote $\{\varphi_i\}$ the corresponding decomposition of unity (i.e. $\varphi_i \in \mathscr{D}(B_i)$, $0 \leqq \varphi_i \leqq 1$, $\sum_{i=0}^{r} \varphi_i(x) = 1\ \forall x \in \bar{\Omega}$). Assume that $\bar{B}_0 \subset \Omega$ and $\bigcup_{i=1}^{r} B_i$ covers the boundary Γ. Denoting $\boldsymbol{u}^j = \boldsymbol{u}\varphi_j$, we have

$$\boldsymbol{u} = \sum_{j=0}^{r} \boldsymbol{u}^j\,, \quad \boldsymbol{u}^j \in [H^1(\Omega)]^2, \quad \operatorname{supp} \boldsymbol{u}^j \subset B_j \quad \forall j\,.$$

We say that $P \in \Gamma$ is a singular point, if it is a vertex of Γ or an "end-point" $\bar{\Gamma}_a \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_u \cap \bar{\Gamma}_a$. Suppose that each B_j contains at most one singular point. For brevity, we shall omit the superscript j. The system of B_i, $i = 1, \ldots, r$, can be divided into eight groups as follows.

1. *group.* Let $B_j \cap \Gamma \subset \Gamma_u$. Then $u_k \in H_0^1(B_j \cap \Omega)$, $(k = 1, 2)$ can be approximated by functions $u_{k\varkappa} \in C_0^\infty(B_j \cap \Omega)$ such that

$$\|u_{k\varkappa} - u_k\|_1 \to 0 \quad \text{for} \quad \varkappa \to 0\,.$$

2. *group.* Let $B_j \cap \Gamma \subset \Gamma_a$ and does not contain any vertex of Γ. Consider the local cartesian coordinate system (ξ, η) such that the ξ-axis coincides with Γ. Then we may write $\boldsymbol{u} = u_\xi \mathbf{e}_\xi + u_\eta \mathbf{e}_\eta$, where $\mathbf{e}_\xi$, $\mathbf{e}_\eta$ are unit basis vectors, and $u_\eta = -u_n \geqq \geqq 0$ for $\eta = 0$.

There exists a function $v \in H^1(B_j \cap \Omega)$ such that $v \geqq 0$ in $B_j \cap \Omega$, $\operatorname{supp} v \subset B_j$, $v = u_\eta$ on Γ (see [5], chpt. 2. Th. 5.7 for the construction of that function). Let us define the extension Pv of v by means of the relation

$$Pv(\xi, \eta) = Pv(\xi, -\eta)\,. \tag{3.6}$$

Then $Pv \in H^1(B_j)$. Using the regularization operator $R_\varkappa$ with the kernel

$$\omega(x, \varkappa) = \begin{cases} A\varkappa^{-2} \exp\left(\dfrac{|x^2|}{|x|^2 - \varkappa^2}\right) & \text{for} \quad |x| < \varkappa\,, \\ 0 \quad \text{for} \quad |x| \geqq \varkappa\,, \end{cases}$$

where $\varkappa$ and A are positive constants, $x \equiv (\xi, \eta)$, we define

$$R_\varkappa Pv(x) = \int_{B_j} \omega(x - x', \varkappa)\, Pv(x')\, \mathrm{d}x'\,, \quad x' = (\xi', \eta')\,. \tag{3.7}$$

Since both ω and Pv are non-negative, we obtain $R_\varkappa Pv \geqq 0$ on Γ, moreover $R_\varkappa Pv \in C^\infty(\bar{\Omega})$ and

$$\|R_\varkappa Pv - v\|_1 \to 0 \tag{3.8}$$

holds for $\varkappa \to 0$. We have

$$u_\eta = v + z\,, \quad z \in H_0^1(B_j \cap \Omega)\,.$$

Setting

$$u_{\eta\varkappa} = R_\varkappa Pv + z_\varkappa\,,$$

where $z_\varkappa \in C_0^\infty(B_j \cap \Omega)$ is an approximation of z, we obtain

$$\|u_{\eta\varkappa} - u_\eta\|_1 \leqq \|R_\varkappa Pv - v\|_1 + \|z_\varkappa - z\|_1 \to 0 \tag{3.9}$$

for $\varkappa \to 0$ and $u_{\eta\varkappa} \in C^\infty(\bar{\Omega})$, $u_{\eta\varkappa} \geqq 0$ on Γ.

We extend also the component u_ξ like v in (3.6) and regularize. Then $u_{\xi\varkappa} \equiv R_\varkappa Pu_\xi \in C^\infty(\bar{\Omega})$,

$$\|u_{\xi\varkappa} - u_\xi\|_1 \to 0 \quad \text{for} \quad \varkappa \to 0\,. \tag{3.10}$$

From (3.9) and (3.10) we conclude that for $\varkappa \to 0$

$$\|\boldsymbol{u}_\varkappa - \boldsymbol{u}\|_1^2 = \sum_{k=1}^{2} \|u_{k\varkappa} - u_k\|_1^2 \leqq 4(\|u_{\xi\varkappa} - u_\xi\|_1^2 + \|u_{\eta\varkappa} - u_\eta\|_1^2) \to 0\,. \tag{3.11}$$

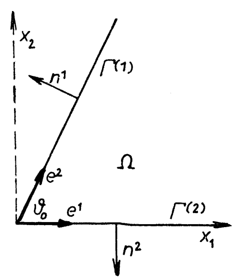

Fig. 1

3. *group*. Let $B_j \cap \Gamma \subset \Gamma_a$ contain a vertex of Γ. In general, we use a "skew" coordinate basis for the components of $\boldsymbol{u}$. Thus let $\mathbf{e}^1$, $\mathbf{e}^2$ be unit tangential and $\boldsymbol{n}^1$, $\boldsymbol{n}^2$ unit outward normal vectors (see Fig. 1).

We may write

$$\boldsymbol{u} = \sum_{p=1}^{2} u^{(p)}\, \mathbf{e}^p / \mathbf{e}^p \,.\, \boldsymbol{n}^p\,,$$

where the dot denotes the scalar product, $u^{(p)} = \boldsymbol{u} \,.\, \boldsymbol{n}^p$. Hence

$$u^{(p)} = u_n \leqq 0 \quad \text{on} \quad \Gamma^{(p)}\,.$$

Let us consider the component $u^{(2)}$. We transform the angular domain $B_j \cap \Omega$ into the upper halfplane $\{(\xi, \eta) \mid \eta > 0\}$ by means of a proper Lipschitz mapping T such that $\Gamma^{(2)}$ is mapped into the positive ξ-axis and $\Gamma^{(1)}$ into the negative ξ-axis. Let us extend the trace $u^{(2)}$ from the positive onto the negative ξ-axis by means of the relation

$$Pu^{(2)}(-\xi) = Pu^{(2)}(\xi) .$$

There exists a function $\hat{v} \in H^1(\hat{B}_{j0})$, where $\hat{B}_{j0} = T(B_j \cap \Omega)$, such that $\hat{v} \leqq 0$ in $\hat{B}_{j0}$, $\hat{v} = Pu^{(2)}$ on the ξ-axis, supp $\hat{v} \subset T(B_j)$ (see again [5], Th. 2.5.7). If we define $v(x_1, x_2) \equiv \hat{v}(T(x_1, x_2))$, then $v \in H^1(B_j \cap \Omega)$, $v \leqq 0$ in $B_j \cap \Omega$, supp $v \subset B_j$ and $v = u^{(2)}$ on $\Gamma^{(2)}$. Consequently, we may write

$$u^{(2)} = v + z ,$$

where $z \in H^1(B_j \cap \Omega)$, supp $z \subset B_j$, $z = 0$ on $\Gamma^{(2)}$.

We define

$$v_\lambda = v(x + \lambda) ,$$

where $\lambda \in R^2$ is a vector in the direction of the axis of the internal angle at the vertex. Then v_λ is an extension of v and it holds $R_\varkappa v_\lambda \in C^\infty(\bar{\Omega})$, $R_\varkappa v_\lambda \leqq 0$ on $\Gamma^{(2)}$,

$$\|R_\varkappa v_\lambda - v\|_1 \leqq \|R_\varkappa v_\lambda - v_\lambda\|_1 + \|v_\lambda - v\|_1 \to 0$$

for $|\lambda| \to 0$ and $\varkappa \to 0$, $\varkappa < C|\lambda|$.

There exists a function $w \in H^1(B_j \cap \Omega)$ such that $w = z$ on Γ, supp $w \subset B_j$, $w = 0$ in the angular domain $0 < \vartheta < \frac{1}{2}\vartheta_0$, where ϑ_0 denotes the internal angle and ϑ the polar coordinate. Choosing the direction of a vector λ properly, we find a "shifted" function $w_\lambda(x) = w(x + \lambda)$ such that $R_\varkappa w_\lambda = 0$ on $\Gamma^{(2)}$ and

$$\|R_\varkappa w_\lambda - w\|_1 \to 0 \quad \text{for}$$
$$|\lambda| \to 0 , \quad \varkappa \to 0 , \quad \varkappa < C|\lambda| .$$

Since it holds

$$z = w + z_0 ,$$

where $z_0 \in H_0^1(B_j \cap \Omega)$, we have

$$u_{\varkappa\lambda}^{(2)} \equiv R_\varkappa v_\lambda + R_\varkappa w_\lambda + R_\varkappa z_0 \in C^\infty(\bar{\Omega}) ,$$
$$u_{\varkappa\lambda}^{(2)} \leqq 0$$

on $\Gamma^{(2)}$ and

$$\|u_{\varkappa\lambda}^{(2)} - u^{(2)}\|_1 \to 0 \quad \text{for} \tag{3.12}$$
$$\varkappa < C|\lambda| , \quad |\lambda| \to 0 .$$

An analogous approach can be applied to $u^{(1)}$. For any vector $\mathbf{w}$ it holds $w_k = a_1 w^{(1)} + a_2 w^{(2)}$, where a_i are constants. Consequently, we have

$$\|w_k\|_1^2 \leqq C \sum_{p=1}^{2} \|w^{(p)}\|_1^2 , \quad k = 1, 2 .$$

Defining

$$\mathbf{u}_{\varkappa\lambda} = \sum_{p=1}^{2} u_{\varkappa\lambda}^{(p)} \mathbf{e}^p / \mathbf{e}^p \,.\, \mathbf{n}^p$$

and using (3.12) together with an analogous result for $p = 1$, we arrive at

$$\|\mathbf{u}_{\varkappa\lambda}^j - \mathbf{u}^j\|_1 \to 0 \quad \text{for} \quad \varkappa < C|\lambda| , \quad |\lambda| \to 0 . \tag{3.13}$$

4. *group.* Let $B_j \cap \Gamma$ contain a singular point $\bar{\Gamma}_a \cap \bar{\Gamma}_\tau$, which may coincide with a vertex of Γ. We transform $B_j \cap \Omega$ into the upper halfplane, mapping Γ_a into the positive ξ-axis. We apply the approach of the 3. group, used for $u^{(2)}$, setting $\Gamma^{(2)} = \Gamma_a$, $\Gamma^{(1)} = \Gamma_\tau$. If $B_j \cap \Gamma$ is straight, we use the same approach, substituting only $-u_\eta$ for $u^{(2)}$.

5. *group.* Let $B_j \cap \Gamma$ contain a point $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, which may coincide with a vertex of Γ. Let Γ_u coincide with the positive ξ-axis of the local coordinate system. We may apply the approach used for approximating the function z in the 3. group, to both components u_k, $k = 1, 2$, substituting Γ_u for $\Gamma^{(2)}$.

6. *group.* Let $B_j \cap \Gamma$ contain a point $\bar{\Gamma}_u \cap \bar{\Gamma}_a$ which is not a vertex. Let Γ_u coincide with the positive ξ-axis. The component u_ξ can be approximated as the function z in the 3. group. For u_η, we may write $u_\eta = v + z$, where $v \in H^1(B_j \cap \Omega)$, $v = u_\eta$ on Γ, $v \geqq 0$ in $B_j \cap \Omega$, supp $v \subset B_j$ and $v = 0$ in the first quadrant $\xi > 0$, $\eta > 0$, $z \in H_0^1(B_j \cap \Omega)$. Defining $v_\lambda(x) = v(x + \lambda)$, where $\lambda = (b, b)$, $b > 0$, and regularizing, we obtain $R_\varkappa v_\lambda = 0$ on Γ_u, $R_\varkappa v_\lambda \geqq 0$ on Γ_a. The remaining steps are obvious.

7. *group.* Let $B_j \cap \Gamma$ contain a point $\bar{\Gamma}_u \cap \bar{\Gamma}_a$, coinciding with a vertex of Γ. Using the "skew" coordinate system, we obtain

$$u^{(1)} = u_n \leqq 0 \quad \text{on} \quad \Gamma^{(1)} \equiv \Gamma_a , \quad u^{(1)} = u^{(2)} = 0 \quad \text{on} \quad \Gamma^{(2)} \equiv \Gamma_u .$$

The component $u^{(2)}$ can be approximated like the function z in the 3. group. $u^{(1)}$ can be written in the form $u^{(1)} = v + z$, where $v \leqq 0$ in $B_j \cap \Omega$, $v = 0$ for $0 < \vartheta < \frac{1}{2}\vartheta_0$, supp $v \subset B_j$ and $v = u^{(1)}$ on Γ. Then "shifting" v properly (as for z in the 3. group) and regularizing, we obtain $R_\varkappa v_\lambda \leqq 0$ on Γ_a, $R_\varkappa v_\lambda = 0$ on Γ_u.

8. *group.* Let $B_j \cap \Gamma \subset \Gamma_\tau$. Since no boundary conditions are imposed, there exist approximations $u_{k\varkappa} \in C^\infty(\bar{\Omega})$, supp $u_{k\varkappa} \subset B_j$ such that

$$\|\mathbf{u}_\varkappa - \mathbf{u}\|_1 \to 0 \quad \text{for} \quad \varkappa \to 0 .$$

For B_0 we define $u_{k\varkappa}^0 = R_\varkappa u_k^0$. Finally, adding $\mathbf{u}_\varkappa^j$ or $\mathbf{u}_{\varkappa\lambda}^j$, respectively, from all the sets B_j, we are led to the assertion of the theorem.

Theorem 3.3. *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of the problem* (1.9) *and* (2.1), *respectively. Let the assumptions of Theorem* 3.2 *be satisfied. Then*

$$\lim_{h \to 0} \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 = 0 . \tag{3.17}$$

Proof. From Theorem 3.2 it follows that a $\boldsymbol{u}_\varkappa \in K \cap [C^\infty(\bar{\Omega})]^2$ exists such that

$$\|\boldsymbol{u} - \boldsymbol{u}_\varkappa\|_1 < \varepsilon/2 .$$

As $\boldsymbol{u}_\varkappa$ is smooth, we can define the Lagrange linear interpolate $\boldsymbol{u}_{\varkappa I}$ over the triangulation $\mathscr{T}_h$ and the estimate (cf. (2.10))

$$\|\boldsymbol{u}_{\varkappa I} - \boldsymbol{u}_\varkappa\|_1 < Ch\|\boldsymbol{u}_\varkappa\|_2$$

holds. We have $\boldsymbol{u}_{\varkappa I} \in K_h$ and for sufficiently small h

$$\|\boldsymbol{u} - \boldsymbol{u}_{\varkappa I}\|_1 \leqq \|\boldsymbol{u} - \boldsymbol{u}_\varkappa\|_1 + \|\boldsymbol{u}_\varkappa - \boldsymbol{u}_{\varkappa I}\| < \varepsilon .$$

Setting $\boldsymbol{v}_h = \boldsymbol{u}_{\varkappa I}$, the assumption (3.1) is satisfied. With $\mathscr{J} \equiv \mathscr{L}$, Theorem 3.1 implies the convergence of $\boldsymbol{u}_h$.

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References

[1] *Signorini, A.:* Questioni di elasticità non linearizzata o semi-linearizzata. Rend. di Matem. e delle sue appl. 18 (1959).

[2] *Fichera, G.:* Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Mem. Acc. Naz. Lincei, s. VIII., 7, f. 5 (1964).

[3] *Fichera, G.:* Boundary Value Problems of Elasticity with Unilateral Constraints. Encycl. of Physics (ed. by S. Flügge), vol. VIa/2. Springer, Berlin 1972.

[4] *Hlaváček, I.:* Dual Finite Element Analysis for Unilateral Boundary Value Problems. Apl. Mat. 22 (1977), 14—51.

[5] *Nečas, J.:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.

[6] *Scarpini, F.* and *Vivaldi, M.:* Error Estimates for the Approximation of Some Unilateral Problems. R.A.I.R.O. (1977).

[7] *Céa, J.:* Optimisation, théorie et algorithmes. Dunod, Paris 1971.

[8] *Frémond, M.*: Dual Formulations for Potential and Complementary Energies. Mathematics of Finite Elements and Appl. (ed. by J. R. Whiteman), Academic Press, 1973.

[9] *Panagiotopoulos, P. D.:* Unilateral Problems in the Theory of Elasticity. Ing. Archiv. 44, (1975), 421—432.

[10] *Mosco, U.* and *Strang, G.:* One-sided Approximations and Variational Inequalities. Bull. Amer. Math. Soc. 80 (1974), 308—312.

Souhrn

ANALÝZA SIGNORINIHO ÚLOHY V ROVINNÉ PRUŽNOSTI METODOU KONEČNÝCH PRVKŮ

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Spočívá-li pružné těleso na dokonale tuhé a hladké opěře, pak jeho rovnováhu lze popsat pomocí Signoriniho jednostranné úlohy (viz [1]). Systematický matematický rozbor podal Fichera ([2], [3]). K numerickému řešení se hodí též metoda konečných prvků (viz [8], [9]).

V této práci se odvozují některé apriorní asymptotické odhady chyb metody konečných prvků za předpokladu jisté regularity řešení. V závěrečném odstavci je dokázána konvergence i k řešení, které není regulární.

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