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QUADRATIC FUNCTIONALS AND BILINEAR FORMS

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Let X be a vector space over the complex field. Let B be a bilinear form on X, i.e. a function defined on $X \times X$ which is linear in the first variable and conjugate-linear in the second variable. Let Q be the function on X defined by the formula $Q(x) = B(x, x)$; it is easy to see that the function Q possesses the following two properties

$$1^\circ \quad Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \text{ for all } x, y \in X$$

$$2^\circ \quad Q(\lambda x) = |\lambda|^2 \, Q(x) \text{ for all } x \in X \text{ and all complex } \lambda .$$

A function defined on X which satisfies conditions 1° and 2° will be called a quadratic functional on X. An obvious question presents itself: are the properties 1° and 2° characteristic for quadratic forms generated by bilinear forms? In other words, given a quadratic functional Q, does there exist a bilinear form B such that $Q(x) = B(x, x)$? In the papers [1] and [2], S. Kurepa proved that the answer is affirmative and that the corresponding result for vector spaces over the real field is false.

In the present note we prove a lemma concerning a certain functional equation; this lemma is then used to obtain a simple proof of the fact that every quadratic functional is generated by a bilinear form.

Lemma. *Let f be an additive complex-valued function of a complex variable which satisfies $f(\lambda) = - |\lambda|^2 f(1/\lambda)$ for all $\lambda \neq 0$. Then $f(\lambda) = f(i) \operatorname{Im} \lambda$.*

Proof. Let t be a real number such that $|t| \leqq 1$. Let us show that $f(t) = 0$. Choose a real s such that $t^2 + s^2 = 1$ and set $\lambda = t + is$. It follows that $f(t) + f(is) = f(\lambda) = - |\lambda|^2 f(\bar{\lambda}) = -f(t) + f(is)$ whence $f(t) = 0$. If $|t| > 1$, we have $|1/t| < 1$ whence $f(t) = -t^2 f(1/t) = 0$. Consider now a real number s with $0 < s \leqq 1$. Choose a real number t such that $t^2 + s^2 = s$ and set $\lambda = t + is$. It follows that $f(is) = f(\lambda) = - |\lambda|^2 f(1/\lambda) = -sf(t/s - i) = sf(i)$. If $s > 1$ we have $f(is) = -s^2 f(1/is) = s^2 f(i/s) = sf(i)$. Since f is additive, the equation $f(is) = sf(i)$ holds for $s \leqq 0$ as well. The proof is complete.

Theorem. *Let Q be a quadratic functional defined on a vector space X over the complex field. Then there exists a (unique) bilinear form B on X such that $Q(x) = B(x, x)$ for all $x \in X$.*

Proof. Set $\varphi(x, y) = Q(x + y) - Q(x - y)$ and let us prove that φ is additive in the first variable. Using the relation 1° four times, we obtain

$$\begin{aligned}\varphi(x_1 + x_2, y) &= Q(x_1 + x_2 + y) - Q(x_1 + x_2 - y) = 2Q(x_1 + y) + 2Q(x_2) - \\ &\quad - Q(x_1 + y - x_2) - Q(x_1 + x_2 - y) = \\ &= Q(x_1 + y) + Q(x_1 + y) + 2Q(x_2) - (Q(x_1 + y - x_2) + Q(x_1 + x_2 - y)) = \\ &= Q(x_1 + y) + (2Q(x_1) + 2Q(y) - Q(x_1 - y)) + 2Q(x_2) - (2Q(x_1) + \\ &\quad + 2Q(x_2 - y)) = \varphi(x_1, y) + (2Q(y) + 2Q(x_2)) - 2Q(x_2 - y) = \\ &= \varphi(x_1, y) + (Q(x_2 + y) + Q(x_2 - y)) - 2Q(x_2 - y) = \varphi(x_1, y) + \varphi(x_2, y)\,.\end{aligned}$$

We observe next that condition 1° alone implies $Q(-x) = Q(x)$ for all $x \in X$. Indeed, it suffices to write down and subtract the equating 1° for the pair x, 0 and 0, x. This implies $\varphi(y, x) = \varphi(x, y)$ for all x and y so that φ is additive in the second variable as well. Now set $B(x, y) = \frac{1}{4}(\varphi(x, y) + i\varphi(x, iy))$ so that $B(x, x) = Q(x)$. Since φ is additive in both variables, the function B is additive in both variables as well.

Now we use condition 2°. First of all, it follows that φ satisfies the relation $\varphi(\lambda x, y) = |\lambda|^2\, \varphi(x, y/\lambda)$ for all $\lambda \neq 0$.

Let us prove now that B satisfies the following relations

$$3°\ B(ix, y) = \quad iB(x, y)$$

$$4°\ B(x, iy) = -iB(x, y)$$

Indeed, $4B(ix, y) = \varphi(ix, y) + i\varphi(ix, iy) = \varphi(x, -iy) + i\varphi(x, y) = i(\varphi(x, y) - i\varphi(x, -iy) = i(\varphi(x, y) + i\varphi(x, iy)) = 4iB(x, y)$ which proves 3°. Furthermore, $4B(x, iy) = \varphi(x, iy) + i\varphi(x, -y) = \varphi(x, iy) - i\varphi(x, y) = -i(\varphi(x, y) + i\varphi(x, iy)) = -4iB(x, y)$.

With view to 3° and 4°, the proof will be complete if we show that

$$5°\ B(tx, y) = B(x, ty) = tB(x, y)\ \text{for real}\ t\,.$$

Let x and y be fixed elements of X. Define a complex-valued function f of a complex variable as follows $f(\lambda) = B(\lambda x, y) - B(x, \lambda y)$. Clearly f is additive. Also, it is easy to check the relation

$$6°\ f(\lambda) = -|\lambda|^2 f\left(\frac{1}{\lambda}\right) \quad \text{for} \quad \lambda \neq 0\,.$$

Indeed,

$$4f(\lambda) = 4(B(\lambda x, y) - B(x, \lambda y)) = \varphi(\lambda x, y) + i\varphi(\lambda x, iy) - (\varphi(x, \lambda y) + $$
$$+ i\varphi(x, i\lambda y)) =$$
$$= \varphi(\lambda x, y) + i\varphi(\lambda x, iy) - (\varphi(\lambda y, x) + i\varphi(i\lambda y, x)) =$$
$$= |\lambda|^2 \left[\varphi\left(x, \frac{y}{\lambda}\right) + i\varphi\left(x, i\frac{y}{\lambda}\right) - \left(\varphi\left(y, \frac{x}{\lambda}\right) + i\varphi\left(iy, \frac{x}{\lambda}\right)\right)\right] =$$
$$= |\lambda|^2 \left[4B\left(x, \frac{y}{\lambda}\right) - \left(\varphi\left(\frac{x}{\lambda}, y\right) + i\varphi\left(\frac{x}{\lambda}, iy\right)\right)\right] =$$
$$= |\lambda|^2 \left(4B\left(x, \frac{y}{\lambda}\right) - 4B\left(\frac{x}{\lambda}, y\right)\right) = -4|\lambda|^2 f\left(\frac{1}{\lambda}\right).$$

According to our lemma $f(\lambda) = f(i) \operatorname{Im} \lambda$. In particular, $f(t) = 0$ for real t so that $B(tx, y) = B(x, ty)$ for all real t. If $\lambda = it$, t real, we obtain

$$B(itx, y) - B(x, ity) = f(it) = tf(i) = t(B(ix, y) - B(x, iy));$$

using 3° and 4°, this yields $i(B(tx, y) + B(x, ty)) = 2itB(x, y)$ whence $2iB(tx, y) = 2itB(x, y)$ which proves 5° and completes the proof.

References

[1] *S. Kurepa:* The Cauchy functional equation and scalar product in vector spaces, Glasnik matematičko-fizički i astronomski *19* (1964), 23—35.

[2] *S. Kurepa:* Quadratic and sesquilinear functionals, Glasnik matematičko-fizički i astronomski *20* (1965), 79—92.

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