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Diagonals of separately continuous functions of n variables with values in strongly σ-metrizable spaces

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Abstract. We prove the result on Baire classification of mappings $f : X \times Y \to Z$ which are continuous with respect to the first variable and belongs to a Baire class with respect to the second one, where X is a PP-space, Y is a topological space and Z is a strongly σ-metrizable space with additional properties. We show that for any topological space X, special equiconnected space Z and a mapping $g : X \to Z$ of the $(n-1)$-th Baire class there exists a strongly separately continuous mapping $f : X^n \to Z$ with the diagonal g. For wide classes of spaces X and Z we prove that diagonals of separately continuous mappings $f : X^n \to Z$ are exactly the functions of the $(n-1)$-th Baire class. An example of equiconnected space Z and a Baire-one mapping $g : [0,1] \to Z$, which is not a diagonal of any separately continuous mapping $f : [0,1]^2 \to Z$, is constructed.

Keywords: diagonal of a mapping; separately continuous mapping; Baire-one mapping; equiconnected space; strongly σ-metrizable space

Classification: Primary 54C08, 54C05; Secondary 26B05

1. Introduction

Let $f : X^n \to Y$ be a mapping. Then the mapping $g : X \to Y$ defined by $g(x) = f(x, \dots, x)$ is called *a diagonal of f*.

Investigations of diagonals of separately continuous functions $f : X^n \to \mathbb{R}$ were started in classical works of R. Baire [1], H. Lebesgue [14], [15] and H. Hahn [6]. They showed that diagonals of separately continuous functions of n real variables are exactly the functions of the $(n-1)$-th Baire class. Baire classification of separately continuous functions and their analogs is intensively studied by many mathematicians (see [17], [21], [25], [16], [2],[3], [9]).

In [16] the problem on a construction of separately continuous functions of n variables with a given diagonal of the $(n-1)$-th Baire class was solved. It was proved in [18] that for any topological space X and a function $g : X \to \mathbb{R}$ of the $(n-1)$-th Baire class there exists a separately continuous function $f : X^n \to \mathbb{R}$ with the diagonal g. Further development of these investigations deals with the changing of the range space $\mathbb{R}$ by a more general space, in particular, by a metrizable space. Notice that conditions on spaces similar to the arcwise connectedness

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(i.e., the equiconnectedness) serve as a convenient tool in a construction of separately continuous mappings (see [10, 20]).

In the given paper we study mappings $f : X^n \to Z$ with values in a space Z from a wide class of spaces which contains metrizable equiconnected spaces and strict inductive limits of sequences of closed locally convex metrizable subspaces. We first generalize a result from [10] concerning mappings of two variables with values in a metrizable equiconnected space to the case of mappings of n variables with values in spaces from wider class. Namely, we prove a theorem on the existence of a separately continuous mapping $f : X^n \to Z$ with the given diagonal $g : X \to Z$ of the $(n-1)$-th Baire class in case X is a topological space and (Z, λ) is a strongly σ-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^{\infty}$ assigned with a mapping λ (Theorem 6). We also obtain a result on a Baire classification of separately continuous mappings and their analogs defined on a product of a PP-space and a topological space and with values in a strongly σ-metrizable space with some additional properties (Theorem 15). In order to prove this theorem we apply the technics of σ-discrete mappings introduced in [7] and developed in [5], [26]. For PP-spaces X using Theorem 15 we generalize Theorem 3.3 from [10] and get a characterization of diagonals of separately continuous mappings $f : X^n \to Z$ (Theorem 16). Finally, we give an example of an equiconnected space Z and a Baire-one mapping $g : [0, 1] \to Z$ which is not a diagonal of any separately continuous mapping $f : [0, 1]^2 \to Z$ (Proposition 18).

2. Preliminaries

Let X, Y be topological spaces and $C(X, Y) = B_0(X, Y)$ be the collection of all continuous mappings between X and Y. For $n \geq 1$ we say that a mapping $f : X \to Y$ belongs to *the n-th Baire class* if f is a pointwise limit of a sequence $(f_k)_{k=1}^{\infty}$ of mappings $f_k : X \to Y$ from the $(n-1)$-th Baire class. By $B_n(X, Y)$ we denote the collection of all mappings $f : X \to Y$ of the n-th Baire class.

For a mapping $f : X \times Y \to Z$ and a point $(x, y) \in X \times Y$ we write $f^x(y) = f_y(x) = f(x, y)$. By $CB_n(X \times Y, Z)$ we denote the collection of all mappings $f : X \times Y \to Z$ which are continuous with respect to the first variable and belongs to the n-th Baire class with respect to the second one. If $n = 0$, then we use the symbol $CC(X \times Y, Z)$ for the class of all separately continuous mappings. Now let $CC_0(X \times Y, Z) = CC(X \times Y, Z)$ and for $n \geq 1$ let $CC_n(X \times Y, Z)$ be the class of all mappings $f : X \times Y \to Z$ which are pointwise limits of a sequence of mappings from $CC_{n-1}(X \times Y, Z)$.

For a metric space X with a metric $|\cdot - \cdot|_X$, a set $\emptyset \neq A \subseteq X$ and a point $x_0 \in X$ we write $|x_0 - A|_X = \inf\{|x_0 - a|_X : a \in A\}$. If $\delta > 0$, then we put $B(A, \delta) = \{x \in X : |x - A|_X < \delta\}$ and $B[A, \delta] = \{x \in X : |x - A|_X \leq \delta\}$. If $A = \emptyset$, then $B(A, \delta) = B[A, \delta] = \emptyset$.

Let X be a set and $n \in \mathbb{N}$. We denote $\Delta_n = \{(x, \ldots, x) \in X^n : x \in X\}$.

Let X be a topological space and $\Delta = \Delta_2 = \{(x, x) : x \in X\}$. A set $A \subseteq X$ is called *equiconnected in X* if there exists a continuous mapping $\lambda : ((X \times X) \cup \Delta) \times [0, 1] \to X$ such that $\lambda(A \times A \times [0, 1]) \subseteq A$, $\lambda(x, y, 0) = \lambda(y, x, 1) = x$

for all $x, y \in A$ and $\lambda(x, x, t) = x$ for all $x \in X$ and $t \in [0,1]$. A space is *equiconnected* if it is equiconnected in itself. Notice that any topological vector space is equiconnected, where a mapping λ is defined by $\lambda(x, y, t) = (1-t)x + ty$. If (X, λ) is an equiconnected space, then we denote $\lambda_1 = \lambda$ and for every $n \geq 2$ we define a continuous function $\lambda_n : X^{n+1} \times [0,1]^n \to X$,

$$\lambda_n(x_1, \ldots, x_{n+1}, t_1, \ldots, t_n) = \lambda(x_1, \lambda_{n-1}(x_2, \ldots, x_{n+1}, t_2, \ldots, t_n), t_1). \tag{1}$$

A topological space X is called *strongly σ-metrizable* if there exists an increasing sequence $(X_n)_{n=1}^{\infty}$ of closed metrizable subspaces X_n of X such that $X = \bigcup_{n=1}^{\infty} X_n$ and for any convergent sequence $(x_n)_{n=1}^{\infty}$ in X there exists a number $m \in \mathbb{N}$ such that $\{x_n : n \in \mathbb{N}\} \subseteq X_m$; the sequence $(X_n)_{n=1}^{\infty}$ is called *a stratification of* X.

We say that a family $\mathcal{A} = (A_i : i \in I)$ of sets A_i *refines* a family $\mathcal{B} = (B_j : j \in J)$ of sets B_j and denote it by $\mathcal{A} \prec \mathcal{B}$ if for every $i \in I$ there exists $j \in J$ such that $A_i \subseteq B_j$. By $\cup\mathcal{A}$ we denote the set $\bigcup_{i\in I} A_i$.

The following notion was introduced in [23]. A space X is said to be *a PP-space* if there exists a sequence $((h_{n,i} : i \in I_n))_{n=1}^{\infty}$ of locally finite partitions of unity $(h_{n,i} : i \in I_n)$ on X and sequence $(\alpha_n)_{n=1}^{\infty}$ of families $\alpha_n = (x_{n,i} : i \in I_n)$ of points $x_{n,i} \in X$ such that for any $x \in X$ and a neighborhood U of x there exists $n_0 \in \mathbb{N}$ such that $x_{n,i} \in U$ if $n \geq n_0$ and $x \in \operatorname{supp} h_{n,i}$, where $\operatorname{supp} h = \{x \in X : h(x) \neq 0\}$. Notice that the notion of a PP-space is close to the notion of a quarter-stratifiable space introduced in [2]. In particular, Hausdorff PP-spaces are exactly metrically quarter-stratifiable spaces [19].

Let $\mathcal{A}$ be a family of functionally closed subsets of a topological space X. Define classes $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ as the following: $\mathcal{F}_0 = \mathcal{A}$, $\mathcal{G}_0 = \{X \setminus A : A \in \mathcal{A}\}$ and for all $1 \leq \alpha < \omega_1$ we put $\mathcal{F}_\alpha = \{\bigcap_{n=1}^{\infty} A_n : A_n \in \bigcup_{\beta<\alpha} \mathcal{G}_\beta,\ n = 1, 2, \ldots\}$, $\mathcal{G}_\alpha = \{\bigcup_{n=1}^{\infty} A_n : A_n \in \bigcup_{\beta<\alpha} \mathcal{F}_\beta,\ n = 1, 2, \ldots\}$. Element of families $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are called *sets of the functionally multiplicative class* α or *sets of the functionally additive class* α, respectively; elements of the family $\mathcal{F}_\alpha \cap \mathcal{G}_\alpha$ are called *functionally ambiguous sets of the class* α.

A family $\mathcal{A} = (A_i : i \in I)$ of subsets of a topological space X is called: *strongly functionally discrete* if there exists a discrete family $(U_i : i \in I)$ of functionally open subsets of X such that $\overline{A_i} \subseteq U_i$ for every $i \in I$; *σ-strongly functionally discrete* if there exists a sequence of strongly functionally discrete families $\mathcal{A}_n$ such that $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$; *a base* for a mapping $f : X \to Y$ if the preimage $f^{-1}(V)$ of any open set V in Y is a union of sets from $\mathcal{A}$. By $\Sigma^f_\alpha(X, Y)$ we denote the collection of all mappings between X and Y with σ-strongly functionally discrete bases which consist of functionally ambiguous sets of the class α in X.

3. A construction of functions with a given diagonal

A general construction of separately continuous mapping of two variables with a given diagonal can be found in [20]:

Theorem 1. *Let X be a topological space, Z be a Hausdorff space, (Z_1, λ) be an equiconnected subspace of Z, $g : X \to Z$, $(G_n)_{n=0}^{\infty}$ and $(F_n)_{n=0}^{\infty}$ be sequences* *of functionally open sets G_n and functionally closed sets F_n in X^2, let $(\varphi_n)_{n=1}^\infty$ be a sequence of separately continuous functions $\varphi_n : X^2 \to [0,1]$, $(g_n)_{n=1}^\infty$ be a sequence of continuous mappings $g_n : X \to Z_1$ satisfying the conditions*

1) $G_0 = F_0 = X^2$ *and* $\Delta = \{(x,x) : x \in X\} \subseteq G_{n+1} \subseteq F_n \subseteq G_n$ *for every* $n \in \mathbb{N}$;
2) $X^2 \setminus G_n \subseteq \varphi_n^{-1}(0)$ *and* $F_n \subseteq \varphi_n^{-1}(1)$ *for every* $n \in \mathbb{N}$;
3) $\lim_{n\to\infty} \lambda(g_n(x_n), g_{n+1}(x_n), t_n) = g(x)$ *for arbitrary* $x \in X$, *any sequence* $(x_n)_{n=1}^\infty$ *of points* $x_n \in X$ *with* $(x_n, x) \in F_{n-1}$ *for all* $n \in \mathbb{N}$, *and any sequence* $(t_n)_{n=1}^\infty$ *of points* $t_n \in [0,1]$.

Then the mapping $f : X^2 \to Z$,

$$f(x,y) = \begin{cases} \lambda(g_n(x), g_{n+1}(x), \varphi_n(x,y)), & (x,y) \in F_{n-1} \setminus F_n \\ g(x), & (x,y) \in E = \bigcap_{n=1}^\infty G_n \end{cases} \tag{2}$$

is separately continuous.

Let X be a strongly σ-metrizable space. A stratification $(X_n)_{n=1}^\infty$ of a space X is said to be *perfect* if for every $n \in \mathbb{N}$ there exists a continuous mapping $\pi_n : X \to X_n$ with $\pi_n(x) = x$ for every $x \in X_n$. A stratification $(X_n)_{n=1}^\infty$ of an equiconnected strongly σ-metrizable space X is *assigned with* λ if $\lambda(X_n \times X_n \times [0,1]) \subseteq X_n$ for every $n \in \mathbb{N}$. It follows from the Dieudonne-Schwartz Theorem (see [24, Proposition II.6.5]) that a strict inductive limit of a sequence of locally convex metrizable spaces X_n, such that X_n is closed in X_{n+1}, is strongly σ-metrizable space with the perfect stratification $(X_n)_{n=1}^\infty$ assigned with an equiconnected function $\lambda(x,y,t) = (1-t)x + ty$.

Proposition 2. *Let X be a topological space, (Z,λ) be a strongly σ-metrizable space with a perfect stratification $(Z_n)_{n=1}^\infty$ assigned with a mapping $\lambda, m \in \mathbb{N}$ and $f \in B_m(X,Z)$. Then there exists a sequence $(f_n)_{n=1}^\infty$ of mappings $f_n \in B_{m-1}(X, Z_n)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$.*

PROOF: It is sufficient to put $f_n = \pi_n \circ g_n$, where $(\pi_n)_{n=1}^\infty$ is a sequence of retractions $\pi_n : Z \to Z_n$ and $(g_n)_{n=1}^\infty$ is a sequence of mappings $g_n \in B_{m-1}(X,Z)$ which is pointwise convergent to f. □

Proposition 3. *Let X be a metrizable space, (Z,λ) be a strongly σ-metrizable equiconnected space with a perfect stratification $(Z_n)_{n=1}^\infty$ assigned with a mapping λ and $g \in B_1(X,Z)$. Then there exists a sequence $(g_n)_{n=1}^\infty$ of continuous mappings $g_n : X \to Z_n$ and a sequence $(W_n)_{n=1}^\infty$ of open sets $W_n \subseteq X^2$ such that*

1) $\Delta_2 \subseteq W_n$ *for every* $n \in \mathbb{N}$;
2) $\lim_{n\to\infty} g_n(x_n) = g(x)$ *for every* $x \in X$ *and for any sequence* $(x_n)_{n=1}^\infty$ *of points* $x_n \in X$ *such that* $(x_n, x) \in W_n$ *for all* $n \in \mathbb{N}$.

PROOF: Let $(h_n)_{n=1}^\infty$ be a sequence of continuous mappings $h_n : X \to Z$ which is pointwise convergent to g on X. For every $n \in \mathbb{N}$ we put $f_n = \pi_n \circ h_n$, where $(\pi_n)_{n=1}^\infty$ is a sequence of retractions $\pi_n : Z \to Z_n$. Clearly, $f_n \in C(X, Z_n)$.

Since Z is a strongly σ-metrizable space with the stratification $(Z_n)_{n=1}^{\infty}$, $f_n \to g$ pointwise on X.

For every $n \in \mathbb{N}$ we set

$$A_n = \{x \in X : f_k(x) \in Z_n \ \forall k \in \mathbb{N}\}.$$

Since every f_k is continuous and Z_n is closed in Z, A_n is closed in X for every n. Moreover, $X = \bigcup_{n=1}^{\infty} A_n$, since Z is strongly σ-metrizable.

We firstly construct a sequence $(g_n)_{n=1}^{\infty}$ of continuous mappings $g_n : X \to Z$ and an increasing sequence $(C_n)_{n=1}^{\infty}$ of closed sets $C_n \subseteq A_n$ such that $(g_n)_{n=1}^{\infty}$ pointwise converges to g on X, $X = \bigcup_{n=1}^{\infty} C_n$ and

$$(\forall n, k \in \mathbb{N})(\forall x \in C_k)(\exists U \in \mathcal{U}_x)|(g_n(U) \subseteq Z_k), \tag{3}$$

where by $\mathcal{U}_x$ we denote a system of all neighborhoods of x in X.

Let $n \in \mathbb{N}$. Define $A_0 = C_0 = \emptyset$, $F_{k,n} = A_k \setminus B\Big(A_{k-1}, \frac{1}{n}\Big)$ for every $k \in \{1, \ldots, n\}$ and $C_n = \bigcup_{k=1}^{n} F_{k,n}$. Observe that every set $F_{k,n}$ is closed, for every n the sets $F_{1,n}, \ldots, F_{n,n}$ are disjoint, every set C_n is closed, $C_n \subseteq A_n \cap C_{n+1}$ for every n and

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{k=1}^{\infty} \bigcup_{n=k}^{\infty} F_{k,n} = \bigcup_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_k \setminus B\Big(A_{k-1}, \frac{1}{n}\Big) = \bigcup_{k=1}^{\infty} A_k \setminus A_{k-1} = X.$$

For every $n \in \mathbb{N}$ we choose a family $(G_{k,n} : 1 \le k \le n)$ of open sets such that $F_{k,n} \subseteq G_{k,n}$ and sets $\overline{G}_{1,n}, \ldots, \overline{G}_{n,n}$ are mutually disjoint. Moreover, we take a family $(\varphi_{k,n} : 1 \le k \le n)$ of continuous mappings $\varphi_{k,n} : X \to [0,1]$ such that $\varphi_{k,n}(G_{k,n}) \subseteq \{0\}$ and $\varphi_{k,n}(G_{i,n}) \subseteq \{1\}$ for $i \neq k$. Let

$$g_n(x) = \lambda_{n-1}(\pi_1(f_n(x)), \ldots, \pi_n(f_n(x)), \varphi_1(x), \ldots, \varphi_{n-1}(x)).$$

Notice that every g_n is continuous and $g_n \in C(X, Z_n)$ since the stratification $(Z_k)_{k=1}^{\infty}$ is assigned with λ. Moreover, $g_n(G_{k,n}) = \pi_k(f_n(G_{k,n})) \subseteq Z_k$ for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, n-1\}$. Since $C_k = \bigcup_{i=1}^{k} F_{i,k} \subseteq \bigcup_{i=1}^{k} F_{i,n} \subseteq \bigcup_{i=1}^{k} G_{i,n}$ and $g_n(\bigcup_{i=1}^{k} G_{i,n}) \subseteq Z_k$ for every $1 \le k \le n$, $(g_n)_{n=1}^{\infty}$ satisfies (3).

Now we show that $g_n \to g$ pointwise on X. Let $x_0 \in X$. Choose $k_0, n_0 \in \mathbb{N}$ such that $x_0 \in A_{k_0} \setminus A_{k_0-1}$ and $x_0 \notin B(A_{k_0-1}, \frac{1}{n_0})$. For every $n \ge \max\{k_0, n_0\}$ we have $x_0 \in F_{k_0,n}$ and $g_n(x_0) = f_n(x_0)$. In particular, $\lim_{n\to\infty} g_n(x_0) = \lim_{n\to\infty} f_n(x_0) = g(x_0)$.

By the Hausdorff Theorem on extension of metric [4, 4.5.20(c)] we choose a metric $|\cdot - \cdot|_Z$ on Z such that the restriction of this metric on every space Z_n generates its topology. Fix $n \in \mathbb{N}$. According to (3) for every $x \in C_k \setminus C_{k-1}$ we find an open neighborhood $U_{n,x}$ of x in X such that

(a) $U_{n,x} \cap C_{k-1} = \emptyset$;
(b) $g_n(u) \in Z_k$ for every $u \in U_{n,x}$;
(c) $|g_n(u) - g_n(x)|_Z < \frac{1}{n}$ for every $u \in U_{n,x}$.

Set $W_n = \bigcup_{x\in X}(U_{n,x} \times U_{n,x})$. Clearly, $(W_n)_{n=1}^\infty$ satisfies the condition 1). We verify 2). Let $x \in C_k \setminus C_{k-1}$ and $(x_n)_{n=1}^\infty$ be a sequence of points $x_n \in X$ such that $(x_n, x) \in W_n$ for every $n \in \mathbb{N}$. We choose $u_n \in X$ such that $(x_n, x) \in U_{n,u_n} \times U_{n,u_n}$, i.e. $x, x_n \in U_{n,u_n}$ for every $n \in \mathbb{N}$. It follows from (a) that $u_n \in C_k$ and the condition (b) implies that $g_n(x_n) \in Z_k$. Moreover, by (c) we have $|g_n(x_n) - g_n(x)|_Z < \frac{2}{n}$. Hence, $\lim_{n\to\infty} |g_n(x_n) - g_n(x)|_Z = 0$. It remains to observe that the restriction of $|\cdot - \cdot|_Z$ on Z_k generates its topological structure. □

A schema of the proof of the following theorem was proposed by H. Hahn for functions of n real variables and was applied in [16, Theorem 3.24] for mappings $f : X^n \to \mathbb{R}$.

Theorem 4. *Let X be a metrizable space, (Z, λ) be a strongly σ-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^\infty$ assigned with λ, $n \in \mathbb{N}$ and $g \in B_{n-1}(X, Z)$. Then there exists a separately continuous mapping $f : X^n \to Z$ with the diagonal g.*

Proof: Let $|\cdot - \cdot|_X$ be a metric on X which generates its topological structure.

We will argue by the induction on n. Let $n = 2$. By Proposition 3 there exists a sequence $(g_n)_{n=1}^\infty$ of continuous mappings $g_n : X \to Z$ and a sequence $(W_n)_{n=1}^\infty$ of open sets $W_n \subseteq X^2$ which satisfy conditions 1) and 2) of Proposition 3. Now we choose sequences $(G_n)_{n=0}^\infty$ and $(F_n)_{n=0}^\infty$ of functionally open sets G_n and functionally closed sets F_n in X^2, and a sequence $(\varphi_n)_{n=1}^\infty$ of continuous functions $\varphi_n : X^2 \to [0, 1]$ which satisfy the first two conditions of Theorem 1 and $F_{n-1} \subseteq W_n \cap W_{n+1}$ for every $n \geq 2$. It remains to check the condition 3) of Theorem 1.

Let $x \in X$, $(x_n)_{n=1}^\infty$ be a sequence of points $x_n \in X$ such that $(x_n, x) \in F_{n-1}$ for every $n \in \mathbb{N}$ and $(t_n)_{n=1}^\infty$ be a sequence of points $t_n \in [0, 1]$. Denote $z_0 = g(x)$ and fix a neighborhood W_0 of z_0 in Z. Since λ is continuous and $\lambda(z_0, z_0, t) = z_0$ for every $t \in [0, 1]$, there exists a neighborhood W of z_0 such that $\lambda(z_1, z_2, t) \in W_0$ for any $z_1, z_2 \in W$ and $t \in [0, 1]$. By the condition 2) of Proposition 3 the equality $\lim_{n\to\infty} g_n(x_n) = \lim_{n\to\infty} g_{n+1}(x_n) = z_0$ holds. Hence, there exists $n_0 \in \mathbb{N}$ such that $g_n(x_n), g_{n+1}(x_n) \in W$ for every $n \geq n_0$. Therefore, $\lambda(g_n(x_n), g_{n+1}(x_n), t_n) \in W_0$ and $\lim_{n\to\infty} \lambda(g_n(x_n), g_{n+1}(x_n), t_n) = g(x)$. The theorem is proved for $n = 2$.

Now assume that $n \geq 3$ and suppose that the theorem is true for mappings of $(n-1)$ variables with diagonals of the $(n-2)$ – th Baire class. We will prove that the theorem is true for mappings of n variables with diagonals of the $(n-1)$ – th Baire class.

Take a sequence $(g_k)_{k=1}^\infty$ of mappings $g_k \in B_{n-2}(X, Z)$ such that $g_k \to g$ pointwise on X. By the inductive assumption for every $k \in \mathbb{N}$ there exists a separately continuous mapping $f_k : X^{n-1} \to Z$ with the diagonal g_k. We put $G_0 = F_0 = X^n$,

$$G_k = \left\{(x_1, \ldots, x_n) \in X^n : \max_{1\leq i,j\leq n} |x_i - x_j|_X < \frac{1}{k}\right\}$$

and

$$F_k = \Big\{(x_1,\dots,x_n)\in X^n : \max_{1\le i,j\le n} |x_i - x_j|_X \le \frac{1}{k+1}\Big\}.$$

Notice that every G_k is open, every F_k is closed,

$$F_k \subseteq G_k \subseteq \overline{G_k} \subseteq F_{k-1}$$

for every $k\in\mathbb{N}$ and $\bigcap_{k=0}^{\infty} F_k = \bigcap_{k=0}^{\infty} G_k = \Delta_n$. Moreover, we choose a sequence $(\varphi_k)_{k=1}^{\infty}$ of continuous mappings $\varphi_k : X^n \to [0,1]$ such that $X^n\setminus G_k \subseteq \varphi_k^{-1}(0)$ and $F_k \subseteq \varphi_k^{-1}(1)$ for every $k\in\mathbb{N}$.

Fix $i\in\{1,\dots,n\}$. For any $x=(x_1,...,x_n)\in X^n$ we put

$$\tilde{x}_i = (x_1,...,x_{i-1},x_{i+1},...,x_n).$$

Denote

$$D_i = \{x\in X^n : \tilde{x}_i \in \Delta_{n-1}\}.$$

Notice that a function $\psi_i : X^n\setminus\Delta_n \to [0,1]$ defined by

$$\psi_i(x_1,\dots x_n) = \frac{\max\{|x_j - x_k|_X : 1\le j<k\le n,\ j,k\ne i\}}{\max\{|x_j-x_k|_X : 1\le j<k\le n\}}$$

is continuous, $\psi_i(x)=0$ if $x\in D_i\setminus\Delta_n$ and $\psi_i(x)=1$ if $x\in D_j\setminus\Delta_n$ for $j\ne i$.

Consider a mapping $h_i : X^n\to Z$,

$$h_i(x) = \begin{cases} \lambda(f_k(\tilde{x}_i), f_{k+1}(\tilde{x}_i), \varphi_k(x)), & x\in F_{k-1}\setminus F_k \\ g(u), & x=(u,\dots,u)\in\Delta_n. \end{cases} \tag{4}$$

It is easy to see that

$$h_i(x) = \lambda(\lambda(f_k(\tilde{x}_i), f_{k+1}(\tilde{x}_i), \varphi_k(x)), f_{k+2}(\tilde{x}_i), \varphi_{k+1}(x)) \tag{5}$$

for all $k\in\mathbb{N}$ and $x\in F_{k-1}\setminus F_{k+1}$.

Since the mappings λ, φ_k and φ_{k+1} are continuous and the mappings f_k, f_{k+1} and f_{k+2} are separately continuous, we get that h_i is separately continuous on the open set $G_k\setminus F_{k+1}$ for every $k\in\mathbb{N}$. Moreover, h_i is separately continuous on the open set $G_0\setminus F_1 = F_0\setminus F_1$. Then h_i is separately continuous on the open set $X^n\setminus\Delta_n = \bigcup_{k=1}^{\infty}(G_{k-1}\setminus F_k)$.

We show that the mapping h_i is continuous with respect to the i–th variable at every point of the set Δ_n. Let $u\in X$, $x=(u,\dots,u)\in\Delta_n$, $z_0=h_i(x)=g(u)$ and W_0 be a neighborhood of z_0 in Z. Since λ is continuous and $\lambda(z_0,z_0,t)=z_0$ for every $t\in[0,1]$, there exists a neighborhood W of z_0 such that $\lambda(z_1,z_2,t)\in W_0$ for any $z_1,z_2\in W$ and $t\in[0,1]$. Taking into consideration that $\lim_{k\to\infty} g_k(u) = g(u) = z_0$ we obtain that there exists a number k_0 such that $g_k(u)\in W$ for every $k\ge k_0$. Now we take any $v\in X$ such that $v\ne u$, $y=(x_1,\dots,x_n)\in F_{k_0-1}$, where $x_j=u$ for $j\ne i$ and $x_i=v$. Choose $k\ge k_0$ with $y\in F_{k-1}\setminus F_k$. Then

$$h_i(y) = \lambda(f_k(\tilde{y}_i), f_{k+1}(\tilde{y}_i), \varphi_k(y)) = \lambda(g_k(u), g_{k+1}(u), \varphi_k(y)) \in W_0.$$

Consider a mapping $f : X^n \to Z$,

(6)
$$f(x) = \begin{cases} \lambda_{n-1}(h_1(x), \dots, h_n(x), \psi_1(x), \dots, \psi_{n-1}(x)), & x \in X^n \setminus \Delta_n \\ g(u), & x = (u, \dots, u) \in \Delta_n. \end{cases}$$

Since the mappings $h_1, \dots, h_n$ are separately continuous and the mappings λ_{n-1}, $\psi_1, \dots, \psi_{n-1}$ are continuous, the mapping f is separately continuous on the set $X^n \setminus \Delta_n$. It remains to prove that f is continuous with respect to every variable x_i at each point of Δ_n.

Fix $i \in \{1, \dots, n\}$ and take any $x \in D_i \setminus \Delta_n$. Since $\psi_i(x) = 0$ and $\psi_j(x) = 1$ for $j \neq i$, properties (i) and (ii) of the function λ and the definition (1) of the functions λ_k imply the equality

$$f(x) = \lambda_{n-1}(h_1(x), \dots, h_n(x), \psi_1(x), \dots, \psi_{n-1}(x)) = h_i(x).$$

Hence, $f|_{D_i} = h_i|_{D_i}$. Therefore, the continuity of f with respect to the i–th variable at every point of Δ_n follows from the similar property of the mapping h_i. □

Theorem 5. *Let X be a metrizable space, (Z, λ) be a strongly σ-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^{\infty}$ assigned with λ, $n \in \mathbb{N}$ and $g \in B_n(X, Z)$. Then there exists a mapping $f \in CB_{n-1}(X \times X, Z) \cap CC_{n-1}(X \times X, Z)$ with the diagonal g.*

Proof: For a multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ we denote $|\alpha| = \alpha_1 + \cdots + \alpha_m$. For $n = 1$ the theorem is a particular case of Theorem 4.

Assume $n \geq 2$. Inductively for $m = 1, \dots, n-1$ we choose families $(g_\alpha : \alpha \in \mathbb{N}^m)$ of mappings $g_\alpha \in B_{n-m}(X, Z)$ such that

$$g_\alpha(x) = \lim_{k\to\infty} g_{\alpha,k}(x) \tag{7}$$

for all $x \in X$, $0 \leq m \leq n-2$ and $\alpha \in \mathbb{N}^m$. Notice that according to [16, Lemma 3.27] these families can be chosen such that

$$g_\alpha = g_\beta, \tag{8}$$

if $\alpha = (\alpha_1, \dots, \alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in \mathbb{N}^m$ and $\beta = (\alpha_1, \dots, \alpha_{m-2}, \alpha_m, \alpha_{m-1})$.

For every $\alpha \in \mathbb{N}^{n-1}$ by Proposition 3 we take sequences $(\tilde{g}_{\alpha,k})_{k=1}^{\infty}$ of continuous mappings $\tilde{g}_{\alpha,k} : X \to Z_k$ and $(W_{\alpha,k})_{k=1}^{\infty}$ of open neighborhoods of the diagonal Δ_2 which satisfy the condition 2) of Proposition 3 which we will denote by (2_α). For every $\alpha = (\alpha_1, \dots, \alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in \mathbb{N}^m$ we put $g_\alpha = \tilde{g}_\alpha$ if $\alpha_m \geq \alpha_{m-1}$, and $g_\alpha = \tilde{g}_\beta$, where $\beta = (\alpha_1, \dots, \alpha_{m-2}, \alpha_m, \alpha_{m-1})$ if $\alpha_m < \alpha_{m-1}$. Notice that the family $(g_\alpha : \alpha \in \mathbb{N}^n)$ satisfies (8), and the sequences $(g_{\alpha,k})_{k=1}^{\infty}$ satisfy (2_α). Moreover, $g_\alpha(X) \subseteq Z_k$, where $k = \max\{\alpha_{m-1}, \alpha_m\}$ for $\alpha = (\alpha_1, \dots, \alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in \mathbb{N}^m$.

Let $|\cdot - \cdot|_X$ be a metric on X which generates its topological structure.

For every $\alpha \in \mathbb{N}^n$ we choose a closed neighborhood $V_\alpha \subseteq W_\alpha$ of Δ_2. Put $G_0 = F_0 = X^2$. Inductively for $k \in \mathbb{N}$ we put

$$G_k = \{(x,y) \in X^2 : |x-y|_X < \frac{1}{k}\} \cap \operatorname{int}(F_{k-1}) \cap \bigcap_{\alpha \in \mathbb{N}^n,\, |\alpha| \le 2k} \{(x,y) : (y,x) \in W_\alpha\}$$

and choose a closed neighborhood F_k of $\triangle$ in X^2 such that

$$F_k \subseteq \{(x,y) \in X^2 : |x-y|_X \le \frac{1}{k+1}\} \cap \bigcap_{\alpha \in \mathbb{N}^n,\, |\alpha| \le 2k} \{(x,y) : (y,x) \in V_\alpha\} \cap G_k.$$

Every set G_k is open and

$$F_k \subseteq G_k \subseteq \overline{G_k} \subseteq F_{k-1}$$

for every $k \in \mathbb{N}$ and $\bigcap_{k=0}^\infty F_k = \bigcap_{k=0}^\infty G_k = \Delta_2$. Similarly as in the proof of Theorem 4 we choose a sequence $(\varphi_k)_{k=1}^\infty$ of continuous functions $\varphi_k : X^2 \to [0,1]$ such that $X^2 \setminus G_k \subseteq \varphi_k^{-1}(0)$ and $F_k \subseteq \varphi_k^{-1}(1)$ for every $k \in \mathbb{N}$.

For any $m \in \{0, 1, \dots n-1\}$ and $\alpha \in \mathbb{N}^m$ we consider a mapping $f_\alpha : X^2 \to Z$,

$$(9) \qquad f_\alpha(x,y) = \begin{cases} \lambda(g_{\alpha,k}(y), g_{\alpha,k+1}(y), \varphi_k(x,y)), & (x,y) \in F_{k-1} \setminus F_k \\ g_\alpha(x), & (x,y) \in \Delta_2. \end{cases}$$

In the same manner as in the proof of the continuity of h_i with respect to the i–th variable in Theorem 4, by condition (7) and by the continuity of λ and φ_k, we obtain that every f_α is continuous with respect to the first variable. For $\alpha \in \mathbb{N}^{n-1}$ we observe that every f_α is continuous with respect to the second variable on the set $X^2 \setminus \Delta_2$, since $g_{\alpha,k}$ is continuous with respect to the second variable.

Let $0 \le m \le n-2$, $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ and $l \in \mathbb{N}$. It follows from (8) that

$$f_{\alpha,l}(x,y) = \begin{cases} \lambda(g_{\alpha,k,l}(y), g_{\alpha,k+1,l}(y), \varphi_k(x,y)), & (x,y) \in F_{k-1} \setminus F_k \\ g_{\alpha,l}(x), & (x,y) \in \Delta_2. \end{cases}$$

Letting $l \to \infty$, applying continuity of λ and conditions (7), (9), we get

$$f_\alpha(x,y) = \lim_{l \to \infty} f_{\alpha,l}(x,y).$$

It remains to check that the mappings f_α, $\alpha \in \mathbb{N}^{n-1}$, are continuous with respect to the second variable on the set Δ_2. Fix $\alpha \in \mathbb{N}^{n-1}$ and $x \in X$. Let $z_0 = g_\alpha(x)$ and W_0 be a neighborhood of z_0 in Z. Since $\lambda(z_0, z_0, t) = z_0$ for every $t \in [0,1]$ and the mapping λ is continuous, there exists a neighborhood W of z_0 such that $\lambda(z_1, z_2, t) \in W_0$ for any $z_1, z_2 \in W$ and $t \in [0,1]$. We show that there exists $k_0 \in \mathbb{N}$ such that $\lambda(g_{\alpha,k}(y), g_{\alpha,k+1}(y), \varphi_k(x,y)) \in W_0$ for all $y \in X$ with $(x,y) \in F_{k-1} \setminus F_k$ for $k \ge k_0$. It is sufficient to prove that $g_{\alpha,k}(y), g_{\alpha,k+1}(y) \in W$ for all $y \in X$ with $(x,y) \in F_{k-1} \setminus F_k$ for $k \ge k_0$.

Assume the contrary. Then there exists a strictly increasing sequence $(k_i)_{i=1}^\infty$ of numbers k_i and a sequence $(y_i)_{i=1}^\infty$ of points $y_i \in X$ such that $(x, y_i) \in F_{k_i-1} \setminus F_{k_i}$,

$g_{\alpha,k_i}(y_i) \notin W$ or $g_{\alpha,k_i+1}(y_i) \notin W$ for all $i \in \mathbb{N}$. Let $g_{\alpha,k_i}(y_i) \notin W$ for all $i \in \mathbb{N}$. We choose $i_0 \in \mathbb{N}$ such that $|\alpha, k_i| \le 2(k_i - 1)$ for all $i \ge i_0$. Since $(x, y_i) \in F_{k_i-1}$, by the definition of F_{k_i-1} it follows that $(y_i, x) \in V_{\alpha,k_i} \subseteq W_{\alpha,k_i}$. Then by condition (2_α) we have $\lim_{i\to\infty} g_{\alpha,k_i}(y_i) = g_\alpha(x) = z_0$, which contradicts to the condition $g_{\alpha,k_i}(y_i) \notin W$ for all $i \in \mathbb{N}$. We apply this argument again when $g_{\alpha,k_i+1}(y_i) \notin W$ for all $i \in \mathbb{N}$.

Hence, f_α is continuous with respect to the second variable at the point (x, x), which completes the proof. □

The following theorem generalizes Corollary 3.2 from [10] and Theorem 3.28 from [16].

Theorem 6. *Let X be a topological space, (Z, λ) be a strongly σ-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^\infty$ assigned with λ, $n \in \mathbb{N}$ and $g \in B_n(X, Z)$. Then there exists a separately continuous mapping $f : X^{n+1} \to Z$ with the diagonal g and a mapping $\tilde{f} \in CB_{n-1}(X \times X, Z) \cap CC_{n-1}(X \times X, Z)$ with the diagonal g.*

Proof: Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $\alpha_{m+1} \in \mathbb{N}$. Then we will identify the multi-index $(\alpha_1, \ldots, \alpha_{m+1}) \in \mathbb{N}^{m+1}$ with the pair α, α_{m+1}. For $m = 0$ we suppose that $\mathbb{N}^0 = \{\emptyset\}$ and $h_\alpha = h$ for any mapping h and $\alpha \in \mathbb{N}^0$.

Successively for $m = 1, \ldots, n$ we choose families $(g_\alpha : \alpha \in \mathbb{N}^m)$ of mappings $g_\alpha \in B_{n-m}(X, Z)$ such that

$$g_\alpha(x) = \lim_{k\to\infty} g_{\alpha,k}(x) \tag{10}$$

for all $x \in X$, $0 \le i \le n-1$ and $\alpha \in \mathbb{N}^i$. According to Proposition 2 we may assume without loss of generality that $g_{\alpha,k} \in C(X, Z_k)$ for any $\alpha \in \mathbb{N}^{n-1}$ and $k \in \mathbb{N}$.

Consider a continuous mapping

$$\varphi = \mathop{\Delta}_{\alpha\in\mathbb{N}^n} g_\alpha : X \to Z^{\mathbb{N}^n},$$

$\varphi(x) = (g_\alpha(x))_{\alpha\in\mathbb{N}^n}$. Denote $Y = \varphi(X)$. Since $g_\alpha(X)$ is a metrizable subspace of Z for every $\alpha \in \mathbb{N}^n$, Y is metrizable. For every $\alpha \in \mathbb{N}^n$ we consider a continuous mapping $h_\alpha : Y \to Z$, $h_\alpha(y) = g_\alpha(x)$, where $y = \varphi(x)$, i.e.,

$$h_\alpha(\varphi(x)) = g_\alpha(x). \tag{11}$$

Passaging to the limit in the last equality and using (10) we obtain for $m = 1, \ldots, n$ families $(h_\alpha : \alpha \in \mathbb{N}^m)$ of mappings $h_\alpha \in B_{n-m}(Y, Z)$ such that

$$h_\alpha(y) = \lim_{k\to\infty} h_{\alpha,k}(y) \tag{12}$$

and

$$h_\alpha(\varphi(x)) = g_\alpha(x) \tag{13}$$

for all $x \in X$, $y \in Y$, $0 \leq i \leq n-1$ and $\alpha \in \mathbb{N}^i$.

In particular, $h \in B_n(Y,Z)$. By Theorem 4 there exists a separately continuous mapping $\tilde{h} : Y^{n+1} \to Z$ with the diagonal h. Now it remains to put $f(x_1,\dots x_{n+1}) = \tilde{h}(\varphi(x_1),\dots,\varphi(x_{n+1}))$.

The existence of $\tilde{f}$ can be proved similarly using Theorem 5. □

Corollary 7. *Let X be a topological space, (Z,λ) be a metrizable equiconnected space, $n \in \mathbb{N}$ and $g \in B_{n-1}(X,Z)$. Then there exists a separately continuous mapping $f : X^n \to Z$ with the diagonal g and a mapping $h \in CB_{n-1}(X \times X,Z) \cap CC_{n-1}(X \times X,Z)$ with the diagonal g.*

4. Baire classification of CB_n-mappings

Proposition 8. *Let X,Y be topological spaces and $(f_i)_{i\in I}$ be at most countable family of continuous mappings $f_i : X \to Y$ such that each space $f_i(X)$ is metrizable. Then there exists a metrizable space Z, a continuous surjective mapping $\varphi : X \to Z$ and a family $(g_i)_{i\in I}$ of continuous mappings $g_i : Z \to Y$ such that $f_i(x) = g_i(\varphi(x))$ for all $i \in I$ and $x \in X$.*

Proof: Consider a continuous mapping

$$\varphi = \mathop{\Delta}_{i\in I} f_i : X \to Y^I,$$

$\varphi(x) = (f_i(x))_{i\in I}$, and denote $Z = \varphi(X)$. Since each space $f_i(X)$ is metrizable, Z is metrizable. It remains to put $g_i(z) = z_i$, where $z = (z_j)_{j\in I} \in Z$. □

Proposition 9. *Let X be a topological space and Y be a metrizable space. Then*

$$B_n(X,Y) \subseteq \Sigma_n^f(X,Y)$$

for every $n \in \mathbb{N}$.

Proof: Consider a mapping $f \in B_n(X,Y)$ and let $(f_{k_1k_2\dots k_n} : k_1,k_2,\dots,k_n \in \mathbb{N})$ be a family of continuous mappings $f_{k_1k_2\dots k_n} : X \to Y$ such that

$$\lim_{k_1\to\infty}\lim_{k_2\to\infty}\dots\lim_{k_n\to\infty} f_{k_1k_2\dots k_n}(x) = f(x)$$

for every $x \in X$. According to Proposition 8 we choose a metrizable space Z, a continuous surjective mapping $\varphi : X \to Z$ and a family $(g_{k_1k_2\dots k_n} : k_1,k_2,\dots,k_n \in \mathbb{N})$ of continuous mappings $g_{k_1k_2\dots k_n} : Z \to Y$ such that

$$f_{k_1k_2\dots k_n}(x) = g_{k_1k_2\dots k_n}(\varphi(x))$$

for all $x \in X$ and $k_1,\dots k_n \in \mathbb{N}$. Now for every $z = \varphi(x) \in Z$ we put

$$g(z) = \lim_{k_1\to\infty}\lim_{k_2\to\infty}\dots\lim_{k_n\to\infty} g_{k_1k_2\dots k_n}(z)$$
$$= \lim_{k_1\to\infty}\lim_{k_2\to\infty}\dots\lim_{k_n\to\infty} f_{k_1k_2\dots k_n}(x) = f(x).$$

Hence, $g \in B_n(Z,Y)$. If follows from [8] that $g \in \Sigma_n^f(Z,Y)$. Since φ is continuous, $f \in \Sigma_n^f(X,Y)$. □

Proposition 10. *Let X be a PP-space, Y be a topological space, Z be a metrizable space and $n \in \mathbb{N} \cup \{0\}$. Then*

$$CB_n(X \times Y, Z) \subseteq \Sigma_{n+1}^f(X \times Y, Z).$$

Proof: Let $f \in CB_n(X \times Y, Z)$. Consider a homeomorphic embedding $\psi : Z \to \ell_\infty$ and denote $g = \psi \circ f$. Then $g \in CB_n(X \times Y, \psi(Z)) \subseteq B_{n+1}(X \times Y, \ell_\infty)$ by [22, Theorem 1]. Applying Proposition 9 we obtain that $g \in \Sigma_{n+1}^f(X \times Y, \psi(Z))$. Since $\psi : Z \to \psi(Z)$ is a homeomorphism, $f \in \Sigma_{n+1}^f(X \times Y, Z)$. □

Proposition 11. *Let X be a topological space, $(Y, |\cdot - \cdot|_Y)$ be a metric arcwise connected space, $f : X \to Y$ be a mapping, $(\mathcal{F}_k : 1 \le k \le n)$ be a family of strongly functionally discrete families $\mathcal{F}_k = (F_{i,k} : i \in I_k)$ of functionally closed sets $F_{i,k}$ in X such that $\mathcal{F}_{k+1} \prec \mathcal{F}_k$ and for every $i \in I_k$ and $x_1, x_2 \in F_{i,k}$ there exists a continuous mapping $\gamma : [0,1] \to Y$ with $\gamma(0) = f(x_1)$, $\gamma(1) = f(x_2)$ and $\operatorname{diam}(\gamma([0,1])) < \frac{1}{2^{k+2}}$ for every k. Then there exists a continuous mapping $g : X \to Y$ such that the inclusion $x \in \cup\mathcal{F}_k$ for $k = 1, \dots, n$ implies*

$$|f(x) - g(x)|_Y < \frac{1}{2^k}. \tag{14}$$

Proof: Take a discrete family $(U_{i,1} : i \in I_1)$ of functionally open sets in X such that $F_{i,1} \subseteq U_{i,1}$, $F_{i,1} = \varphi_{i,1}^{-1}(0)$ and $X \setminus U_{i,1} = \varphi_{i,1}^{-1}(1)$, where $\varphi_{i,1} : X \to [0,1]$ is a continuous function, and put $V_{i,1} = \varphi_{i,1}^{-1}([0, \frac{1}{2}))$ for every $i \in I_1$. Then $F_{i,1} \subseteq \overline{V_{i,1}} \subseteq U_{i,1}$. Now choose a discrete family $(G_{i,2} : i \in I_2)$ of functionally open sets such that $F_{i,2} \subseteq G_{i,2}$ for every $i \in I_2$. Since $\mathcal{F}_2 \prec \mathcal{F}_1$, for every $i \in I_2$ we fix a unique $j \in I_1$ such that $F_{i,2} \subseteq F_{j,1}$. Let $U_{i,2} = G_{i,2} \cap V_{j,1}$. Then $F_{i,2} = \varphi_{i,2}^{-1}(0)$ and $X \setminus U_{i,1} = \varphi_{i,2}^{-1}(1)$ for some continuous function $\varphi_{i,2} : X \to [0,1]$. Denote $V_{i,2} = \varphi_{i,2}^{-1}([0, \frac{1}{2}))$. Then $F_{i,2} \subseteq \overline{V_{i,2}} \subseteq U_{i,2} \subseteq V_{j,1}$. Proceeding analogously we get discrete families $(U_{i,k} : i \in I_k)$ and $(V_{i,k} : i \in I_k)$ of functionally open subsets of X for $k = 1, \dots, n$ such that for every $k = 1, \dots, n-1$ and $i \in I_{k+1}$ there is a unique $j = j_k(i) \in I_k$ with

$$F_{i,k+1} \subseteq \overline{V_{i,k+1}} \subseteq U_{i,k+1} \subseteq V_{j,k}. \tag{15}$$

For every k we put

$$U_k = \bigcup_{i \in I_k} U_{i,k}$$

and observe that the sets

$$H_k = \bigcup_{i \in I_k} \varphi_{i,k}^{-1}([0, \tfrac{1}{2}]) \quad \text{and} \quad E_k = X \setminus U_k$$

are disjoint and functionally closed in X. Take a continuous function $h_k : X \to [0,1]$ such that $H_k = h_k^{-1}(1)$ and $E_k = h_k^{-1}(0)$.

Fix arbitrary points $y_0 \in f(X)$ and $y_{i,k} \in f(F_{i,k})$ for every k and $i \in I_k$, and for all $x \in X$ put $g_0(x) = y_0$. Since Y is arcwise connected, for every $i \in I_1$ there exists a continuous function $\gamma_{i,1} : [0,1] \to Y$ such that $\gamma_{i,1}(0) = y_0$ and $\gamma_{i,1}(1) = y_{i,1}$. Now for every $1 < k \leq n$ and $i \in I_k$ there exists a continuous function $\gamma_{i,k} : [0,1] \to Y$ such that $\gamma_{i,k}(0) = y_{j,k-1}$, where $j \in I_{k-1}$ satisfies $F_{i,k} \subseteq F_{j,k-1}$, $\gamma_{i,k}(1) = y_{i,k}$ and

$$\text{diam}(\gamma_{i,k}([0,1])) < \frac{1}{2^{k+1}}. \tag{16}$$

Inductively for $k = 0, \dots n-1$ we define a continuous mapping $g_{k+1} : X \to Y$,

$$g_{k+1}(x) = \begin{cases} g_k(x), & x \in E_{k+1}, \\ \gamma_{i,k+1}(h_{k+1}(x)), & i \in I_{k+1}, x \in U_{i,k+1}. \end{cases}$$

Notice that $g_{k+1}(x) = y_{i,k+1}$ for all $x \in \overline{V_{i,k+1}}$ and $i \in I_{k+1}$.

We show that for all $x \in X$ the inequality

$$|g_{k+1}(x) - g_k(x)|_Y < \frac{1}{2^{k+2}} \tag{17}$$

holds for $k \geq 1$. Clearly, (17) is valid if $x \in E_{k+1}$. Let $x \in U_{i,k+1}$ for $i \in I_{k+1}$. Then $g_{k+1}(x) = \gamma_{i,k+1}(h_{k+1}(x))$ and $g_k(x) = y_{j,k} = \gamma_{i,k+1}(0)$, since $x \in V_{j,k}$ for $j = j_k(i) \in I_k$. Taking into account (16) we obtain (17).

We put $g = g_n$. Let $1 \leq k \leq n$ and $x \in \cup\mathcal{F}_k$. Then $x \in F_{i,k}$ for some $i \in I_k$. It follows that $g_k(x) = y_{i,k} \in f(F_{i,k})$. Then $|f(x) - g_k(x)|_Y \leq \frac{1}{2^{k+1}}$. The inequality (17) implies that

$$|f(x) - g(x)|_Y \leq |f(x) - g_k(x)|_Y + \sum_{i=k}^{n-1} |g_i(x) - g_{i+1}(x)|_Y < \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}.$$

□

The similar result to the following theorem was obtained also in [13, Theorem 4.1], but we include its proof for the sake of completeness.

Theorem 12. *Let X be a topological space, Y be a metrizable arcwise connected and locally arcwise connected space. Then $\Sigma_1^f(X,Y) \subseteq B_1(X,Y)$.*

Proof: Fix a metric $|\cdot - \cdot|_Y$ on Y which generates its topological structure. For every $k \in \mathbb{N}$ and $y \in Y$ we take an open neighborhood $U_k(y)$ of y such that any points from $U_k(y)$ can be joined with an arc of a diameter $< \frac{1}{2^{k+1}}$.

Let $f \in \Sigma_1^f(X,Y)$. It is easy to see that f has a σ-strongly functionally discrete base $\mathcal{B}$ which consists of functionally closed sets in X. For every $k \in \mathbb{N}$ we put

$$\mathcal{B}_k = (B \in \mathcal{B} : \exists y \in Y \mid B \subseteq f^{-1}(U_k(y))).$$

Then $\mathcal{B}_k$ is a σ-strongly functionally discrete family and $X = \cup\mathcal{B}_k$ for every k. According to [12, Lemma 13] for every $k \in \mathbb{N}$ there exists a sequence $(\mathcal{B}_{k,n})_{n=1}^{\infty}$ of strongly functionally discrete families $\mathcal{B}_{k,n} = (B_{k,n,i} : i \in I_{k,n})$ of functionally closed subsets of X such that $\mathcal{B}_{k,n} \prec \mathcal{B}_k$ and $\mathcal{B}_{k,n} \prec \mathcal{B}_{k,n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} \bigcup \mathcal{B}_{k,n} = X$. For all $k, n \in \mathbb{N}$ we put

$$\mathcal{F}_{k,n} = (B_{1,n,i_1} \cap \cdots \cap B_{k,n,i_k} : i_m \in I_{m,n}, 1 \le m \le k).$$

Notice that every family $\mathcal{F}_{k,n}$ is strongly functionally discrete, consists of functionally closed sets and

(a) $\mathcal{F}_{k+1,n} \prec \mathcal{F}_{k,n}$,
(b) $\mathcal{F}_{k,n} \prec \mathcal{F}_{k,n+1}$,
(c) $\bigcup_{n=1}^{\infty} \bigcup \mathcal{F}_{k,n} = X$.

For every $n \in \mathbb{N}$ we apply Proposition 11 to the function f and the families $\mathcal{F}_{1,n}$, $\mathcal{F}_{2,n}$,…,$\mathcal{F}_{n,n}$. We obtain a sequence of continuous mappings $g_n : X \to Y$ such that

$$|f(x) - g_n(x)|_Y < \frac{1}{2^k}$$

if $x \in \mathcal{F}_{k,n}$ for $k \le n$.

Now conditions (b) and (c) imply that $g_n \to f$ pointwise on X. Hence, $f \in B_1(X, Y)$. □

Let Z be a topological space and $(Z_k)_{k=1}^{\infty}$ be a sequence of sets $Z_k \subseteq Z$ such that $Z = \bigcup_{k=1}^{\infty} Z_k$. We say that the pair $(Z, (Z_k)_{k=1}^{\infty})$ *has the property* $(*)$ if for every convergent sequence $(x_m)_{m=1}^{\infty}$ in Z there exists a number k such that $\{x_m : m \in \mathbb{N}\} \subseteq Z_k$.

Proposition 13. *Let X be a PP-space, Y be a topological space, $n \in \mathbb{N} \cup \{0\}$, $(Z, (Z_k)_{k=1}^{\infty})$ have the property $(*)$, Z_k be functionally closed in Z and $f \in CB_n(X \times Y, Z)$. Then there exists a sequence $(B_k)_{k=1}^{\infty}$ of sets of the functionally multiplicative class n in $X \times Y$ such that $\bigcup_{k=1}^{\infty} B_k = X \times Y$ and $f(B_k) \subseteq Z_k$ for every $k \in \mathbb{N}$.*

Proof: Take a sequence $(\mathcal{U}_m = (U_{i,m} : i \in I_m))_{m=1}^{\infty}$ of locally finite functionally open coverings of X and a sequence $((x_{i,m} : i \in I_m))_{m=1}^{\infty}$ of families of points from X such that

$$(\forall x \in X)((\forall m \in \mathbb{N}\ \ x \in U_{i_m,m}) \Longrightarrow (x_{i_m,m} \to x)). \tag{18}$$

By [19, Corollary 3.1] there exists a weaker metrizable topology $\mathcal{T}$ on X in which every $U_{i,m}$ is open. Since $(X, \mathcal{T})$ is paracompact, for every m there exists a locally finite open covering $\mathcal{V}_m = (V_{s,m} : s \in S_m)$ which refines $\mathcal{U}_m$. It follows from [4, Theorem 1.5.18] that for every m there exists a locally finite closed covering $(F_{s,m} : s \in S_m)$ of $(X, \mathcal{T})$ such that $F_{s,m} \subseteq V_{s,m}$ for every $s \in S_m$. Now for every $s \in S_m$ we choose $i_m(s) \in I_m$ such that $F_{s,m} \subseteq U_{i_m(s),m}$.

For all $m,k\in\mathbb{N}$ and $s\in S_m$ we denote $i=i_m(s)$ and put

$$A_{s,m,k}=(f^{x_{i,m}})^{-1}(Z_k),\quad B_{m,k}=\bigcup_{s\in S_m}(F_{s,m}\times A_{s,m,k}),\quad B_k=\bigcap_{m=1}^{\infty}B_{m,k}.$$

Since f belongs to the n–th Baire class with respect to the second variable, for every k the set $A_{s,m,k}$ is of the functionally multiplicative class n in Y for all $m\in\mathbb{N}$ and $s\in S_m$. Then the set $B_{m,k}$ is of the functionally multiplicative class n in $(X,\mathcal{T})\times Y$ as a locally finite union of sets of the n–th functionally multiplicative class. Hence, B_k is of the n–th functionally multiplicative class in $(X,\mathcal{T})\times Y$, and, consequently, in $X\times Y$ for every k.

We show that $f(B_k)\subseteq Z_k$ for every k. Fix $k\in\mathbb{N}$ and $(x,y)\in B_k$. Take a sequence $(s_m)_{m=1}^{\infty}$ of indexes $s_m\in S_m$ such that $x\in F_{s_m,m}\subseteq U_{i_m(s_m),m}$ and $f(x_{i_m(s_m),m},y)\in Z_k$. Then $x_{i_m(s_m),m}\to_{m\to\infty}x$. Since f is continuous with respect to the first variable, $f(x_{i_m(s_m),m},y)\to_{m\to\infty}f(x,y)$. Since Z_k is closed, $f(x,y)\in Z_k$.

It remains to show that $\bigcup_{k=1}^{\infty}B_k=X\times Y$. Let $(x,y)\in X\times Y$. Then there exists a sequence $(s_m)_{m=1}^{\infty}$ such that $s_m\in S_m$ and $x\in F_{s_m,m}\subseteq U_{i_m(s_m),m}$. Notice that $f(x_{i_m(s_m),m},y)\to_{m\to\infty}f(x,y)$. Since $(Z,(Z_k)_{k=1}^{\infty})$ satisfies $(*)$, there exists a number k such that the set $\{f(x_{i_m(s_m),m},y):m\in\mathbb{N}\}$ is contained in Z_k, i.e. $y\in A_{s_m,m,k}$ for every $m\in\mathbb{N}$. Hence, $(x,y)\in B_k$. $\square$

The following result will be useful (see [11, Proposition 5.2]).

Proposition 14. *Let $0<\alpha<\omega_1$, X be a topological space, $Z=\bigcup_{k=1}^{\infty}Z_k$ be a contractible space, $f:X\to Z$ be a mapping, $(X_k)_{k=1}^{\infty}$ be a sequence of sets of the α-th functionally additive class in X such that $X=\bigcup_{k=1}^{\infty}X_k$, $f(X_k)\subseteq Z_k$ and assume that there exists a function $f_k\in B_\alpha(X,Z_k)$ with $f_k|_{X_k}=f|_{X_k}$ for every $k\in\mathbb{N}$. Then $f\in B_\alpha(X,Z)$.*

Theorem 15. *Let $n\in\mathbb{N}$, X be a PP-space, Y be a topological space and Z be a contractible space. Then*

$$CB_n(X\times Y,Z)\subseteq B_{n+1}(X\times Y,Z).$$

If, moreover, Z is a strongly σ-metrizable space with a perfect stratification $(Z_k)_{k=1}^{\infty}$, where every Z_k is an arcwise connected and locally arcwise connected subspace of Z, then

$$CC(X\times Y,Z)\subseteq B_1(X\times Y,Z).$$

Proof: By the definition of a PP-space we choose a sequence $((h_{n,i}:i\in I_n))_{n=1}^{\infty}$ of locally finite partitions of unity $(h_{n,i}:i\in I_n)$ on X and a sequence $(\alpha_n)_{n=1}^{\infty}$ of families $\alpha_n=(x_{n,i}:i\in I_n)$ of points $x_{n,i}\in X$ such that for any $x\in X$ the condition $x\in\operatorname{supp}h_{n,i}$ implies that $x_{n,i}\to x$. According to [19, Proposition 3.2] there exists a continuous pseudo-metric p on X such that each function $h_{n,i}$ is continuous with respect to p. Then the first inclusion $CB_n(X\times Y,Z)\subseteq B_{n+1}(X\times$

$Y, Z)$ in fact was proved in [2, Theorem 5.3], where X is a metrically quarter-stratifiable space (i.e., Hausdorff PP-space [19]). Another proof of this inclusion can be obtained analogously to the proof of Theorem 6.6 from [9].

Now we prove the second inclusion. Let $f \in CC(X\times Y, Z)$. For every $k \in \mathbb{N}$ we consider a retraction $\pi_k : Z \to Z_k$. Notice that every subspace Z_k is functionally closed in Z as the preimage of closed set under a continuous mapping $\varphi : Z \to \prod_{k=1}^{\infty} Z_k$, $\varphi(z) = (\pi_k(z))_{k=1}^{\infty}$. By Proposition 13 we take a sequence $(B_k)_{k=1}^{\infty}$ of functionally closed subsets of $X\times Y$ such that $\bigcup_{k=1}^{\infty} B_k = X\times Y$ and $f(B_k) \subseteq Z_k$ for every $k \in \mathbb{N}$. Observe that

$$f_k = \pi_k \circ f \in CC(X \times Y, Z_k) \subseteq \Sigma_1^f(X \times Y, Z_k)$$

by Proposition 10. According to Theorem 12, $f_k \in B_1(X \times Y, Z_k)$. Moreover, $f_k|_{B_k} = f|_{B_k}$. It remains to notice that every set B_k belongs to the first functionally additive class in $X \times Y$ and to apply Proposition 14. □

The following result generalizes Theorem 3.3 from [10] and gives a characterization of diagonals of separately continuous mappings.

Theorem 16. *Let X be a topological space, (Z, λ) be a strongly σ-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^{\infty}$ assigned with λ, $n \in \mathbb{N}$, $g : X \to Z$ and at least one of the following conditions holds:*

(1) *every separately continuous mapping $h : X^{n+1} \to Z$ belongs to the n–th Baire class;*
(2) *X is a PP-space (in particular, X is a metrizable space).*

Then the following conditions are equivalent:

(i) *$g \in B_n(X, Z)$;*
(ii) *there exists a separately continuous mapping $f : X^{n+1} \to Z$ with the diagonal g.*

Proof: In the case (1) the theorem is a corollary from Theorem 6.

In the case (2) the theorem follows from Theorem 15 and case (1). □

The following characterizations of diagonals of separately continuous mappings can be proved similarly.

Theorem 17. *Let X be a topological space, (Z, λ) be a strongly σ-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^{\infty}$ assigned with λ, $n \in \mathbb{N}$, $g : X \to Z$ and at least one of the following conditions holds:*

(1) *every separately continuous mapping $h : X^2 \to Z$ belongs to the first Baire class;*
(2) *X is a PP-space (in particular, X is a metrizable space).*

Then the following conditions are equivalent:

(i) *$g \in B_n(X, Z)$;*
(ii) *there exists a mapping $f \in CB_{n-1}(X \times X, Z) \cap CC_{n-1}(X \times X, Z)$ with the diagonal g.*

5. Examples and questions

For a topological space Y by $\mathcal{F}(Y)$ we denote the space of all nonempty closed subsets of Y with the Vietoris topology.

A multi-valued mapping $f : X \to \mathcal{F}(Y)$ is said to be *upper (lower) continuous at* $x_0 \in X$ if for any open set $V \subseteq Y$ with $f(x_0) \subseteq V$ ($f(x_0) \cap V \neq \emptyset$) there exists a neighborhood U of x_0 in X such that $f(x) \subseteq V$ ($f(x) \cap V \neq \emptyset$) for every $x \in U$. If a multi-valued mapping f is upper and lower continuous at x_0 simultaneously, then it is called *continuous at* x_0.

Proposition 18. *There exists an equiconnected space* (Z, λ) *with a metrizable equiconnected subspace* Z_1 *and a mapping* $g \in B_1([0,1], Z)$ *such that*

(1) *there exists a sequence* $(g_n)_{n=1}^{\infty}$ *of continuous mappings* $g_n : [0,1] \to Z_1$ *which is pointwise convergent to* g*;*
(2) g *is not a diagonal of any separately continuous mapping* $f : [0,1]^2 \to Z$*.*

Proof: Let $Y = [0,1] \times [0,1)$ and

$$Z = \{\{x\} \times [0,y] : x \in [0,1], y \in [0,1)\} \cup \{\{x\} \times [0,1) : x \in [0,1]\}$$

be a subspace of $\mathcal{F}(Y)$. Notice that $Z_1 = \{\{x\} \times [0,y] : x \in [0,1], y \in [0,1)\}$ is dense metrizable subspace of Z, since Z_1 consists of compacts subsets of a metrizable space Y.

We show that Z is equiconnected. Firstly we consider the space $Q = [0,1]^2$. For $q_1 = (x_1, y_1), q_2 = (x_2, y_2) \in Q$ we set

$$\theta(q_1, q_2) = \min\{y_1, y_2, 1 - |x_1 - x_2|\},$$

$\alpha_1(q_1, q_2) = y_1 - \theta(q_1, q_2)$, $\alpha_2(q_1, q_2) = |x_1 - x_2|$, $\alpha_3(q_1, q_2) = y_2 - \theta(q_1, q_2)$ and $\alpha(q_1, q_2) = \alpha_1(q_1, q_2) + \alpha_2(q_1, q_2) + \alpha_3(q_1, q_2)$. We denote $\theta = \theta(q_1, q_2)$, $\alpha_1 = \alpha_1(q_1, q_2)$, $\alpha_2 = \alpha_2(q_1, q_2)$, $\alpha_3 = \alpha_3(q_1, q_2)$, $\alpha = \alpha(q_1, q_2)$ and set

(19)
$$\mu(q_1, q_2, t) = \begin{cases} (x_1, y_1 - t\alpha), & q_1 \neq q_2, t \in [0, \frac{\alpha_1}{\alpha}]; \\ (x_1 + (t\alpha - \alpha_1)\mathrm{sign}(x_2 - x_1), \theta), & q_1 \neq q_2, t \in [\frac{\alpha_1}{\alpha}, \frac{\alpha_1 + \alpha_2}{\alpha}]; \\ (x_2, \theta + t\alpha - \alpha_1 - \alpha_2), & q_1 \neq q_2, t \in [\frac{\alpha_1 + \alpha_2}{\alpha}, \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha}]; \\ q_1, & q_1 = q_2, t \in [0,1]. \end{cases}$$

The function $\mu : Q^2 \times [0,1] \to Q$ is continuous and the space (Q, μ) is equiconnected.

Consider the continuous bijection $\varphi : Z \to Q$,

$$\varphi(z) = \begin{cases} (x, y), & z = x \times [0, y]; \\ (x, 1), & z = x \times [0, 1). \end{cases} \tag{20}$$

Note that the inverse mapping $\psi = \varphi^{-1} : Q \to Z$ is lower continuous on Q and continuous on $[0,1] \times [0,1)$. For every $z_1, z_2 \in Z$ and $t \in [0,1]$ we set

$$\lambda(z_1, z_2, t) = \psi\left(\mu(\varphi(z_1), \varphi(z_2), t)\right).$$

Obviously, the mapping $\lambda : Z^2 \times [0,1] \to Z$ is lower continuous and continuous at a point (z_1, z_2, t) if $\lambda(z_1, z_2, t) \in Z_1$.

We show that λ is upper continuous at a point (z_1, z_2, t) if $\lambda(z_1, z_2, t) \in Z \setminus Z_1$. Let $\lambda(z_1, z_2, t_0) \in Z \setminus Z_1$. Then $\lambda(z_1, z_2, t_0) = z_1$ or $\lambda(z_1, z_2, t_0) = z_2$. Suppose that $\lambda(z_1, z_2, t_0) = z_1 = x_1 \times [0,1)$ and $z_2 \subseteq x_2 \times [0,1)$. Fix a set G open in Y such that $z_1 \subseteq G$.

Let $x_1 \neq x_2$. Note that $t_0 = 0$. Choose a neighborhood U_1 of z_1, a neighborhood U_2 of z_2 and $\delta > 0$ such that $z \subseteq G$ for every $z \in U_1$ and

$$\frac{\alpha_1(\varphi(z'), \varphi(z''))}{\alpha(\varphi(z'), \varphi(z''))} \geq \delta$$

for every $z' \in U_1$ and $z'' \in U_2$. According to (19), $\lambda(z', z'', t) \subseteq G$ for every $z' \in U_1$, $z'' \in U_2$ and $t \in [0, \delta)$.

Now let $x_1 = x_2$. Choose a set G_0 open in Y such that $z_1 \subseteq G_0 \subseteq G$ and if $(x', y), (x'', y) \in G_0$ then $(\{x'\} \times [0, y]) \cup ([x', x''] \times \{y\}) \subseteq G_0$. It follows from (19) that $\lambda(z', z'', t) \subseteq G_0$ for every $z', z'' \subseteq G_0$ and $t \in [0,1]$.

In the case of $\lambda(z_1, z_2, t_0) = z_2 = x_2 \times [0,1)$ we argue analogously. Thus the mapping λ is continuous and, consequently, (Z, λ) is equiconnected. Moreover, $\lambda(Z_1 \times Z_1 \times [0,1]) \subseteq Z_1$. Hence, Z_1 is an equiconnected subspace of Z.

We define a mapping $g : [0,1] \to Z$,

$$g(x) = \{x\} \times [0,1)$$

and for every $n \in \mathbb{N}$ we consider a continuous mapping $g_n : [0,1] \to Z_1$,

$$g_n(x) = \{x\} \times \left[0, 1 - \frac{1}{n}\right].$$

It is easy to see that $\lim_{n\to\infty} g_n(x) = g(x)$ for every $x \in [0,1]$, i.e. the condition (1) of the proposition holds.

Now we verify (2). Assume to the contrary that there exists a separately continuous mapping $f : [0,1]^2 \to Z$ such that $f(x,x) = g(x)$ for every $x \in X$. Since f is separately upper continuous on the set $\Delta = \{(x,x) : x \in [0,1]\}$, for every $x \in [0,1]$ there exists $\delta_x \in (0,1)$ such that

$$(f(x,y) \cup f(y,x)) \cap ([0,1] \times [1 - \delta_x, 1)) \subseteq g(x)$$

for every $y \in [0,1]$ with $|x - y| < \delta_x$.

Take $\delta > 0$, an open nonempty set $U \subseteq [0,1]$ and a set A dense in U such that $\delta_x \geq \delta$ for every $x \in A$. Without loss of generality we may suppose that

$\operatorname{diam}(U) < \delta$. Then

$$f(x,y) \cap ([0,1] \times [1-\delta, 1)) \subseteq g(x) \cap g(y)$$

for any $x, y \in A$. Since $g(x) \cap g(y) = \emptyset$ for any distinct $x, y \in [0,1]$, $f(x,y) \subseteq [0,1] \times [0, 1-\delta]$ for any distinct $x, y \in A$. Since f is separately lower continuous and A is dense in U, $f(x,y) \subseteq [0,1] \times [0, 1-\delta]$ for any $x, y \in U$, which leads to a contradiction, provided g is a diagonal of f. □

Question 1. *Let Z be a topological vector space and $g \in B_1([0,1], Z)$. Does there exist a separately continuous mapping $f : [0,1]^2 \to Z$ with the diagonal g?*

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