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Applications of Mathematics, Vol. 68 (2023), No. 4, 425–439

Persistent URL: `http://dml.cz/dmlcz/151703`

A NEW APPROACH TO SOLVING A QUASILINEAR BOUNDARY VALUE PROBLEM WITH p-LAPLACIAN USING OPTIMIZATION

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Received August 21, 2022. Published online June 9, 2023.

Abstract. We present a novel approach to solving a specific type of quasilinear boundary value problem with p-Laplacian that can be considered an alternative to the classic approach based on the mountain pass theorem. We introduce a new way of proving the existence of nontrivial weak solutions. We show that the nontrivial solutions of the problem are related to critical points of a certain functional different from the energy functional, and some solutions correspond to its minimum. This idea is new even for $p = 2$. We present an algorithm based on the introduced theory and apply it to the given problem. The algorithm is illustrated by numerical experiments and compared with the classic approach.

Keywords: p-Laplacian operator; quasilinear elliptic PDE; critical point and value; optimization algorithm; gradient method

MSC 2020: 35J92, 65N30, 35B38

1. Introduction

In the paper, we focus on nontrivial weak solutions of a special case of Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u = u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \tag{2}$$

is the p-Laplace operator, $p \in (\frac{4}{3}, 4)$, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary.

The work was supported by the following grants: VŠB – Technical University of Ostrava (SGS SP2023/067, SP2023/011); Ministry of Education, Youth and Sports of the Czech Republic through the e-INFRA CZ (ID:90140); European Regional Development Fund at Research Centre of Advanced Mechatronic Systems (CZ.02.1.01/0.0/0.0/16019/0000867).

DOI: 10.21136/AM.2023.0194-22

For the choice $p = 2$, (1) describes a behaviour of the density of a gas sphere in hydrostatic equilibrium (see, for example, [12], [14]) for main sequence stars, including the Sun.

Another reason why this particular equation was chosen is the fact that the problem is very well known and serves as a model example for a particular type of nonlinearity. Problem (1) has been addressed by many authors, both for the semilinear variant (where $p = 2$ and $\Delta_p u = \Delta u$), see [2], [6], [7], [9], [10], [13], [4], [17], [15], [8] and the quasilinear variant (where $p \neq 2$), see [8], [18]. Therefore, there is an opportunity to verify and compare our results. At the same time, the topic is still interesting and provides plenty of room for further research.

The existence of nontrivial weak solutions of (1) has always been proven using the mountain pass theorem (see [1]) and its modifications that focus on nontrivial critical points of the type 'minimax'. This approach has also inspired all numerical algorithms for finding such solutions (see for example [6], [7], [10], [5], [13], [16], [3]).

It is known that the weak solutions of (1) correspond to critical points of a certain functional J (see below). In this paper, we show that they are also related to critical points of a different functional F introduced in the following section. This result is new not only for $p \neq 2$ but also for $p = 2$. We also show that some solutions of (1) correspond to a minimum of F. Consequently, we introduce an algorithm based on our new approach and apply it to the given problem (1).

The paper has the following organization. In Section 2, we formulate the analytical background and crucial theorems that represent the main results of this paper, including complete proofs of these theorems. In Section 3, we introduce an optimization algorithm based on the proposed approach. In Section 4, we apply the algorithm to the given problem and compare it to a mountain pass type algorithm.

2. Main results

In the paper, we are interested in finding nontrivial weak solutions of problem (1). Thus, let us first remind the definition.

Definition 1. A function $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1) if for all $v \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \,\mathrm{d}x = \int_\Omega u^3 v \,\mathrm{d}x.$$

It is known that weak solutions of the boundary value problem (1) correspond to critical points of the functional $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ given by

$$J(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{1}{4} \int_\Omega u^4 \,\mathrm{d}x. \tag{3}$$

We consider a Banach space $W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_p := \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{1/p}.$$

Let us define a functional $F \in C^1(W_0^{1,p}(\Omega) \setminus \{0\}, \mathbb{R})$ by

$$F(u) := \frac{\|u\|_p}{\|u\|_{L^4(\Omega)}}.$$

We can now formulate our main results.

Theorem 2.

(a) *Let $u \neq 0$ be a critical point of J. Then u is a critical point of F.*

(b) *Let u be a critical point of F. Then*

- ▷ *for all $c \in \mathbb{R}^+$, cu is a critical point of F,*
- ▷ *$\tilde{u} := \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x / \int_\Omega u^4 \,\mathrm{d}x\right)^{1/(4-p)} u$ is a critical point of J.*

Theorem 3. *There exists*

$$\min_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} F(v).$$

Corollary 4. *If*

$$F(u) = \min_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} F(v),$$

then $\tilde{u} := \tilde{c}u \neq 0$, where

$$\tilde{c} = \left(\frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega u^4 \,\mathrm{d}x}\right)^{1/(4-p)}$$

is a weak solution of (1).

Now, we can proceed to the proofs of the above theorems.

P r o o f of Theorem 2. (a) It is not difficult to compute that for $u \neq 0$

$$F'(u) = 0 \Leftrightarrow (\forall\, v \in W_0^{1,p}(\Omega)) : \left(\int_\Omega u^4 \,\mathrm{d}x\right)\left(\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x\right) = \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\left(\int_\Omega u^3 v \,\mathrm{d}x\right). \tag{4}$$

Since u is a critical point of J, for all $v \in W_0^{1,p}(\Omega) \setminus \{0\}$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x = \int_\Omega u^3 v \,\mathrm{d}x. \tag{5}$$

Therefore, for the choice of $v = u$,

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x = \int_\Omega u^4 \,\mathrm{d}x. \tag{6}$$

Hence, the proven statement simply follows from (4).

(b) First, we need to show that if $F'(u) = 0$, then also $F'(cu) = 0$ for all $c \in \mathbb{R}^+$. It can be done simply by substituting into (4).

Let us now take $\tilde{u} = \tilde{c}u$, where $F'(u) = 0$ and

$$\tilde{c} = \left(\frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega u^4 \,\mathrm{d}x}\right)^{1/(4-p)}.$$

We already know that $F'(\tilde{u}) = 0$. Hence, see (4),

$$\left(\int_\Omega \tilde{u}^4 \,\mathrm{d}x\right)\left(\int_\Omega |\nabla \tilde{u}|^{p-2}\nabla\tilde{u}\nabla v \,\mathrm{d}x\right) = \left(\int_\Omega |\nabla \tilde{u}|^p \,\mathrm{d}x\right)\left(\int_\Omega \tilde{u}^3 v \,\mathrm{d}x\right)$$

for all $v \in W_0^{1,p}(\Omega)$. We are left to show that

$$\int_\Omega \tilde{u}^4 \,\mathrm{d}x = \int_\Omega |\nabla \tilde{u}|^p \,\mathrm{d}x.$$

However, this is clear because

$$\begin{aligned}\int_\Omega \tilde{u}^4 \,\mathrm{d}x &= \left(\frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega u^4 \,\mathrm{d}x}\right)^{4/(4-p)} \int_\Omega u^4 \,\mathrm{d}x \\ &= \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{4/(4-p)} \left(\int_\Omega u^4 \,\mathrm{d}x\right)^{-p/(4-p)}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |\nabla \tilde{u}|^p \,\mathrm{d}x &= \left(\frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega u^4 \,\mathrm{d}x}\right)^{p/(4-p)} \int_\Omega |\nabla u|^p \,\mathrm{d}x \\ &= \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{4/(4-p)} \left(\int_\Omega u^4 \,\mathrm{d}x\right)^{-p/(4-p)}.\end{aligned}$$

$\square$

P r o o f of Theorem 3. Since

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega), \tag{7}$$

it holds that

$$0 < c := \sup_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|v\|_{L^4(\Omega)}}{\|v\|_p} \in \mathbb{R}.$$

If we manage to prove that there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ satisfying

$$\frac{\|u\|_{L^4(\Omega)}}{\|u\|_p} = c,$$

then clearly

$$\frac{1}{c} = F(u) = \min_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} F(v).$$

Let us consider a sequence $(u_n) \subset W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\frac{\|u_n\|_{L^4(\Omega)}}{\|u_n\|_p} \to c.$$

Then, for the sequence (v_n), $v_n := u_n/\|u_n\|_p$, we have

$$\|v_n\|_p = 1, \quad \|v_n\|_{L^4(\Omega)} \to c.$$

From the reflexivity of $W_0^{1,p}(\Omega)$ it follows that there exists a subsequence (v_n) (labelled in the same way) and $u \in W_0^{1,p}(\Omega)$ such that

$$\|v_n\|_p = 1, \quad v_n \rightharpoonup u$$

in $W_0^{1,p}(\Omega)$. The compact embedding (7) yields

$$v_n \to u \quad \text{in } L^4(\Omega), \quad \|v_n\|_{L^4(\Omega)} \to \|u\|_{L^4(\Omega)}.$$

However, the sequence (v_n) was chosen to satisfy

$$\|v_n\|_{L^4(\Omega)} \to c.$$

Hence,

$$\|u\|_{L^4(\Omega)} = c > 0.$$

Now all that remains is to use a weak lower semicontinuity of a norm and the definition of c. We get

$$c = \|u\|_{L^4(\Omega)} \leqslant c\|u\|_p \leqslant c \liminf \|v_n\|_p = c \lim \|v_n\|_p = c.$$

From that we obtain

$$\frac{\|u\|_{L^4(\Omega)}}{\|u\|_p} = c.$$

□

The corollary follows from part (b) of Theorem 2 and the fact that u satisfying

$$F(u) = \min_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} F(v)$$

must be a critical point of F.

Let us now mention one more interesting observation and formulate the following lemma.

Lemma 5. *Let $0 \neq u_1 \in W_0^{1,p}(\Omega)$ be a weak solution of* (1) *and*

$$F(u_1) = \min_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} F(u).$$

Then

$$0 < J(u_1) = \min\{J(u)\colon\ u \textit{ is a nontrivial weak solution of } (1)\}.$$

P r o o f. Let $0 \neq u_2 \in W_0^{1,p}(\Omega)$ be a weak solution of (1). From (6) it follows that

$$\begin{aligned}\Big(\int_\Omega |\nabla u_1|^p \,\mathrm{d}x\Big)^{(4-p)/(4p)} &= \frac{\big(\int_\Omega |\nabla u_1|^p \,\mathrm{d}x\big)^{1/p}}{\big(\int_\Omega |\nabla u_1|^p \,\mathrm{d}x\big)^{1/4}} = F(u_1) \leqslant F(u_2)\\ &= \frac{\big(\int_\Omega |\nabla u_2|^p \,\mathrm{d}x\big)^{1/p}}{\big(\int_\Omega |\nabla u_2|^p \,\mathrm{d}x\big)^{1/4}} = \Big(\int_\Omega |\nabla u_2|^p \,\mathrm{d}x\Big)^{(4-p)/(4p)}.\end{aligned}$$

Since $p < 4$, it also holds

$$\Big(\int_\Omega |\nabla u_1|^p \,\mathrm{d}x\Big) \leqslant \Big(\int_\Omega |\nabla u_2|^p \,\mathrm{d}x\Big).$$

Thus, considering (6) again, we can write

$$0 < J(u_1) = \frac{4-p}{4p}\int_\Omega |\nabla u_1|^p \,\mathrm{d}x \leqslant \frac{4-p}{4p}\int_\Omega |\nabla u_2|^p \,\mathrm{d}x = J(u_2),$$

which completes the proof. □

The introduced approach and main theorems can also be used for a generalized version of (1) with an exponent different from 3 and corresponding values of p. However, the main goal of this paper was to introduce an alternative to the method introduced in [2] that was tailored to problem (1).

3. Numerical solution of problem (1)

The numerical approach presented in this section is based on the introduced theory and the fact that F has a minimum. The minimum is sought using a classical gradient method, which forms the core of our algorithm. Since the algorithm actually returns a point with zero derivative, the result we get is a stationary point of F (that is not necessarily a minimum) corresponding to the solution of (1). The most difficult part is finding a descent direction and computing $\|F'(u)\|$, which will be discussed below. In this section, we present two novel algorithms.

Algorithm 1

1: Choose an arbitrary vector $w \in W_0^{1,p}(\Omega)$, $w \neq 0$, as an initial guess and take $u := w$.

2: If $\|F'(u)\| \leqslant \varepsilon$, where $\varepsilon > 0$ is the desired precision, return u and stop.

3: Find $v_0 \in W_0^{1,p}(\Omega)$ such that $\|v_0\|_p = 1$ and

$$\langle F'(u), v_0 \rangle < 0$$

(see below).

4: Set

$$\tilde{u} = u + \delta v_0,$$

where $0 < \delta = \min\{\hat{\delta}; \|u\|_p/2\}$, where $\hat{\delta} > 0$ is given apriori.

5: If the value of F is not lower, i.e.,

$$F(\tilde{u}) \geqslant F(u),$$

then take

$$v_0 := \frac{v_0}{2}$$

and return to step 4. Else, go to step 6.

6: Take $u := \tilde{u}$ and go to step 2.

Algorithm 2

1: Choose an arbitrary vector $w \in W_0^{1,p}(\Omega)$, $w \neq 0$ as an initial guess and set $u := w$.

2: If $\|F'(u)\| \leqslant \varepsilon$, where $\varepsilon > 0$ is the desired precision, return u and stop.

3: Find $v_0 \in W_0^{1,p}(\Omega)$ such that $\|v_0\|_p = 1$ and

$$\langle F'(u), v_0 \rangle < 0$$

(see below).

4: Find $\tilde{t} \in (0, \delta]$, $0 < \delta = \min\{\hat{\delta}; \|u\|_p/2\}$, where $\hat{\delta} > 0$ is given apriori, such that

$$F(u + \tilde{t} v_0) = \min_{t \in [0,\delta]} F(u + t v_0).$$

(The value $\tilde{t}$ can be computed using any method for one-dimensional optimization).

5: Set

$$u = u + \tilde{t} v_0$$

and go to step 2.

Now, focus on finding the descent direction v_0 in $u \neq 0$ and computing the dual norm

$$\|F'(u)\| := \sup_{v \in W_0^{1,p}(\Omega), \|v\|_p = 1} \langle F'(u), v \rangle.$$

For $u \in W_0^{1,p}(\Omega)$, $F'(u) \neq 0$, we take the descent direction v_0 in the form

$$v_0 = -\frac{b}{\|b\|_p},$$

where $b \in W_0^{1,p}(\Omega)$ is a weak solution of the boundary value problem

$$\begin{cases} -\Delta_p b = F'(u) & \text{in } \Omega, \\ b = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

It can be shown, see [2], that the weak solution of (8) is unique and v_0 is even the steepest descent direction in u, i.e.

$$\langle F'(u), v_0 \rangle = \min_{v \in W_0^{1,p}(\Omega), \|v\|_p = 1} \langle F'(u), v \rangle < 0,$$

and

$$\|F'(u)\| = \left\langle F'(u), \frac{b}{\|b\|_p} \right\rangle = \|b\|_p^{p-1}.$$

Problem (8) can be solved using an approach and a gradient method introduced in [11]. The authors of [11] also suggested some methods for preconditioning.

We use a standard finite element method with a finite-dimensional subspace V^h of $W_0^{1,p}(\Omega)$ containing piecewise linear and continuous functions (h is the discretization step).

It is difficult to say anything about the convergence of the introduced algorithms. The computation stops if the norm of the derivative is sufficiently small. The received solution does not necessarily have the critical value, which, in fact, is not even known. Moreover, problem (1) has infinitely many solutions. In the following section, two experiments resulting in multiple solutions were made and illustrated. Although there are no analytical results about the convergence, the numerical experiments presented in the following section suggest that the introduced algorithms converge to solutions of problem (1).

A crucial part of the algorithms is finding the descent direction as a solution of (8). As mentioned before, the problem has a unique solution that can be found using an algorithm presented in [11]. Furthermore, the authors of [11] also formulated convergence theorems for the preconditioned version of the algorithm.

4. Results

In this section, we demonstrate the algorithms proposed in Section 3. They return $u \in V^h$ and compute $\overline{u} = \tilde{c}u \in V^h$, which is an approximation of the weak solution of (1). For the numerical validation of the results, we compare $\overline{u}$ with $\underline{u} \in V^h$ that is an approximation of the weak solution u of the problem

$$
\begin{cases} -\Delta_p u = \overline{u}^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}
$$

(For this purpose, we can use a solver introduced in [11].) In other words, we compute

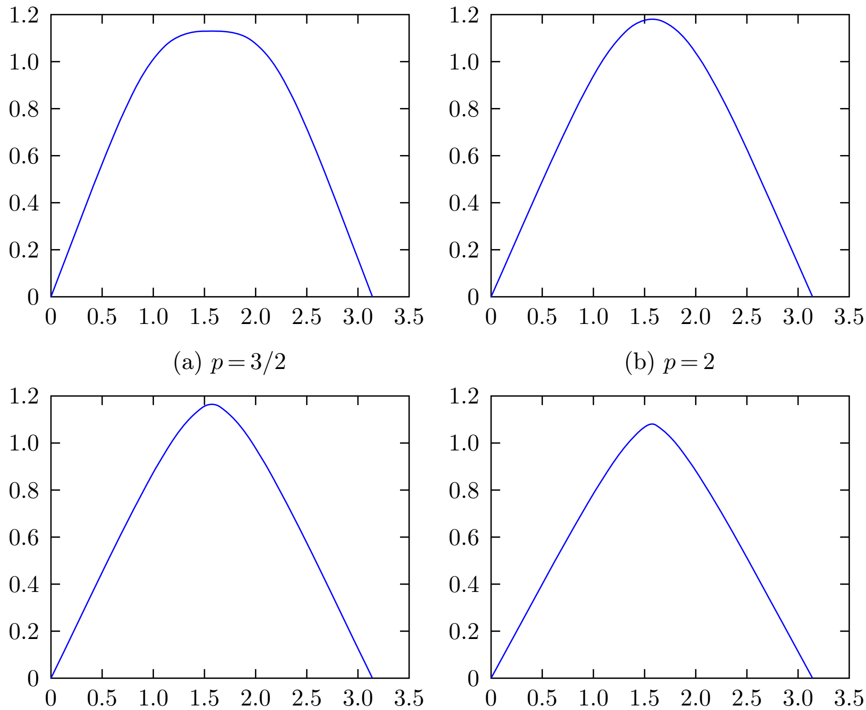
$$
\|\overline{u} - \underline{u}\|_p = \left(\int_\Omega |\nabla(\overline{u} - \underline{u})|^p \, \mathrm{d}x \right)^{1/p}. \tag{10}
$$

For the numerical experiments, we mainly use the first introduced algorithm (Algorithm 1) which requires less time to find the solution. On the other hand, Algorithm 2 can find solutions different from the minimum of F. First, focus on the one-dimensional version of the given problem. We choose $\overline{\Omega} = [0, \pi]$ and finite-element discretization with equidistant nodes, discretization step h, and continuous piecewise linear basis functions. We take $w = \sin(x)$ as the initial guess. For numerical computation, we consider a finite element interpolation of w.

In Table 1, we can see the results of Algorithm 1 with a fixed precision $\varepsilon = 10^{-5}$, various values of p, and three values of h. As we can see, the error is significantly reduced if h decreases. Fig. 1 illustrates solutions for some chosen values of p.

	$h = \pi/2^6$		$h = \pi/2^8$		$h = \pi/2^{10}$	
p	$J(\overline{u})$	$\Vert\overline{u} - \underline{u}\Vert_p$	$J(\overline{u})$	$\Vert\overline{u} - \underline{u}\Vert_p$	$J(\overline{u})$	$\Vert\overline{u} - \underline{u}\Vert_p$
9/6	1.38675	$8.31046 \cdot 10^{-4}$	1.38662	$6.66252 \cdot 10^{-5}$	1.38661	$1.87289 \cdot 10^{-5}$
10/6	1.30945	$6.62698 \cdot 10^{-4}$	1.30932	$3.31420 \cdot 10^{-5}$	1.30931	$1.40259 \cdot 10^{-5}$
11/6	1.24619	$5.59338 \cdot 10^{-4}$	1.24606	$3.39869 \cdot 10^{-5}$	1.24605	$5.35008 \cdot 10^{-6}$
12/6	1.19417	$4.91589 \cdot 10^{-4}$	1.19404	$2.96421 \cdot 10^{-5}$	1.19403	$3.15774 \cdot 10^{-6}$
13/6	1.15091	$4.40157 \cdot 10^{-4}$	1.15078	$2.73057 \cdot 10^{-5}$	1.15078	$6.15550 \cdot 10^{-6}$
14/6	1.11452	$3.99088 \cdot 10^{-4}$	1.11439	$2.96935 \cdot 10^{-5}$	1.11438	$5.79427 \cdot 10^{-6}$
15/6	1.08354	$3.68951 \cdot 10^{-4}$	1.08342	$2.23293 \cdot 10^{-5}$	1.08341	$1.74400 \cdot 10^{-5}$
16/6	1.05690	$3.38161 \cdot 10^{-4}$	1.05678	$3.25783 \cdot 10^{-5}$	1.05677	$2.50345 \cdot 10^{-5}$
17/6	1.03377	$3.20484 \cdot 10^{-4}$	1.03365	$3.99744 \cdot 10^{-5}$	1.03365	$2.58841 \cdot 10^{-5}$
18/6	1.01351	$2.93890 \cdot 10^{-4}$	1.01340	$4.61777 \cdot 10^{-5}$	1.01339	$4.14522 \cdot 10^{-5}$

Table 1. Algorithm 1: Results for $\overline{\Omega} = [0, \pi]$, $\varepsilon = 10^{-5}$, $w = \sin(x)$.



(a) $p = 3/2$

(b) $p = 2$

(c) $p = 5/2$

(d) $p = 3$

Figure 1. Algorithm 1: The solutions $\overline{u}$ of (1) for $\overline{\Omega} = [0, \pi]$, $\varepsilon = 10^{-5}$, $h = 2^{-9}\pi$, $w = \sin(x)$ and different values of p.

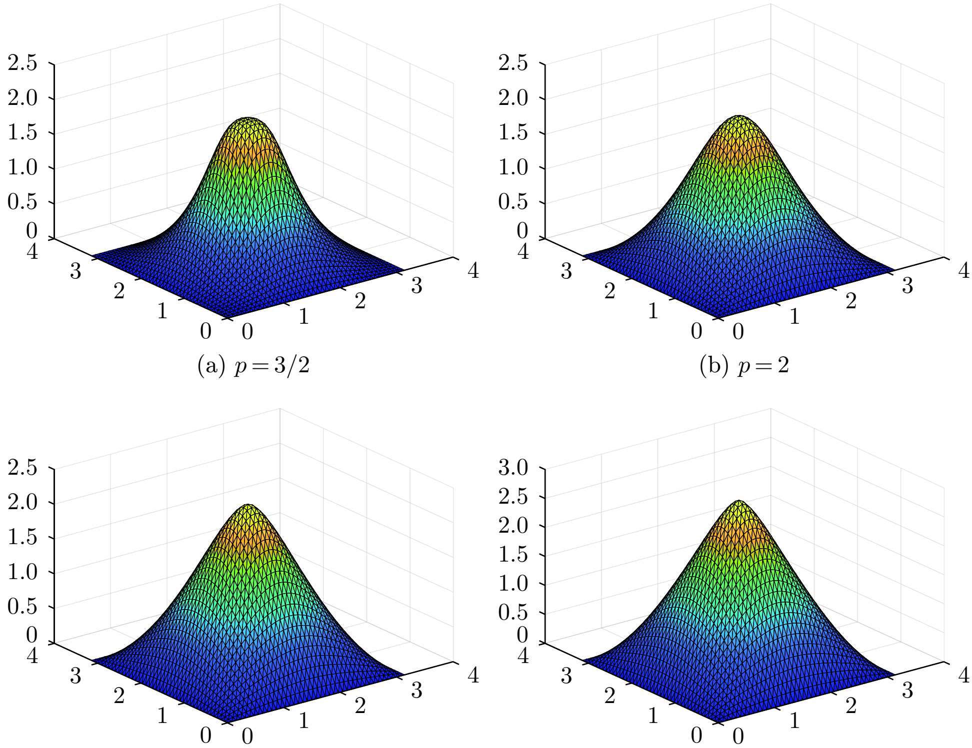
Consequently, we focus on problem (1) in two dimensions. We take $\overline{\Omega} = [0, \pi] \times [0, \pi]$ and finite-element discretization with equidistant nodes, right-angled triangles, discretization step h (corresponding to the length of the legs of the triangles), and continuous piecewise linear basis functions. As the initial guess, we use a finite element interpolation of $w = \sin(x)\sin(y)$.

First, we take a fixed precision $\varepsilon = 10^{-5}$ and various values of p. In Table 2, we can find the results (where $c = 1/F(u)$ represents an estimate of the embedding constant from (7) at the solution obtained by Algorithm 1). We can also see how the error (10) depends on the discretization step. As we can see, the error is reduced if h decreases to zero, and the values of J decrease. Fig. 2 illustrates solutions for some chosen values of p.

Now, we show, see Table 3, the performance of Algorithm 1 by comparing it with the minimax algorithm introduced in [2] (with the descent direction from [2], Subsection 4.2). Since finding the descent directions for both algorithms is computationaly expensive, especially if $p < 2$, we restrict ourselves on $p \in [11/6, 3]$.

$h = \pi/2^5$				
p	$J(\overline{u})$	$F(u)$	c	$\Vert\overline{u} - \underline{u}\Vert_p$
9/6	4.90223	2.79306	0.35803	$1.36291 \cdot 10^{-2}$
10/6	4.49311	2.44326	0.40929	$8.27395 \cdot 10^{-3}$
11/6	4.13619	2.18083	0.45854	$6.11904 \cdot 10^{-3}$
12/6	3.85099	1.98110	0.50477	$4.90581 \cdot 10^{-3}$
13/6	3.63882	1.82545	0.54781	$4.10995 \cdot 10^{-3}$
14/6	3.50226	1.70140	0.58775	$3.56354 \cdot 10^{-3}$
15/6	3.45145	1.60061	0.62476	$3.14602 \cdot 10^{-3}$
16/6	3.51145	1.51731	0.65906	$2.78719 \cdot 10^{-3}$
17/6	3.73839	1.44743	0.69088	$2.62947 \cdot 10^{-3}$
18/6	4.26398	1.38808	0.72042	$2.34993 \cdot 10^{-3}$
$h = \pi/2^6$				
9/6	4.88230	2.78839	0.35863	$3.25788 \cdot 10^{-3}$
10/6	4.47603	2.43998	0.40984	$2.02514 \cdot 10^{-3}$
11/6	4.11895	2.17817	0.45910	$1.52130 \cdot 10^{-3}$
12/6	3.83242	1.97875	0.50537	$1.23035 \cdot 10^{-3}$
13/6	3.61801	1.97875	0.54847	$1.02796 \cdot 10^{-3}$
14/6	3.47814	1.69932	0.58847	$9.02870 \cdot 10^{-4}$
15/6	3.42249	1.59859	0.62555	$8.00853 \cdot 10^{-4}$
16/6	3.47514	1.51534	0.65992	$7.28314 \cdot 10^{-4}$
17/6	3.69011	1.44550	0.69180	$6.95355 \cdot 10^{-4}$
18/6	4.19400	1.38617	0.72141	$6.87146 \cdot 10^{-4}$
$h = \pi/2^7$				
9/6	4.87724	2.78715	0.35879	$7.26888 \cdot 10^{-4}$
10/6	4.47174	2.43920	0.40997	$4.67036 \cdot 10^{-4}$
11/6	4.11463	2.17746	0.45925	$3.68437 \cdot 10^{-4}$
12/6	3.82778	1.97812	0.50553	$3.07873 \cdot 10^{-4}$
13/6	3.61281	1.82269	0.54864	$2.54766 \cdot 10^{-4}$
14/6	3.47211	1.69880	0.58865	$2.45503 \cdot 10^{-4}$
15/6	3.41525	1.59808	0.62575	$2.32033 \cdot 10^{-4}$
16/6	3.46606	1.51483	0.66014	$3.64296 \cdot 10^{-4}$
17/6	3.67803	1.44500	0.69204	$5.01201 \cdot 10^{-4}$
18/6	4.19400	1.38617	0.72141	$6.87146 \cdot 10^{-4}$

Table 2. Algorithm 1: Results for $\overline{\Omega} = [0, \pi] \times [0, \pi]$, $\varepsilon = 10^{-5}$, $w = \sin(x)\sin(y)$.



(a) $p=3/2$ (b) $p=2$

(c) $p=5/2$ (d) $p=3$

Figure 2. Algorithm 1: The solutions $\overline{u}$ of (1) for $\overline{\Omega}=[0,\pi]\times[0,\pi]$, $\varepsilon=10^{-5}$, $h=2^{-5}\pi$, $w=\sin(x)\sin(y)$ and different values of p.

	Algorithm 1			Minimax algorithm		
p	$J(\overline{u})$	$\\|\overline{u}-\underline{u}\\|_p$	it	$J(\overline{u})$	$\\|\overline{u}-\underline{u}\\|_p$	it
11/6	4.11895	$1.52130\cdot10^{-3}$	18	4.11895	$1.52047\cdot10^{-3}$	20
12/6	3.83242	$1.23036\cdot10^{-3}$	9	3.83243	$1.23044\cdot10^{-3}$	9
13/6	3.61802	$1.02796\cdot10^{-3}$	12	3.61802	$1.02793\cdot10^{-3}$	13
14/6	3.47814	$9.02870\cdot10^{-4}$	10	3.47811	$9.03389\cdot10^{-4}$	11
15/6	3.42250	$8.00854\cdot10^{-4}$	8	3.42250	$7.98002\cdot10^{-4}$	12
16/6	3.47514	$7.28314\cdot10^{-4}$	7	3.47514	$7.23060\cdot10^{-4}$	12
17/6	3.69012	$6.95355\cdot10^{-4}$	7	3.69012	$6.90178\cdot10^{-4}$	16
18/6	4.19400	$7.30576\cdot10^{-4}$	6	4.19400	$6.64144\cdot10^{-4}$	32

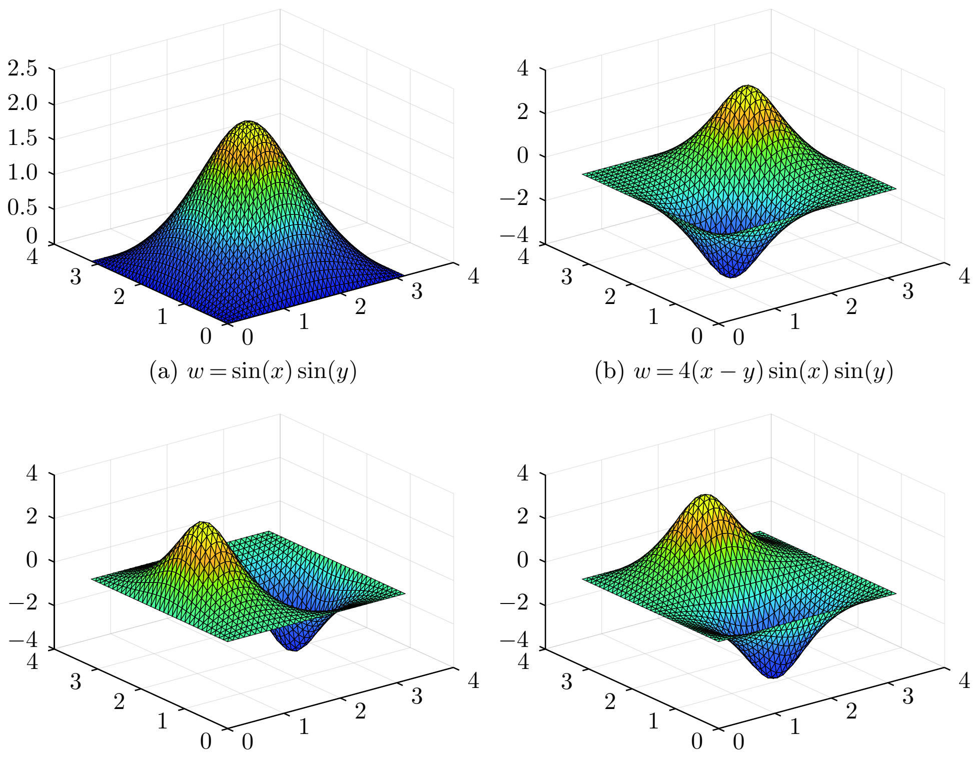
Table 3. Results for $\overline{\Omega}=[0,\pi]\times[0,\pi]$, $\varepsilon=10^{-5}$, $h=2^{-6}\pi$, $w=\sin(x)\sin(y)$.

As we can see in the table, both algorithms find the numerical solution with similar accuracy. The proposed algorithm needs fewer iterations to find the numerical solution, while the cost of one iteration is the same for both algorithms.

Finally, we focus on the other introduced algorithm (Algorithm 2). We take $p = 2$ and choose five different functions as the initial guess w. In Table 4, we can find results given by Algorithm 1 and Algorithm 2. As the table shows, for some w, Algorithm 2 found solutions of (1) that do not correspond to the minimum of F, while Algorithm 1 always returned the same solution. The results of Algorithm 2 are illustrated in Fig. 3.

	Algorithm 1			Algorithm 2		
Initial guess w	$F(u)$	$J(\overline{u})$	$\lVert\overline{u}-\underline{u}\rVert_p$	$F(u)$	$J(\overline{u})$	$\lVert\overline{u}-\underline{u}\rVert_p$
$\sin(x)\sin(y)$	1.97812	3.82778	$3.08072 \cdot 10^{-4}$	1.97812	3.82778	$3.07415 \cdot 10^{-4}$
$10\sin(2x)\sin(y)$	1.97812	3.82778	$3.09902 \cdot 10^{-4}$	3.05387	21.74424	$1.44633 \cdot 10^{-3}$
$10\sin(x)\sin(2y)$	1.97812	3.82778	$3.10213 \cdot 10^{-4}$	3.05387	21.74424	$1.44832 \cdot 10^{-3}$
$4(x-y)\sin(x)\sin(y)$	1.97812	3.82778	$3.08963 \cdot 10^{-4}$	2.98337	19.80472	$8.34581 \cdot 10^{-4}$

Table 4. Algorithm 1, Algorithm 2: Results for $\overline{\Omega} = [0,\pi]\times[0,\pi]$, $\varepsilon = 10^{-5}$, $p = 2$, $h = 2^{-7}\pi$ and different choices of w.



(a) $w = \sin(x)\sin(y)$

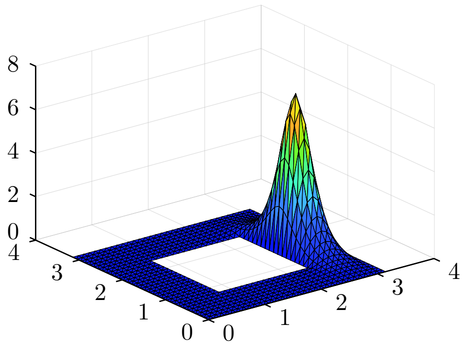
(b) $w = 4(x-y)\sin(x)\sin(y)$

(c) $w = 10\sin(2x)\sin(y)$

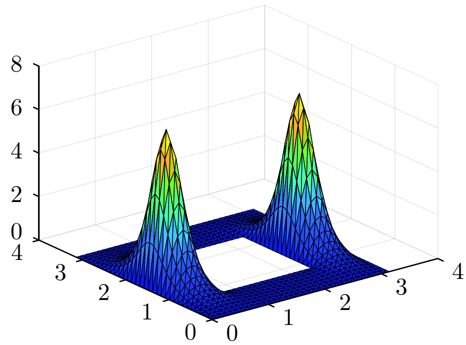
(d) $w = 10\sin(x)\sin(2y)$

Figure 3. Algorithm 2: The solutions $\overline{u}$ of (1) for $\overline{\Omega} = [0,\pi]\times[0,\pi]$, $\varepsilon = 10^{-4}$, $h = 2^{-5}\pi$, $p = 2$ and different choices of w.

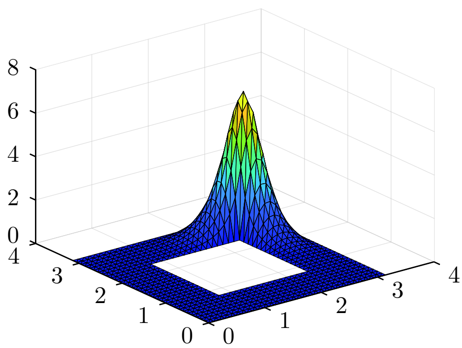
The last experiment focuses on finding multiple solutions on a nonconvex domain. Consider $\overline{\Omega} = ([0,\pi] \times [0,\pi]) \setminus ((\pi/4, 3\pi/4) \times (\pi/4, 3\pi/4))$, $h = 2^{-5}\pi$, $\varepsilon = 10^{-4}$ and four choices of the initial guess. The results are illustrated in Figure 4.



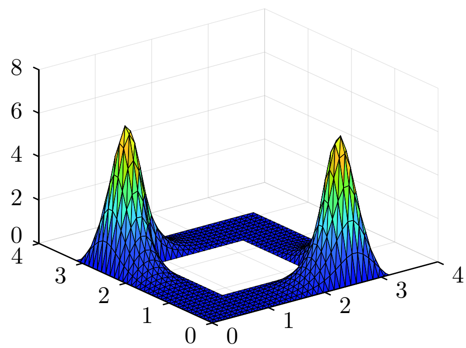
(a) $w = \sin(x)\sin(y)$, $J = 45.182134$, $F = 3.666535$



(b) $w = 4|\sin(x)\sin(2y)|$, $J = 90.364415$, $F = 4.360281$



(c) $w = 4(x-y)\sin(x)\sin(y)$, $J = 36.035313$, $F = 3.464955$



(d) $w = |4\sin(3x)\sin(3y)|$, $J = 71.220508$, $F = 4.108345$

Figure 4. Algorithm 2: The solutions $\overline{u}$ of (1) for $\overline{\Omega} = ([0,\pi] \times [0,\pi]) \setminus ((\pi/4, 3\pi/4) \times (\pi/4, 3\pi/4))$, $\varepsilon = 10^{-4}$, $h = 2^{-5}\pi$, $p = 2$ and different choices of w.

Again, the algorithm returned different solutions depending on the initial guess. The solution with the minimal value of F among the received ones is illustrated in Subfigure (c). One can clearly get another three solutions with the same value of F just by rotating solution (c). In [14], the authors took a domain with an annulus shape and received a solution with a similar quality as (c). Thanks to the shape of the domain, even infinite many solutions can arise by rotating the original solution.

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