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DING PROJECTIVE AND DING INJECTIVE MODULES OVER TRIVIAL RING EXTENSIONS

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Abstract. Let $R \ltimes M$ be a trivial extension of a ring R by an R-R-bimodule M such that M_R, ${}_RM$, $(R,0)_{R\ltimes M}$ and ${}_{R\ltimes M}(R,0)$ have finite flat dimensions. We prove that (X,α) is a Ding projective left $R \ltimes M$-module if and only if the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{\alpha} X$ is exact and $\operatorname{coker}(\alpha)$ is a Ding projective left R-module. Analogously, we explicitly describe Ding injective $R \ltimes M$-modules. As applications, we characterize Ding projective and Ding injective modules over Morita context rings with zero bimodule homomorphisms.

Keywords: trivial extension; Ding projective module; Ding injective module

MSC 2020: 16D40, 16D50, 16E05

1. Introduction

The origin of Gorenstein homological algebra may date back to the 1960s when Auslander and Bridger introduced the concept of G-dimension for finitely generated modules over a two-sided Noetherian ring, see [1]. In the 1990s, Enochs and Jenda extended the ideas of Auslander and Bridger and introduced the concepts of Gorenstein projective and Gorenstein injective modules over arbitrary rings, see [4], [5]. Ding, Li and Mao considered two special cases of the Gorenstein projective and Gorenstein injective modules, which they called strongly Gorenstein flat and Gorenstein FP-injective modules, respectively, in [2], [18]. These two classes of modules over coherent rings possess many nice properties analogous to Gorenstein projective and Gorenstein injective modules over Noetherian rings, see [2], [8], [16], [18], [25], [26].

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So Gillespie later renamed strongly Gorenstein flat as Ding projective, and Gorenstein FP-injective as Ding injective. He used these modules to produce new model structures in the categories of modules, see [8] for details.

Let R be an associative ring and M be an R-R-bimodule. The Cartesian product $R \times M$, with the natural addition and multiplication, given by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$, becomes a ring. This ring is called the *trivial extension* of the ring R by the bimodule M (see [7], [22]) and denoted by $R \ltimes M$. The notion of trivial extension of a ring by a bimodule is an important extension of rings and has played a crucial role in ring theory and homological algebra. When R is a commutative ring, Nagata also called this construction an *idealization* in [20]. Fossum, Griffith and Reiten studied the categorical aspect and homological properties of trivial ring extensions in [7]. Palmér and Roos gave some explicit formulae for global homological dimensions of trivial ring extensions in [21]. Mao considered (relative) homological behaviours of trivial ring extensions in [17]. Holm and Jørgensen investigated Gorenstein projective, injective and flat modules over trivial ring extensions in [11]. Mahdou and Ouarghi also studied the Gorenstein modules and dimensions over trivial ring extensions in [15].

The present paper is devoted to Ding projective and Ding injective modules over trivial ring extensions.

In Section 2, we describe Ding projective modules over a trivial ring extension $R \ltimes M$. Let M_R, ${}_RM$, $\mathbf{Z}(R)_{R\ltimes M}$ and ${}_{R\ltimes M}\mathbf{Z}(R)$ have finite flat dimensions. It is proven that (X, α) is a Ding projective left $R \ltimes M$-module if and only if the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{\alpha} X$ is exact and $\operatorname{coker}(\alpha)$ is a Ding projective left R-module. As an application, we characterize Ding projective modules over Morita context rings with zero bimodule homomorphisms.

Section 3 is devoted to Ding injective $R \ltimes M$-modules. Let $R \ltimes M$ be a left coherent ring, M_R have finite flat dimension, ${}_RM$ be finitely presented and have finite projective or FP-injective dimension, $\mathbf{Z}(R)_{R\ltimes M}$ have finite flat dimension, ${}_{R\ltimes M}\mathbf{Z}(R)$ have finite projective or FP-injective dimension. We prove that $[Y, \beta]$ is a Ding injective left $R \ltimes M$-module if and only if the sequence

$$Y \xrightarrow{\beta} \operatorname{Hom}_R(M, Y) \xrightarrow{\beta_*} \operatorname{Hom}_R(M, \operatorname{Hom}_R(M, Y))$$

is exact and $\ker(\beta)$ is a Ding injective left R-module. As an application, we characterize Ding injective modules over Morita context rings with zero bimodule homomorphisms.

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring R, we write R-Mod (or Mod-R) for the category of left (or right) R-modules, respectively. The symbol ${}_RX$ (or X_R) denotes a left (or right)

R-module, respectively. For an R-module X, the character module $\mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Q}/\mathbb{Z})$ of X is denoted by X^{+}, $\mathrm{pd}(X)$, $\mathrm{id}(X)$ and $\mathrm{fd}(X)$ denote the projective, injective and flat dimensions of X, respectively.

Next we recall some basic concepts and results on trivial extensions.

Recall from [7] that the classical *right trivial extension* of an abelian category $\underline{\underline{\mathbf{A}}}$ by an additive endofunctor $\mathbf{F}$, denoted by $\underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$, is a category whose objects are couples (X,f) with $X \in \mathrm{Ob}(\underline{\underline{\mathbf{A}}})$ and $f\colon\ \mathbf{F}(X) \to X$ such that $f \cdot \mathbf{F}(f) = 0$, and a morphism $\gamma\colon\ (X,\alpha) \to (Y,\beta)$ is a morphism $\gamma\colon\ X \to Y$ in $\underline{\underline{\mathbf{A}}}$ such that $\beta\mathbf{F}(\gamma) = \gamma\alpha$. If $\mathbf{F}$ is right exact, then $\underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$ is an abelian category. In this case, a sequence in $\underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$ is exact if and only if the sequence of codomains in $\underline{\underline{\mathbf{A}}}$ is exact.

Dually, the *left trivial extension* of an abelian category $\underline{\underline{\mathbf{A}}}$ by an additive endofunctor $\mathbf{G}$, denoted by $\mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$, is a category whose objects are couples $[X,g]$ with $X \in \mathrm{Ob}(\underline{\underline{\mathbf{A}}})$ and $g\colon\ X \to \mathbf{G}(X)$ such that $\mathbf{G}(g) \cdot g = 0$, and a morphism $\gamma\colon\ [X,\alpha] \to [Y,\beta]$ is a morphism $\gamma\colon\ X \to Y$ in $\underline{\underline{\mathbf{A}}}$ such that $\mathbf{G}(\gamma)\alpha = \beta\gamma$. If $\mathbf{G}$ is left exact, then $\mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ is an abelian category. In this case, a sequence in $\mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ is exact if and only if the sequence of domains in $\underline{\underline{\mathbf{A}}}$ is exact.

For a right exact endofunctor $\mathbf{F}\colon\ \underline{\underline{\mathbf{A}}} \to \underline{\underline{\mathbf{A}}}$ and a left exact endofunctor $\mathbf{G}\colon\ \underline{\underline{\mathbf{A}}} \to \underline{\underline{\mathbf{A}}}$, there are some important functors as follows.

The functor $\mathbf{T}\colon\ \underline{\underline{\mathbf{A}}} \to \underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$ is given for every object $X \in \underline{\underline{\mathbf{A}}}$, by $\mathbf{T}(X) = (X \oplus \mathbf{F}(X), \mu)$ with $\mu = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\colon\ \mathbf{F}(X) \oplus \mathbf{F}^2(X) \to X \oplus \mathbf{F}(X)$ and for morphisms by $\mathbf{T}(\alpha) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \mathbf{F}(\alpha) \end{smallmatrix}\right)$.

The functor $\mathbf{U}\colon\ \underline{\underline{\mathbf{A}}} \ltimes \mathbf{F} \to \underline{\underline{\mathbf{A}}}$ is given for every object $(X,f) \in \underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$ by $\mathbf{U}(X,f) = X$ and for morphisms by $\mathbf{U}(\alpha) = \alpha$.

The functor $\mathbf{Z}\colon\ \underline{\underline{\mathbf{A}}} \to \underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$ is given for every object $X \in \underline{\underline{\mathbf{A}}}$ by $\mathbf{Z}(X) = (X,0)$ and for morphisms by $\mathbf{Z}(\alpha) = \alpha$.

The functor $\mathbf{C}\colon\ \underline{\underline{\mathbf{A}}} \ltimes \mathbf{F} \to \underline{\underline{\mathbf{A}}}$ is given for every object $(X,f) \in \underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$, by $\mathbf{C}(X,f) = \mathrm{coker}(f)$ and for morphisms by $\mathbf{C}(\alpha) =$ the induced morphism.

The functor $\mathbf{H}\colon\ \underline{\underline{\mathbf{A}}} \to \mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ is given for every object $X \in \underline{\underline{\mathbf{A}}}$ by $\mathbf{H}(X) = [\mathbf{G}(X) \oplus X, \vartheta]$ with $\vartheta = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\colon\ \mathbf{G}(X) \oplus X \to \mathbf{G}^2(X) \oplus \mathbf{G}(X)$ and for morphisms by $\mathbf{H}(\beta) = \left(\begin{smallmatrix} \mathbf{G}(\beta) & 0 \\ 0 & \beta \end{smallmatrix}\right)$.

The functor $\mathbf{U}\colon\ \mathbf{G} \rtimes \underline{\underline{\mathbf{A}}} \to \underline{\underline{\mathbf{A}}}$ is given for every object $[X,g] \in \mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ by $\mathbf{U}[X,g] = X$ and for morphisms by $\mathbf{U}(\alpha) = \alpha$.

The functor $\mathbf{Z}\colon\ \underline{\underline{\mathbf{A}}} \to \mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ is given for every object $X \in \underline{\underline{\mathbf{A}}}$ by $\mathbf{Z}(X) = [X,0]$ and for morphisms by $\mathbf{Z}(\alpha) = \alpha$.

The functor $\mathbf{K}\colon\ \mathbf{G} \rtimes \underline{\underline{\mathbf{A}}} \to \underline{\underline{\mathbf{A}}}$ is given for every object $[X,g] \in \mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ by $\mathbf{K}[X,g] = \ker(g)$ and for morphisms by $\mathbf{K}(\alpha) =$ the induced morphism.

There exist several important pairs of adjoint functors

$$\underline{\underline{\mathbf{A}}} \underset{\mathbf{U}}{\overset{\mathbf{T}}{\rightleftarrows}} \underline{\underline{\mathbf{A}}} \ltimes \mathbf{F} \underset{\mathbf{Z}}{\overset{\mathbf{C}}{\rightleftarrows}} \underline{\underline{\mathbf{A}}}\,, \quad \underline{\underline{\mathbf{A}}} \underset{\mathbf{K}}{\overset{\mathbf{Z}}{\rightleftarrows}} \mathbf{G} \rtimes \underline{\underline{\mathbf{A}}} \underset{\mathbf{H}}{\overset{\mathbf{U}}{\rightleftarrows}} \underline{\underline{\mathbf{A}}}\,,$$

i.e., $(\mathbf{T},\mathbf{U})$, $(\mathbf{C},\mathbf{Z})$, $(\mathbf{Z},\mathbf{K})$ and $(\mathbf{U},\mathbf{H})$ are adjoint pairs with $\mathbf{CT} = \mathrm{id}_{\underline{\underline{\mathbf{A}}}}$, $\mathbf{UZ} = \mathrm{id}_{\underline{\underline{\mathbf{A}}}}$, $\mathbf{KH} = \mathrm{id}_{\underline{\underline{\mathbf{A}}}}$. We note that the functors $\mathbf{T}$ and $\mathbf{C}$ are right exact, $\mathbf{H}$ and $\mathbf{K}$ are left exact, $\mathbf{U}$ and $\mathbf{Z}$ are exact.

It is known that, when $\underline{\underline{\mathbf{A}}}$ is the category of left R-modules, M is an R-R-bimodule, $\mathbf{F} = M \otimes_R -$ and $\mathbf{G} = \mathrm{Hom}_R(M,-)$, both $\underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$ and $\mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ are isomorphic to the category of left modules over $R \ltimes M$. We will identify $R \ltimes M$-Mod with $\underline{\underline{\mathbf{A}}} \ltimes \mathbf{F}$ and $\mathbf{G} \rtimes \underline{\underline{\mathbf{A}}}$ in what follows.

2. Ding projective modules over trivial ring extensions

Recall that a left R-module X is *Ding projective* (see [2], [8]) if there is an exact sequence of projective left R-modules

$$\Xi\colon \ldots \to P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \to \ldots$$

such that $X \cong \ker(f^0)$ and $\mathrm{Hom}_R(\Xi, Q)$ is also exact for any flat left R-module Q.

Theorem 2.1. *Let (X,α) be a left $R \ltimes M$-module.*

(1) *If M_R and ${}_RM$ have finite flat dimensions, the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{\alpha} X$ is exact and $\mathrm{coker}(\alpha)$ is a Ding projective left R-module, then (X,α) is a Ding projective left $R \ltimes M$-module.*

(2) *If $\mathbf{Z}(R)_{R\ltimes M}$ and ${}_{R\ltimes M}\mathbf{Z}(R)$ have finite flat dimensions, and (X,α) is a Ding projective left $R \ltimes M$-module, then the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{\alpha} X$ is exact and $\mathrm{coker}(\alpha)$ is a Ding projective left R-module.*

Proof. (1) There exists an exact sequence of projective left R-modules

$$\Xi\colon \ldots \to P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \to \ldots$$

such that $\mathrm{coker}(\alpha) \cong \ker(f^0)$ and $\mathrm{Hom}_R(\Xi, Q)$ is also exact for any flat left R-module Q. Since $\mathrm{fd}(M_R) < \infty$, we get the exact sequence of left R-modules

$$M \otimes_R \Xi\colon \ldots \to M \otimes_R P^{-1} \xrightarrow{M\otimes f^{-1}} M \otimes_R P^0 \xrightarrow{M\otimes f^0} M \otimes_R P^1 \xrightarrow{M\otimes f^1} M \otimes_R P^2 \to \ldots$$

with $M \otimes_R \mathrm{coker}(\alpha) \cong \ker(M \otimes f^0)$ by [3], Lemma 2.3.

The exact sequence $M \otimes_R X \xrightarrow{\alpha} X \xrightarrow{\varrho} \operatorname{coker}(\alpha) \to 0$ induces the exact sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{M\otimes\varrho} M \otimes_R \operatorname{coker}(\alpha) \to 0$. Since the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{\alpha} X \xrightarrow{\varrho} \operatorname{coker}(\alpha) \to 0$ is exact, we get the exact sequence

$$0 \to M \otimes_R \operatorname{coker}(\alpha) \xrightarrow{\delta} X \xrightarrow{\varrho} \operatorname{coker}(\alpha) \to 0$$

with $\delta(M \otimes \varrho) = \alpha$. Since $\operatorname{fd}({}_R M) < \infty$, $\operatorname{fd}(M \otimes_R P^i) < \infty$. Thus, $\operatorname{Hom}_R(\Xi, M \otimes_R P^i)$ is exact for any $i \in \mathbb{N}$ by [16], Lemma 3.1. So $\operatorname{Ext}^1_R(\ker(f^i), M \otimes_R P^i) = 0$.

Let $\iota\colon \operatorname{coker}(\alpha) \to P^0$ be the canonical monomorphism and $\pi\colon P^{-1} \to \operatorname{coker}(\alpha)$ be the canonical epimorphism such that $\iota\pi = f^{-1}$. Since $\operatorname{Ext}^1_R(\operatorname{coker}(\alpha), M \otimes_R P^0) = 0$, there is $\psi\colon X \to M \otimes_R P^0$ such that $\psi\delta = M \otimes \iota$. Also there is $\eta\colon P^{-1} \to X$ such that $\varrho\eta = \pi$. Define $\lambda\colon X \to P^0 \oplus (M \otimes_R P^0)$ by $\lambda(x) = (\iota\varrho(x), \psi(x))$ and $\xi\colon P^{-1} \oplus (M \otimes_R P^{-1}) \to X$ by $\xi(x, y) = \eta(x) + \delta(M \otimes \pi)(y)$. It is easy to check that λ is a monomorphism and ξ is an epimorphism. Then by generalized Horseshoe (see Lemma [27], Lemma 1.6), we get the exact sequence of left R-modules

$$\begin{aligned}\ldots \to P^{-1} \oplus (M \otimes_R P^{-1}) \xrightarrow{g^{-1}} P^0 \oplus (M \otimes_R P^0) &\xrightarrow{g^0} P^1 \oplus (M \otimes_R P^1)\\ &\xrightarrow{g^1} P^2 \oplus (M \otimes_R P^2) \to \ldots\end{aligned}$$

with $g^{-1} = \lambda\xi$, $g^i = \begin{pmatrix} f^i & 0 \\ \sigma^i & M\otimes f^i \end{pmatrix}$ $(i \neq -1)$ and $X \cong \ker(g^0)$.

It is easy to verify that the two diagrams

$$\begin{tikzcd}
 & M \otimes_R X \arrow[r, "M\otimes\lambda"] \arrow[d, "\alpha"'] & M \otimes_R (P^0 \oplus (M \otimes_R P^0)) \arrow[r, "M\otimes g^0"] \arrow[d, "\mu_0"] & M \otimes_R (P^1 \oplus (M \otimes_R P^1)) \arrow[r] \arrow[d, "\mu_1"] & \cdots \\
0 \arrow[r] & X \arrow[r, "\lambda"] & P^0 \oplus (M \otimes_R P^0) \arrow[r, "g^0"] & P^1 \oplus (M \otimes_R P^1) \arrow[r] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & M \otimes_R (P^{-2} \oplus (M \otimes_R P^{-2})) \arrow[r, "M\otimes g^{-2}"] \arrow[d, "\mu_{-2}"] & M \otimes_R (P^{-1} \oplus (M \otimes_R P^{-1})) \arrow[r, "M\otimes\xi"] \arrow[d, "\mu_{-1}"] & M \otimes_R X \arrow[d, "\alpha"'] & \\
\cdots \arrow[r] & P^{-2} \oplus (M \otimes_R P^{-2}) \arrow[r, "g^{-2}"] & P^{-1} \oplus (M \otimes_R P^{-1}) \arrow[r, "\xi"] & X \arrow[r] & 0
\end{tikzcd}$$

are commutative. So we obtain two exact sequences of left $R \ltimes M$-modules $0 \to (X, \alpha) \xrightarrow{\lambda} \mathbf{T}(P^0) \xrightarrow{g^0} \mathbf{T}(P^1) \to \ldots$ and $\ldots \to \mathbf{T}(P^{-2}) \xrightarrow{g^{-2}} \mathbf{T}(P^{-1}) \xrightarrow{\xi} (X, \alpha) \to 0$.

By [7], Corollary 1.6 (c), each $\mathbf{T}(P^i)$ is projective. Thus, we get the exact sequence of projective left $R \ltimes M$-modules

$$\Delta\colon \ldots \to \mathbf{T}(P^{-1}) \xrightarrow{g^{-1}} \mathbf{T}(P^0) \xrightarrow{g^0} \mathbf{T}(P^1) \xrightarrow{g^1} \mathbf{T}(P^2) \to \ldots$$

with $(X, \alpha) \cong \ker(g^0)$.

Let (Y,β) be any flat left $R\ltimes M$-module. By [7], Proposition 1.14, $\operatorname{coker}(\beta)$ is a flat left R-module and the sequence $M\otimes_R M\otimes_R Y \xrightarrow{M\otimes\beta} M\otimes_R Y \xrightarrow{\beta} Y$ is exact. So we get the exact sequence $0\to M\otimes_R \operatorname{coker}(\beta)\to Y\to \operatorname{coker}(\beta)\to 0$. By [16], Lemma 3.2, $\operatorname{fd}(M\otimes_R \operatorname{coker}(\beta))<\infty$ since $\operatorname{fd}({}_RM)<\infty$. By [21], Lemma 1, there is an exact sequence $0\to \mathbf{Z}(M\otimes_R \operatorname{coker}(\beta))\to (Y,\beta)\to \mathbf{Z}(\operatorname{coker}(\beta))\to 0$. So we get the exact sequence of complexes

$$\begin{aligned}0\to \operatorname{Hom}_{R\ltimes M}(\Delta,\mathbf{Z}(M\otimes_R \operatorname{coker}(\beta)))&\to \operatorname{Hom}_{R\ltimes M}(\Delta,(Y,\beta))\\ &\to \operatorname{Hom}_{R\ltimes M}(\Delta,\mathbf{Z}(\operatorname{coker}(\beta)))\to 0.\end{aligned}$$

Since $\operatorname{Hom}_{R\ltimes M}(\mathbf{T}(P^i),\mathbf{Z}(\operatorname{coker}(\beta)))\cong \operatorname{Hom}_R(P^i,\operatorname{coker}(\beta))$, then

$$\operatorname{Hom}_{R\ltimes M}(\Delta,\mathbf{Z}(\operatorname{coker}(\beta)))\cong \operatorname{Hom}_R(\Xi,\operatorname{coker}(\beta))$$

is exact. Since $\operatorname{Hom}_{R\ltimes M}(\mathbf{T}(P^i),\mathbf{Z}(M\otimes_R \operatorname{coker}(\beta)))\cong \operatorname{Hom}_R(P^i,M\otimes_R \operatorname{coker}(\beta))$, then $\operatorname{Hom}_{R\ltimes M}(\Delta,\mathbf{Z}(M\otimes_R \operatorname{coker}(\beta)))\cong \operatorname{Hom}_R(\Xi,M\otimes_R \operatorname{coker}(\beta))$ is exact by [16], Lemma 3.1. So $\operatorname{Hom}_{R\ltimes M}(\Delta,(Y,\beta))$ is exact.

It follows that (X,α) is a Ding projective left $R\ltimes M$-module.

(2) By [7], Corollary 1.6 (c), there is an exact sequence of projective left $R\ltimes M$-modules

$$\Delta\colon\ \dots\to \mathbf{T}(P^{-1})\to \mathbf{T}(P^0)\xrightarrow{g^0}\mathbf{T}(P^1)\to \mathbf{T}(P^2)\to\dots$$

such that $(X,\alpha)\cong\ker(g^0)$ and the complex $\operatorname{Hom}_{R\ltimes M}(\Delta,L)$ is exact for any flat left $R\ltimes M$-module L.

Since $\operatorname{fd}(\mathbf{Z}(R)_{R\ltimes M})<\infty$, $\mathbf{Z}(R)\otimes_{R\ltimes M}\Delta$ is exact by [3], Lemma 2.3. Since $\mathbf{Z}(R)\otimes_{R\ltimes M}\mathbf{T}(P^i)\cong R\otimes_R P^i\cong P^i$, we get the exact sequence of projective left R-modules

$$\mathbf{C}(\Delta)\colon\ \dots\to P^{-1}\to P^0\xrightarrow{\mathbf{C}(g^0)}P^1\to P^2\to\dots$$

with $\operatorname{coker}(\alpha)\cong\ker(\mathbf{C}(g^0))$.

Let Q be a flat left R-module. Then $Q=\varinjlim N_i$ with each N_i free by [13], Theorem 3.4. Note that $\mathbf{Z}(Q)=\mathbf{Z}(\varinjlim N_i)=\varinjlim \mathbf{Z}(N_i)$. Since $\operatorname{fd}({}_{R\ltimes M}\mathbf{Z}(R))<\infty$, $\operatorname{fd}({}_{R\ltimes M}\mathbf{Z}(Q))<\infty$. So $\operatorname{Hom}_R(\mathbf{C}(\Delta),Q)\cong \operatorname{Hom}_{R\ltimes M}(\Delta,\mathbf{Z}(Q))$ is exact by [16], Lemma 3.1. Hence, $\operatorname{coker}(\alpha)$ is a Ding projective left R-module.

By [21], Lemma 1, there is an exact sequence $0\to\mathbf{Z}(M)\to\mathbf{T}(R)\to\mathbf{Z}(R)\to 0$, which induces the exact sequence of complexes

$$0\to \mathbf{Z}(M)\otimes_{R\ltimes M}\Delta\to \mathbf{T}(R)\otimes_{R\ltimes M}\Delta\to \mathbf{Z}(R)\otimes_{R\ltimes M}\Delta\to 0.$$

Since $\mathbf{T}(R)\otimes_{R\ltimes M}\Delta$ and $\mathbf{Z}(R)\otimes_{R\ltimes M}\Delta$ are exact, $\mathbf{Z}(M)\otimes_{R\ltimes M}\Delta$ is exact. Note that $M\otimes_R P^i\cong \mathbf{Z}(M)\otimes_{R\ltimes M}\mathbf{T}(P^i)$ by [14], page 295. So $M\otimes_R\mathbf{C}(\Delta)\cong \mathbf{Z}(M)\otimes_{R\ltimes M}\Delta$

is exact. Let $\iota\colon \operatorname{coker}(\alpha) \to P^0$ be the obvious monomorphism. Then $M \otimes \iota\colon M \otimes_R \operatorname{coker}(\alpha) \to M \otimes_R P^0$ is a monomorphism. The exact sequence $M \otimes_R X \xrightarrow{\alpha} X \xrightarrow{\varrho} \operatorname{coker}(\alpha) \to 0$ induces the exact sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{M\otimes\varrho} M \otimes_R \operatorname{coker}(\alpha) \to 0$. Since $\alpha(M \otimes \alpha) = 0$, there is $\delta\colon M \otimes_R \operatorname{coker}(\alpha) \to X$ such that $\delta(M \otimes \varrho) = \alpha$. Let $\lambda\colon X \to P^0 \oplus (M \otimes_R P^0)$ and $\varphi^0\colon M \otimes_R P^0 \to P^0 \oplus (M \otimes_R P^0)$ be the injections. By [7], page 57, we get the following commutative diagram in R-Mod:

$$\begin{tikzcd}
M \otimes_R \operatorname{coker}(\alpha) \arrow[r, "M\otimes\iota"] \arrow[d, "\delta"'] & M \otimes_R P^0 \arrow[d, "\varphi^0"] \\
X \arrow[r, "\lambda"] & P^0 \oplus (M \otimes_R P^0).
\end{tikzcd}$$

Then δ is a monomorphism. Since the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{M\otimes\varrho} M \otimes_R \operatorname{coker}(\alpha) \to 0$ is exact, the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{\alpha} X$ is exact. $\square$

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let M_R, ${}_RM$, $\mathbf{Z}(R)_{R\ltimes M}$ and ${}_{R\ltimes M}\mathbf{Z}(R)$ have finite flat dimensions. Then*

(1) *(X, α) is a Ding projective left $R \ltimes M$-module if and only if the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes\alpha} M \otimes_R X \xrightarrow{\alpha} X$ is exact and* $\operatorname{coker}(\alpha)$ *is a Ding projective left R-module.*

(2) *$\mathbf{T}(X)$ is a Ding projective left $R \ltimes M$-module if and only if X is a Ding projective left R-module.*

(3) *$\mathbf{Z}(X)$ is a Ding projective left $R \ltimes M$-module if and only if $M \otimes_R X = 0$ and X is a Ding projective left R-module.*

Specializing $M = R$ in Theorem 2.1 (1), we have the following statement.

Corollary 2.3. *If $X \xrightarrow{\alpha} X \xrightarrow{\alpha} X$ is an exact sequence in R-Mod and* $\operatorname{coker}(\alpha)$ *is a Ding projective left R-module, then (X, α) is a Ding projective left $R \ltimes R$-module.*

Morita context rings with zero bimodule homomorphisms is one important special case of trivial ring extensions. Let A and B be rings, ${}_BU_A$ and ${}_AV_B$ be bimodules, $\phi\colon U \otimes_A V \to B$ and $\psi\colon V \otimes_B U \to A$ be bimodule homomorphisms. Then $\begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\phi,\psi)}$ becomes a ring with the usual matrix addition and multiplication given by

$$\begin{pmatrix} a_1 & v_1 \\ u_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & v_2 \\ u_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + \psi(v_1 \otimes u_2) & a_1v_2 + v_1b_2 \\ u_1a_2 + b_1u_2 & b_1b_2 + \phi(u_1 \otimes v_2) \end{pmatrix}.$$

The matrix $\begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\phi,\psi)}$ is called a *Morita context ring* or *formal matrix ring*, see [12], [19]. In particular, if $\phi = 0$, $\psi = 0$, then $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$ is called a *Morita context ring with zero bimodule homomorphisms*, which is a generalization of the formal triangular matrix ring $\begin{pmatrix} A & 0 \\ {}_BU_A & B \end{pmatrix}$.

Let $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$. Green in [9] proved that the category Λ-Mod is equivalent to the category Ω whose objects are tuples (X, Y, f, g), where $X \in A$-Mod, $Y \in B$-Mod, $f \in \mathrm{Hom}_B(U \otimes_A X, Y)$ and $g \in \mathrm{Hom}_A(V \otimes_B Y, X)$ such that $g(V \otimes f) = 0$, $f(U \otimes g) = 0$, and whose morphisms from (X_1, Y_1, f_1, g_1) to (X_2, Y_2, f_2, g_2) are pairs (α, β) such that $\alpha \in \mathrm{Hom}_A(X_1, X_2)$, $\beta \in \mathrm{Hom}_B(Y_1, Y_2)$, $f_2(U \otimes \alpha) = \beta f_1$, $g_2(V \otimes \beta) = \alpha g_1$. In view of the well-known adjointness relation, the category Λ-Mod is also equivalent to the category Γ whose objects are tuples $[X, Y, f, g]$, where $X \in A$-Mod, $Y \in B$-Mod, $f \in \mathrm{Hom}_A(X, \mathrm{Hom}_B(U, Y))$ and $g \in \mathrm{Hom}_B(Y, \mathrm{Hom}_A(V, X))$ such that $\mathrm{Hom}_B(U, g)f = 0, \mathrm{Hom}_A(U, f)g = 0$, and whose morphisms from $[X_1, Y_1, f_1, g_1]$ to $[X_2, Y_2, f_2, g_2]$ are pairs $[\alpha, \beta]$ such that $\alpha \in \mathrm{Hom}_A(X_1, X_2), \beta \in \mathrm{Hom}_B(Y_1, Y_2)$ and $f_2\alpha = \mathrm{Hom}_B(U, \beta)f_1$, $g_2\beta = \mathrm{Hom}_A(V, \alpha)g_1$. We will identify Λ-Mod with Ω and Γ in what follows.

It is known that the ring $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$ is isomorphic to the trivial ring extension $(A \times B) \ltimes (U \oplus V)$ under the correspondence $\left(\begin{smallmatrix} a & v \\ u & b \end{smallmatrix}\right) \to ((a, b), (u, v))$, see [7]. Note that $U \oplus V$ attains the $A \times B$-$A \times B$-bimodule structure through the ring homomorphisms $A \times B \to A$ and $A \times B \to B$. It is well known that a left $A \times B$-module is an order pair (X, Y) with $X \in A$-Mod and $Y \in B$-Mod. Similarly, a right $A \times B$-module is an order pair (W_1, W_2) with $W_1 \in$ Mod-A and $W_2 \in$ Mod-B. So $(U \oplus V) \otimes_{A\times B} (X, Y) \cong (V \otimes_B Y, U \otimes_A X)$ and $\mathrm{Hom}_{A\times B}(U \oplus V, (X, Y)) \cong (\mathrm{Hom}_B(U, Y), \mathrm{Hom}_A(V, X))$. Therefore Λ-Mod is isomorphic to $(A \times B) \ltimes (U \oplus V)$-Mod by the functor $\Theta\colon \Lambda\text{-Mod} \to (A \times B) \ltimes (U \oplus V)$-Mod given by $\Theta(X, Y, f, g) = ((X, Y), (g, f))$. Similarly, Mod-$\Lambda$ is isomorphic to Mod-$(A\times B)\ltimes(U\oplus V)$ by the functor $\Upsilon\colon$ Mod-$\Lambda \to$ Mod-$(A \times B) \ltimes (U \oplus V)$ given by $\Upsilon(W, Q, f, g) = ((W, Q), (f, g))$.

Theorem 2.4. *Let $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$ and (X, Y, f, g) be a left Λ-module.*

(1) *If U_A, ${}_BU$, ${}_AV$ and V_B have finite flat dimensions, the sequences $V \otimes_B U \otimes_A X \xrightarrow{V\otimes f} V \otimes_B Y \xrightarrow{g} X$ and $U \otimes_A V \otimes_B Y \xrightarrow{U\otimes g} U \otimes_A X \xrightarrow{f} Y$ are exact,* coker(f) *is a Ding projective left B-module and* coker(g) *is a Ding projective left A-module, then (X, Y, f, g) is a Ding projective left Λ-module.*

(2) *If ${}_\Lambda(A, B, 0, 0)$ and $(A, B, 0, 0)_\Lambda$ have finite flat dimensions and (X, Y, f, g) is a Ding projective left Λ-module, then the sequences $V \otimes_B U \otimes_A X \xrightarrow{V\otimes f} V \otimes_B Y \xrightarrow{g} X$ and $U \otimes_A V \otimes_B Y \xrightarrow{U\otimes g} U \otimes_A X \xrightarrow{f} Y$ are exact,* coker(f) *is a Ding projective left B-module and* coker(g) *is a Ding projective left A-module.*

Proof. (1) Since the sequences $V \otimes_B U \otimes_A X \xrightarrow{V \otimes f} V \otimes_B Y \xrightarrow{g} X$ and $U \otimes_A V \otimes_B Y \xrightarrow{U \otimes g} U \otimes_A X \xrightarrow{f} Y$ are exact, the sequence $(U \oplus V) \otimes_{A \times B} (U \oplus V) \otimes_{A \times B} (X, Y) \xrightarrow{(U \oplus V) \otimes (g,f)} (U \oplus V) \otimes_{A \times B} (X, Y) \xrightarrow{(g,f)} (X, Y)$ is exact. Since $\operatorname{coker}(f)$ is a Ding projective left B-module and $\operatorname{coker}(g)$ is a Ding projective left A-module, $\operatorname{coker}(g, f) = (\operatorname{coker}(g), \operatorname{coker}(f))$ is a Ding projective left $A \times B$-module. By Theorem 2.1, $((X, Y), (g, f))$ is a Ding projective left $(A \times B) \ltimes (U \oplus V)$-module. So (X, Y, f, g) is a Ding projective left Λ-module.

(2) Since $\operatorname{fd}({}_\Lambda(A, B, 0, 0)) < \infty$ and $\operatorname{fd}((A, B, 0, 0)_\Lambda) < \infty$,

$$\operatorname{fd}({}_{(A \times B) \ltimes (U \oplus V)}\mathbf{Z}(A \times B)) < \infty \quad \text{and} \quad \operatorname{fd}(\mathbf{Z}(A \times B)_{(A \times B) \ltimes (U \oplus V)}) < \infty.$$

Since (X, Y, f, g) is a Ding projective left Λ-module, $((X, Y), (g, f))$ is a Ding projective left $(A \times B) \ltimes (U \oplus V)$-module. By Theorem 2.1, the sequence $(U \oplus V) \otimes_{A \times B} (U \oplus V) \otimes_{A \times B} (X, Y) \xrightarrow{(U \oplus V) \otimes (g,f)} (U \oplus V) \otimes_{A \times B} (X, Y) \xrightarrow{(g,f)} (X, Y)$ is exact and $\operatorname{coker}(g, f)$ is a Ding projective left $A \times B$-module. So the sequences $V \otimes_B U \otimes_A X \xrightarrow{V \otimes f} V \otimes_B Y \xrightarrow{g} X$ and $U \otimes_A V \otimes_B Y \xrightarrow{U \otimes g} U \otimes_A X \xrightarrow{f} Y$ are exact, $\operatorname{coker}(f)$ is a Ding projective left B-module and $\operatorname{coker}(g)$ is a Ding projective left A-module. $\square$

Corollary 2.5. *Let* $\Lambda = \begin{pmatrix} A & {}_A V_B \\ {}_B U_A & B \end{pmatrix}_{(0,0)}$, U_A, ${}_B U$, ${}_A V$, V_B, ${}_\Lambda(A, B, 0, 0)$ *and* $(A, B, 0, 0)_\Lambda$ *have finite flat dimensions. Then*

(1) $(X, U \otimes_A X, \mathrm{id}_{U \otimes_A X}, 0)$ *is a Ding projective left* Λ*-module if and only if* X *is a Ding projective left* A*-module.*

(2) $(V \otimes_B Y, Y, 0, \mathrm{id}_{V \otimes_B Y})$ *is a Ding projective left* Λ*-module if and only if* Y *is a Ding projective left* B*-module.*

(3) $(X, Y, 0, 0)$ *is a Ding projective left* Λ*-module if and only if* $U \otimes_A X = 0$, $V \otimes_B Y = 0$, X *is a Ding projective left* A*-module and* Y *is a Ding projective left* B*-module.*

3. Ding injective modules over trivial ring extensions

Recall that a left R-module X is FP-*injective* (see [24]) if $\operatorname{Ext}^1_R(N, X) = 0$ for every finitely presented left R-module N, equivalently, if every exact sequence $0 \to X \to Y \to L \to 0$ of left R-modules is pure by [6], Theorem 3.1. The FP-*injective dimension* of a left R-module X, denoted by $\mathrm{FP}-\mathrm{id}(X)$, is defined to be the smallest integer $n \geqslant 0$ such that $\operatorname{Ext}^{n+1}_R(N, X) = 0$ for every finitely presented left R-module N (if no such n exists, set $\mathrm{FP}-\mathrm{id}(X) = \infty$).

A left R-module X is called *Ding injective* (see [8], [18]) if there is an exact sequence of injective left R-modules

$$\Xi\colon \ldots \to E^{-1} \xrightarrow{f^{-1}} E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2 \to \ldots$$

such that $X \cong \ker(f^0)$ and $\mathrm{Hom}_R(Y, \Xi)$ is exact for any FP-injective left R-module Y. Recall that R is a *left coherent ring* (see [13]) if every finitely generated left ideal is finitely presented.

Lemma 3.1. *Let $R \ltimes M$ be a left coherent ring and ${}_RM$ be finitely presented. Then*

(1) *$[Y, \beta]$ is an* FP-*injective left $R \ltimes M$-module if and only if* $\ker(\beta)$ *is an* FP-*injective left R-module and the sequence $Y \xrightarrow{\beta} \mathrm{Hom}_R(M, Y) \xrightarrow{\beta_*} \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, Y))$ is exact.*

(2) *$\mathbf{H}(Y)$ is an* FP-*injective left $R \ltimes M$-module if and only if Y is an* FP-*injective left R-module.*

(3) *$\mathbf{Z}(Y)$ is an* FP-*injective left $R \ltimes M$-module if and only if Y is an* FP-*injective left R-module and* $\mathrm{Hom}_R(M, Y) = 0$.

Proof. Since $R \ltimes M$ is a left coherent ring, R is also a left coherent ring by [7], Theorem 2.2. Since ${}_RM$ is finitely presented, the natural map $\sigma\colon Y^+ \otimes_R M \to \mathrm{Hom}_R(M, Y)^+$ is an isomorphism by [23], Lemma 3.60.

(1) The exact sequence $0 \to \ker(\beta) \to Y \xrightarrow{\beta} \mathrm{Hom}_R(M, Y)$ induces the exact sequence $\mathrm{Hom}_R(M, Y)^+ \xrightarrow{\beta^+} Y^+ \to (\ker(\beta))^+ \to 0$. Therefore, $[Y, \beta]$ is an FP-injective left $R \ltimes M$-module if and only if $[Y, \beta]^+ \cong (Y^+, \beta^+\sigma)$ is a flat right $R \ltimes M$-module by [6], Theorem 2.2 if and only if $\mathrm{coker}(\beta^+) \cong \mathrm{coker}(\beta^+\sigma)$ is a flat right R-module and the sequence $Y^+ \otimes_R M \otimes_R M \xrightarrow{(\beta^+\sigma)\otimes M} Y^+ \otimes_R M \xrightarrow{\beta^+\sigma} Y^+$ is exact by [7], Proposition 1.14 if and only if $(\ker(\beta))^+$ is a flat right R-module and the sequence $\mathrm{Hom}_R(M, \mathrm{Hom}_R(M, Y))^+ \xrightarrow{(\beta_*)^+} \mathrm{Hom}_R(M, Y)^+ \xrightarrow{\beta^+} Y^+$ is exact if and only if $\ker(\beta)$ is an FP-injective left R-module and the sequence $Y \xrightarrow{\beta} \mathrm{Hom}_R(M, Y) \xrightarrow{\beta_*} \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, Y))$ is exact by [6], Theorem 2.2.

(2) and (3) are immediate consequences of (1). $\square$

Theorem 3.2. *Let $R \ltimes M$ be a left coherent ring and $[Y, \beta]$ be a left $R \ltimes M$-module.*

(1) *If M_R has finite flat dimension, ${}_RM$ is finitely presented and has finite projective or* FP-*injective dimension, the sequence $Y \xrightarrow{\beta} \mathrm{Hom}_R(M, Y) \xrightarrow{\beta_*} \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, Y))$ is exact and* $\ker(\beta)$ *is a Ding injective left R-module, then $[Y, \beta]$ is a Ding injective left $R \ltimes M$-module.*

(2) *If* $\mathbf{Z}(R)_{R\ltimes M}$ *has finite flat dimension,* ${}_{R\ltimes M}\mathbf{Z}(R)$ *has finite projective or* FP*-injective dimension,* $[Y,\beta]$ *is a Ding injective left* $R\ltimes M$*-module, then the sequence* $Y \xrightarrow{\beta} \mathrm{Hom}_R(M,Y) \xrightarrow{\beta_*} \mathrm{Hom}_R(M,\mathrm{Hom}_R(M,Y))$ *is exact and* $\ker(\beta)$ *is a Ding injective left* R*-module.*

P r o o f. (1) There is an exact sequence of injective left R-modules

$$\Xi\colon \ldots \to E^{-1} \xrightarrow{f^{-1}} E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2 \to \ldots$$

such that $\ker(\beta) \cong \ker(f^0)$ and $\mathrm{Hom}_R(Q,\Xi)$ is also exact for any FP-injective left R-module Q.

Since $\mathrm{pd}({}_RM) < \infty$ or $\mathrm{FP}-\mathrm{id}({}_RM) < \infty$, we get the exact sequence of left R-modules

$$\mathrm{Hom}_R(M,\Xi)\colon \ldots \to \mathrm{Hom}_R(M,E^{-1}) \xrightarrow{(f^{-1})_*} \mathrm{Hom}_R(M,E^0) \xrightarrow{(f^0)_*} \mathrm{Hom}_R(M,E^1) \to \ldots$$

with $\mathrm{Hom}_R(M,\ker(\beta)) \cong \mathrm{Hom}_R(M,\ker((f^0)_*))$ by [3], Lemma 2.5 and [16], Lemma 3.2.

The exact sequence $0 \to \ker(\beta) \xrightarrow{\iota} Y \xrightarrow{\beta} \mathrm{Hom}_R(M,Y)$ induces the exact sequence $0 \to \mathrm{Hom}_R(M,\ker(\beta)) \xrightarrow{\iota_*} \mathrm{Hom}_R(M,Y) \xrightarrow{\beta_*} \mathrm{Hom}_R(M,\mathrm{Hom}_R(M,Y))$. Since the sequence $0 \to \ker(\beta) \xrightarrow{\iota} Y \xrightarrow{\beta} \mathrm{Hom}_R(M,Y) \xrightarrow{\beta_*} \mathrm{Hom}_R(M,\mathrm{Hom}_R(M,Y))$ is exact, we get the exact sequence

$$0 \to \ker(\beta) \xrightarrow{\iota} Y \xrightarrow{\gamma} \mathrm{Hom}_R(M,\ker(\beta)) \to 0$$

such that $\beta = \iota_*\gamma$. Since $\mathrm{fd}(M_R) < \infty$, we have $\mathrm{id}(\mathrm{Hom}_R(M,E^i)) < \infty$ by [16], Lemma 3.2. Therefore, the complex $\mathrm{Hom}_R(\mathrm{Hom}_R(M,E^i),\Xi)$ is exact by [16], Lemma 4.1. So $\mathrm{Ext}^1_R(\mathrm{Hom}_R(M,E^i),\ker(f^{i+1})) = 0$. Let $\pi\colon E^{-1} \to \ker(\beta)$ be the canonical epimorphism and $\tau\colon \ker(\beta) \to E^0$ be the canonical monomorphism such that $\tau\pi = f^{-1}$. Since $\mathrm{Ext}^1_R(\mathrm{Hom}_R(M,E^{-1}),\ker(\beta)) = 0$, there is $\psi\colon \mathrm{Hom}_R(M,E^{-1}) \to Y$ such that $\gamma\psi = \pi_*$. Also there is $\phi\colon Y \to E^0$ such that $\phi\iota = \tau$. Define $\xi\colon \mathrm{Hom}_R(M,E^{-1}) \oplus E^{-1} \to Y$ by $\xi(x,y) = \psi(x) + \iota\pi(y)$ and $\lambda\colon Y \to \mathrm{Hom}_R(M,E^0) \oplus E^0$ by $\lambda(x) = (\tau\gamma(x),\phi(x))$. It is easy to check that λ is a monomorphism and ξ is an epimorphism. Then by generalized Horseshoe Lemma (see [27], Lemma 1.6), we get the exact sequence of left R-modules

$$\ldots \to \mathrm{Hom}_R(M,E^{-1}) \oplus E^{-1} \xrightarrow{\partial^{-1}} \mathrm{Hom}_R(M,E^0) \oplus E^0 \xrightarrow{\partial^0} \mathrm{Hom}_R(M,E^1) \oplus E^1 \to \ldots$$

with $\partial^{-1} = \lambda\xi$, $\partial^i = \begin{pmatrix} (f^i)_* & 0 \\ \tau^i & f^i \end{pmatrix}$ $(i \neq -1)$ and $Y \cong \ker(\partial^0)$.

It is easy to check that the two diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}_R(M, E^{-2}) \oplus E^{-2} \arrow[r, "\partial^{-2}"] \arrow[d, "\vartheta_{-2}"'] & \mathrm{Hom}_R(M, E^{-1}) \oplus E^{-1} \arrow[r, "\xi"] \arrow[d, "\vartheta_{-1}"] & Y \arrow[r] \arrow[d, "\beta"] & 0 \\
\cdots \arrow[r] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, E^{-2}) \oplus E^{-2}) \arrow[r, "(\partial^{-2})_*"] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, E^{-1}) \oplus E^{-1}) \arrow[r, "\xi_*"] & \mathrm{Hom}_R(M, Y) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "\lambda"] \arrow[d, "\beta"'] & \mathrm{Hom}_R(M, E^{0}) \oplus E^{0} \arrow[r, "\partial^{0}"] \arrow[d, "\vartheta_{0}"] & \mathrm{Hom}_R(M, E^{1}) \oplus E^{1} \arrow[r] \arrow[d, "\vartheta_{1}"] & \cdots \\
 & \mathrm{Hom}_R(M, Y) \arrow[r, "\lambda_*"] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, E^{0}) \oplus E^{0}) \arrow[r, "(\partial^{0})_*"] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, E^{1}) \oplus E^{1}) \arrow[r] & \cdots
\end{tikzcd}$$

are commutative. So we obtain two exact sequences $\ldots \to \mathbf{H}(E^{-2}) \xrightarrow{\partial^{-2}} \mathbf{H}(E^{-1}) \xrightarrow{\xi} [Y, \beta] \to 0$ and $0 \to [Y, \beta] \xrightarrow{\lambda} \mathbf{H}(E^0) \xrightarrow{\partial^0} \mathbf{H}(E^1) \to \ldots$

By [7], Corollary 1.6 (d), each $\mathbf{H}(E^i)$ is injective. So we get the exact sequence of injective left $R \ltimes M$-modules

$$\Delta\colon \ \ldots \to \mathbf{H}(E^{-2}) \xrightarrow{\partial^{-2}} \mathbf{H}(E^{-1}) \xrightarrow{\partial^{-1}} \mathbf{H}(E^{0}) \xrightarrow{\partial^{0}} \mathbf{H}(E^{1}) \to \ldots$$

with $[Y, \beta] \cong \ker(\partial^0)$.

Let $[X, \zeta]$ be an FP-injective left $R \ltimes M$-module. By Lemma 3.1, $\ker(\zeta)$ is an FP-injective left R-module and the sequence

$$X \xrightarrow{\zeta} \mathrm{Hom}_R(M, X) \xrightarrow{\zeta_*} \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, X))$$

is exact. Thus, we get the exact sequence

$$0 \to \ker(\zeta) \to X \to \mathrm{Hom}_R(M, \ker(\zeta)) \to 0.$$

So there is an exact sequence

$$0 \to \mathbf{Z}(\ker(\zeta)) \to [X, \zeta] \to \mathbf{Z}(\mathrm{Hom}_R(M, \ker(\zeta))) \to 0,$$

which induces the exact sequence of complexes

$$\begin{aligned} 0 \to \mathrm{Hom}_{R \ltimes M}(\mathbf{Z}(\mathrm{Hom}_R(M, \ker(\zeta))), \Delta) &\to \mathrm{Hom}_{R \ltimes M}([X, \zeta], \Delta) \\ &\to \mathrm{Hom}_{R \ltimes M}(\mathbf{Z}(\ker(\zeta)), \Delta) \to 0. \end{aligned}$$

Note that $\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(\ker(\zeta)),\mathbf{H}(E^i))\cong \operatorname{Hom}_R(\ker(\zeta),E^i)$. Therefore, the complex $\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(\ker(\zeta)),\Delta)\cong \operatorname{Hom}_R(\ker(\zeta),\Xi)$ is exact. On the other hand, by [16], Lemma 4.2, $\mathrm{FP}-\mathrm{id}(\operatorname{Hom}_R(M,\ker(\zeta)))<\infty$. Since

$$\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(\operatorname{Hom}_R(M,\ker(\zeta))),\mathbf{H}(E^i))\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M,\ker(\zeta)),E^i),$$

we have the complex

$$\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(\operatorname{Hom}_R(M,\ker(\zeta))),\Delta)\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M,\ker(\zeta)),\Xi)$$

is exact by [16], Lemma 4.1. Therefore, $\operatorname{Hom}_{R\ltimes M}([X,\zeta],\Delta)$ is exact. It follows that $[Y,\beta]$ is a Ding injective left $R\ltimes M$-module.

(2) By [7], Corollary 1.6 (d), there is an exact sequence of injective left $R\ltimes M$-modules

$$\Delta\colon\ \ldots\to\mathbf{H}(E^{-1})\to\mathbf{H}(E^0)\xrightarrow{\partial^0}\mathbf{H}(E^1)\to\mathbf{H}(E^2)\to\ldots$$

such that $[Y,\beta]\cong\ker(\partial^0)$ and $\operatorname{Hom}_{R\ltimes M}(X,\Delta)$ is also exact for any FP-injective left $R\ltimes M$-module X. Since $\operatorname{pd}({}_{R\ltimes M}\mathbf{Z}(R))<\infty$ or $\mathrm{FP}-\mathrm{id}({}_{R\ltimes M}\mathbf{Z}(R))<\infty$, the complex $\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(R),\Delta)$ is exact by [3], Lemma 2.5 and [16], Lemma 4.1. Note that $\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(R),\Delta)\cong\operatorname{Hom}_R(R,\mathbf{K}(\Delta))\cong\mathbf{K}(\Delta)$. So we get the exact sequence of injective left R-modules

$$\mathbf{K}(\Delta)\colon\ \ldots\to E^{-1}\to E^0\xrightarrow{\mathbf{K}(\partial^0)}E^1\to E^2\to\ldots$$

with $\ker(\beta)\cong\ker(\mathbf{K}(\partial^0))$.

Let Q be an FP-injective left R-module. Then there is a pure monomorphism in R-Mod $Q\to\prod R^+$. By [10], Lemma 2.1 (i), we get the pure monomorphism in $R\ltimes M$-Mod

$$\mathbf{Z}(Q)\to\mathbf{Z}\Big(\prod R^+\Big)=\prod\mathbf{Z}(R^+).$$

Since $\operatorname{fd}(\mathbf{Z}(R)_{R\ltimes M})<\infty$, we have $\mathrm{FP}-\mathrm{id}({}_{R\ltimes M}\mathbf{Z}(R^+))=\mathrm{FP}-\mathrm{id}({}_{R\ltimes M}\mathbf{Z}(R)^+)<\infty$ by [6], Theorem 2.1. Thus, $\mathrm{FP}-\mathrm{id}\big({}_{R\ltimes M}\prod\mathbf{Z}(R^+)\big)<\infty$ and so $\operatorname{fd}\big(\mathbf{Z}\big(\prod R^+\big)^+\big)<\infty$ by [6], Theorem 2.2 since $R\ltimes M$ is a left coherent ring. The pure monomorphism $\mathbf{Z}(Q)\to\mathbf{Z}\big(\prod R^+\big)$ induces the split epimorphism $\mathbf{Z}\big(\prod R^+\big)^+\to\mathbf{Z}(Q)^+$. Thus, $\operatorname{fd}(\mathbf{Z}(Q)^+)<\infty$ and so $\mathrm{FP}-\mathrm{id}({}_{R\ltimes M}\mathbf{Z}(Q))<\infty$. By [16], Lemma 4.1, $\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(Q),\Delta)$ is exact. Since $\operatorname{Hom}_R(Q,E^i)\cong\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(Q),\mathbf{H}(E^i))$, $\operatorname{Hom}_R(Q,\mathbf{K}(\Delta))\cong\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(Q),\Delta)$ is exact. So $\ker(\beta)$ is a Ding injective left R-module.

By [21], Lemma 1, there is an exact sequence $0\to\mathbf{Z}(M)\to\mathbf{T}(R)\to\mathbf{Z}(R)\to0$, which induces the exact sequence of complexes

$$0\to\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(R),\Delta)\to\operatorname{Hom}_{R\ltimes M}(\mathbf{T}(R),\Delta)\to\operatorname{Hom}_{R\ltimes M}(\mathbf{Z}(M),\Delta)\to0.$$

Since $\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(R),\Delta)$ and $\mathrm{Hom}_{R\ltimes M}(\mathbf{T}(R),\Delta)$ are exact, $\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(M),\Delta)$ is exact. So $\mathrm{Hom}_R(M,\mathbf{K}(\Delta))\cong\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(M),\Delta)$ is exact. Let $\tau\colon E^{-1}\to\ker(\beta)$ be the canonical epimorphism, then $\tau_*\colon \mathrm{Hom}_R(M,E^{-1})\to\mathrm{Hom}_R(M,\ker(\beta))$ is an epimorphism.

The exact sequence $0\to\ker(\beta)\xrightarrow{\iota}Y\xrightarrow{\beta}\mathrm{Hom}_R(M,Y)$ induces the exact sequence $0\to\mathrm{Hom}_R(M,\ker(\beta))\xrightarrow{\iota_*}\mathrm{Hom}_R(M,Y)\xrightarrow{\beta_*}\mathrm{Hom}_R(M,\mathrm{Hom}_R(M,Y))$. Since $\beta_*\beta=0$, there is $\gamma\colon Y\to\mathrm{Hom}_R(M,\ker(\beta))$ such that $\iota_*\gamma=\beta$. Let $\varrho\colon \mathrm{Hom}_R(M,E^{-1})\oplus E^{-1}\to Y$ be the canonical epimorphism and

$$\varphi^{-1}\colon \mathrm{Hom}_R(M,E^{-1})\oplus E^{-1}\to\mathrm{Hom}_R(M,E^{-1})$$

be the projection. Consider the following commutative diagram in R-Mod:

$$\begin{tikzcd}
\mathrm{Hom}_R(M,E^{-1})\oplus E^{-1} \arrow[r, "\varrho"] \arrow[d, "\varphi^{-1}"] & Y \arrow[d, "\gamma"] \\
\mathrm{Hom}_R(M,E^{-1}) \arrow[r, "\tau_*"] & \mathrm{Hom}_R(M,\ker(\beta)).
\end{tikzcd}$$

Since τ_* and φ^{-1} are epimorphisms, γ is an epimorphism. Hence, the sequence $Y\xrightarrow{\beta}\mathrm{Hom}_R(M,Y)\xrightarrow{\beta_*}\mathrm{Hom}_R(M,\mathrm{Hom}_R(M,Y))$ is exact. $\square$

The following result is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Suppose that $R\ltimes M$ is a left coherent ring, M_R has finite flat dimension, ${}_RM$ is finitely presented and has finite projective or* FP*-injective dimension, $\mathbf{Z}(R)_{R\ltimes M}$ has finite flat dimension, ${}_{R\ltimes M}\mathbf{Z}(R)$ has finite projective or* FP*-injective dimension.*

(1) *$[Y,\beta]$ is a Ding injective left $R\ltimes M$-module if and only if the sequence $Y\xrightarrow{\beta}\mathrm{Hom}_R(M,Y)\xrightarrow{\beta_*}\mathrm{Hom}_R(M,\mathrm{Hom}_R(M,Y))$ is exact and $\ker(\beta)$ is a Ding injective left R-module.*
(2) *$\mathbf{H}(Y)$ is a Ding injective left $R\ltimes M$-module if and only if Y is a Ding injective left R-module.*
(3) *$\mathbf{Z}(Y)$ is a Ding injective left $R\ltimes M$-module if and only if $\mathrm{Hom}_R(M,Y)=0$ and Y is a Ding injective left R-module.*

Specializing $M=R$ in Theorem 3.2 (1), we have:

Corollary 3.4. *If R is a left coherent ring, the sequence $Y\xrightarrow{\beta}Y\xrightarrow{\beta}Y$ is exact in R-Mod and $\ker(\beta)$ is a Ding injective left R-module, then $[Y,\beta]$ is a Ding injective left $R\ltimes R$-module.*

Theorem 3.5. *Let* $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$ *be a left coherent ring and* $[X,Y,f,g]$ *be a left* Λ*-module.*

(1) *If* U_A *and* V_B *have finite flat dimensions,* ${}_BU$ *and* ${}_AV$ *are finitely presented and have finite projective dimensions or* ${}_BU$ *and* ${}_AV$ *have finite* FP*-injective dimensions,* $\ker(f)$ *is a Ding injective left* A*-module,* $\ker(g)$ *is a Ding injective left* B*-module, the two sequences* $X \xrightarrow{f} \mathrm{Hom}_B(U,Y) \xrightarrow{\mathrm{Hom}_B(U,g)} \mathrm{Hom}_B(U,\mathrm{Hom}_A(V,X))$ *and* $Y \xrightarrow{g} \mathrm{Hom}_A(V,X) \xrightarrow{\mathrm{Hom}_A(V,f)} \mathrm{Hom}_A(V,\mathrm{Hom}_B(U,Y))$ *are exact, then* $[X,Y,f,g]$ *is a Ding injective left* Λ*-module.*

(2) *If* $[A,B,0,0]_\Lambda$ *has finite flat dimension,* ${}_\Lambda[A,B,0,0]$ *has finite projective or* FP*-injective dimension and* $[X,Y,f,g]$ *is a Ding injective left* Λ*-module, then* $\ker(f)$ *is a Ding injective left* A*-module and* $\ker(g)$ *is a Ding injective left* B*-module, the two sequences* $X \xrightarrow{f} \mathrm{Hom}_B(U,Y) \xrightarrow{\mathrm{Hom}_B(U,g)} \mathrm{Hom}_B(U,\mathrm{Hom}_A(V,X))$ *and* $Y \xrightarrow{g} \mathrm{Hom}_A(V,X) \xrightarrow{\mathrm{Hom}_A(V,f)} \mathrm{Hom}_A(V,\mathrm{Hom}_B(U,Y))$ *are exact.*

Proof. Obviously, $(A\times B)\ltimes(U\oplus V)$ is a left coherent ring.

(1) Note that $U\oplus V$ is a finitely presented left $A\times B$-module, $\mathrm{fd}((U\oplus V)_{A\times B})<\infty$, $\mathrm{pd}({}_{A\times B}(U\oplus V))<\infty$ or $\mathrm{FP}-\mathrm{id}({}_{A\times B}(U\oplus V))<\infty$. Since the two sequences $X \xrightarrow{f} \mathrm{Hom}_B(U,Y) \xrightarrow{\mathrm{Hom}_B(U,g)} \mathrm{Hom}_B(U,\mathrm{Hom}_A(V,X))$ and $Y \xrightarrow{g} \mathrm{Hom}_A(V,X) \xrightarrow{\mathrm{Hom}_A(V,f)} \mathrm{Hom}_A(V,\mathrm{Hom}_B(U,Y))$ are exact, it follows that the sequence

$$(X,Y) \xrightarrow{(f,g)} \mathrm{Hom}_{A\times B}(U\oplus V,(X,Y)) \xrightarrow{(f,g)_*} \mathrm{Hom}_{A\times B}(U\oplus V,\mathrm{Hom}_{A\times B}(U\oplus V,(X,Y)))$$

is also exact. Since $\ker(f)$ is a Ding injective left A-module and $\ker(g)$ is a Ding injective left B-module, $\ker(f,g)=(\ker(f),\ker(g))$ is a Ding injective left $A\times B$-module. By Theorem 3.2 (1), $[(X,Y),(f,g)]$ is a Ding injective left $(A\times B)\ltimes(U\oplus V)$-module. So $[X,Y,f,g]$ is a Ding injective left Λ-module.

(2) Note that $\mathrm{fd}(\mathbf{Z}(A\times B)_{(A\times B)\ltimes(U\oplus V)})<\infty$, $\mathrm{pd}({}_{(A\times B)\ltimes(U\oplus V)}\mathbf{Z}(A\times B))<\infty$ or $\mathrm{FP}-\mathrm{id}({}_{(A\times B)\ltimes(U\oplus V)}\mathbf{Z}(A\times B))<\infty$. Since $[X,Y,f,g]$ is a Ding injective left Λ-module, $[(X,Y),(f,g)]$ is a Ding injective left $(A\times B)\ltimes(U\oplus V)$-module. By Theorem 3.2 (2), the sequence $(X,Y) \xrightarrow{(f,g)} \mathrm{Hom}_{A\times B}(U\oplus V,(X,Y)) \xrightarrow{(f,g)_*} \mathrm{Hom}_{A\times B}(U\oplus V,\mathrm{Hom}_{A\times B}(U\oplus V,(X,Y)))$ is exact and $\ker(f,g)$ is a Ding injective left $A\times B$-module. So the sequences $X \xrightarrow{f} \mathrm{Hom}_B(U,Y) \xrightarrow{\mathrm{Hom}_B(U,g)} \mathrm{Hom}_B(U,\mathrm{Hom}_A(V,X))$ and $Y \xrightarrow{g} \mathrm{Hom}_A(V,X) \xrightarrow{\mathrm{Hom}_A(V,f)} \mathrm{Hom}_A(V,\mathrm{Hom}_B(U,Y))$ are exact, $\ker(f)$ is a Ding injective left A-module and $\ker(g)$ is a Ding injective left B-module. $\square$

Corollary 3.6. *Let* $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$ *be a left coherent ring,* U_A *and* V_B *have finite flat dimensions,* ${}_BU$ *and* ${}_AV$ *be finitely presented and have finite projective* *dimensions or* ${}_BU$ *and* ${}_AV$ *have finite* FP*-injective dimensions,* $[A,B,0,0]_\Lambda$ *have finite flat dimension,* ${}_\Lambda[A,B,0,0]$ *have finite projective or* FP*-injective dimension.*

(1) $[X, \mathrm{Hom}_A(V,X), 0, \mathrm{id}_{\mathrm{Hom}_A(V,X)}]$ *is a Ding injective left* Λ*-module if and only if* X *is a Ding injective left* A*-module.*

(2) $[\mathrm{Hom}_B(U,Y), Y, \mathrm{id}_{\mathrm{Hom}_B(U,Y)}, 0]$ *is a Ding injective left* Λ*-module if and only if* Y *is a Ding injective left* B*-module.*

(3) $[X,Y,0,0]$ *is a Ding injective left* Λ*-module if and only if* $\mathrm{Hom}_A(V,X)=0$, $\mathrm{Hom}_B(U,Y)=0$, X *is a Ding injective left* A*-module and* Y *is a Ding injective left* B*-module.*

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