Hyejong O; Hakho Hong; Jongsung Kim

Global existence of smooth solutions for the compressible viscous fluid flow with radiation in $\mathbb{R}^3$

Applications of Mathematics, Vol. 68 (2023), No. 5, 535–558

Persistent URL: `http://dml.cz/dmlcz/151831`

GLOBAL EXISTENCE OF SMOOTH SOLUTIONS FOR THE COMPRESSIBLE VISCOUS FLUID FLOW WITH RADIATION IN $\mathbb{R}^3$

Hyejong O, Hakho Hong, Jongsung Kim, Pyongyang

Received March 10, 2022. Published online June 13, 2023.

Abstract. This paper is concerned with the 3-D Cauchy problem for the compressible viscous fluid flow taking into account the radiation effect. For more general gases including ideal polytropic gas, we prove that there exists a unique smooth solutions in $[0,\infty)$, provided that the initial perturbations are small. Moreover, the time decay rates of the global solutions are obtained for higher-order spatial derivatives of density, velocity, temperature, and the radiative heat flux.

Keywords: radiation hydrodynamics; Navier-Stokes system with radiation; existence; convergence rate

MSC 2020: 76N10, 35B40, 35A01

1. Introduction

The importance of thermal radiation in physical problems increases as the temperature is raised. At moderate temperatures, the role of the radiation is primarily the one of transporting energy by radiative process, while at higher temperature, the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding field quantities. In this paper, we consider the following system for the compressible viscous fluid flow taking into account the radiation effect in $\mathbb{R}^3$:

$$\begin{cases} \varrho_t + \operatorname{div}(\varrho\mathbf{u}) = 0, \\ \varrho\mathbf{u}_t + \varrho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}, \\ \varrho e_t + \varrho\mathbf{u}\cdot\nabla e + P\operatorname{div}\mathbf{u} + \operatorname{div} q = \kappa\Delta\theta + 2\mu\mathbf{D} : \mathbf{D} + \lambda(\operatorname{div}\mathbf{u})^2, \\ -\nabla\operatorname{div}\mathbf{q} + \sigma_a\mathbf{q} + \widetilde{\sigma}\nabla(\theta^4) = 0. \end{cases} \tag{1.1}$$

Here, the unknowns are $(\varrho,\mathbf{u},\theta,\mathbf{q})$, where $\varrho = \varrho(\mathbf{x},t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x},t) = (u_1,u_2,u_3)(\mathbf{x},t)$, $\theta = \theta(\mathbf{x},t)$ for $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$, $t > 0$ denote the density, the velocity, the

DOI: 10.21136/AM.2023.0059-22

temperature of the fluid, respectively, and $\mathbf{q} = \mathbf{q}(\mathbf{x}, t) = (q_1, q_2, q_3)(\mathbf{x}, t)$ denotes the radiative heat flux. The pressure $P = P(\varrho, \theta)$ and inertial energy $e = e(\varrho, \theta)$ are the smooth functions of ϱ and θ: μ and λ are the constant viscosity coefficients, $\mu > 0$, $3\lambda + 2\mu \geqslant 0$; $\kappa > 0$ is the heat-conducting coefficient; $\sigma_a > 0$ is the absorption coefficient; $\widetilde{\sigma}$ is the positive constant defined by

$$0 < \widetilde{\sigma} = 4\pi\sigma_a \alpha\beta^{-4} \int_0^\infty \frac{s^3}{e^s - 1}\,\mathrm{d}s < \infty$$

with positive constants α and β, and $\mathbf{D} = \mathbf{D}(\mathbf{u})$ is the deformation tensor

$$D_{ij} = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big) \quad \text{and} \quad \mathbf{D} : \mathbf{D} = \sum_{i,j=1}^{3} D_{ij}^2.$$

In radiation hydrodynamics, due to the presence of radiation, the classical flow has to be coupled with radiation which is an assembly of photons (the photons are massless particles traveling at the speed of light) and need an a priori relativistic treatment. Hence, the whole problem to be considered is then a coupling between the standard Navier-Stokes equations for the fluid flow and a Boltzmann-type transport equation for the photon distribution, regarding the three basic interactions between photons and matter, namely, absorption, scattering and emission (see [32]). For the last decade, there have been many mathematical studies about the compressible Navier-Stokes-Botzmann models in radiation hydrodynamics: we refer to [6], [7], [5], [35], [8], [9], [10] for 1-D global solutions and to [24], [25], [26], [41], [42] for multi-D local solutions.

On the other hand, the compressible Navier-Stokes-Botzmann models include very complicated integro-differential equation, and hence, the physically valid approximate descriptions of radiative transfer need to be introduced. Such approximation models may be classified in two major catalogues: diffusion approximation model and P_1-approximation model. First, for the diffusion approximation (also called the Eddington approximation) model, which is particularly accurate if the specific intensity of radiation is almost isotropic (see [32]), the governing equations can be written as follows ([16], Appendix):

$$(1.2) \qquad \begin{cases} \varrho_t + \operatorname{div}(\varrho\mathbf{u}) = 0, \\ \varrho\mathbf{u}_t + \varrho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}, \\ \varrho e_t + \varrho\mathbf{u}\cdot\nabla e + P\operatorname{div}\mathbf{u} = \kappa\Delta\theta + 2\mu\mathbf{D}:\mathbf{D} + \lambda(\operatorname{div}\mathbf{u})^2 - \widetilde{\sigma}\theta^4 + \sigma_a n, \\ \dfrac{1}{c}n_t - \Delta n = \widetilde{\sigma}\theta^4 - \sigma_a n, \end{cases}$$

where $n = n(\mathbf{x}, t)$ is the radiation field and $c > 0$ is the the light speed. If we assume that the reciprocal of the light speed $1/c$ is very small, then letting $1/c = 0$ in $(1.2)_4$ and denoting $\mathbf{q} = -\nabla n$ yields that

$$\operatorname{div} \mathbf{q} = \widetilde{\sigma}\theta^4 - \sigma_a n \quad \text{and} \quad -\nabla \operatorname{div} \mathbf{q} + \sigma_a \mathbf{q} + \widetilde{\sigma}\nabla(\theta^4) = 0.$$

Therefore, we can formally obtain the limit system (1.1) from (1.2) as $1/c \to 0$. Next, for the P_1 approximation model under the two physical approximations, that is, "gray" approximation and P_1 approximation ([32], Chapter 3), the governing equations can be written as follows (see [19]):

$$\begin{cases} \varrho_t + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \varrho \mathbf{u}_t + \varrho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P = \mu \Delta \mathbf{u} + (\mu + \lambda)\nabla \operatorname{div} \mathbf{u}, \\ \varrho e_t + \varrho \mathbf{u} \cdot \nabla e + P \operatorname{div} \mathbf{u} = \kappa \Delta \theta + 2\mu \mathbf{D} : \mathbf{D} + \lambda(\operatorname{div} \mathbf{u})^2 - \widetilde{\sigma}\theta^4 + \sigma_a n_0, \\ \dfrac{1}{c}(n_0)_t + \operatorname{div} \mathbf{n}_1 = \widetilde{\sigma}\theta^4 - \sigma_a n_0, \\ \dfrac{1}{c}(\mathbf{n}_1)_t + \nabla n_0 = -\mathbf{n}_1, \end{cases} \tag{1.3}$$

which is valid when the distribution of photons is almost isotropic, where $n = n_0 + \mathbf{n}_1 \cdot \omega$ and $\omega \in S^2$ is the direction vector of photon. Taking $1/c = 0$ in $(1.3)_{4,5}$ and setting $\mathbf{q} = -\nabla n_0$, we obtain

$$\operatorname{div} \mathbf{q} = \widetilde{\sigma}\theta^4 - \sigma_a n_0 \quad \text{and} \quad -\nabla \operatorname{div} \mathbf{q} + \sigma_a \mathbf{q} + \widetilde{\sigma}\nabla(\theta^4) = 0.$$

Therefore, we can also formally obtain the limit system (1.1) from (1.3) as $1/c \to 0$. These arguments show that we can regard system (1.1) on the compressible viscous fluid flow with radiation as an infrarelativistic model of diffusion approximation model (1.2) and P_1-approximation model (1.3).

We will review some mathematical results for the approximation models in radiation hydrodynamics. For the diffusion approximation model (1.2), the global existence of solutions was proved in [20], [15] and [16] in the case of the 1-D initial-boundary value problem and 3-D Cauchy problem, respectively. Recently, for a nonequilibrium diffusion approximation model in thermally radiative magnetohydrodynamics, which is a generalization of system (1.3), the global existence of solutions was proved in [17] and [21] for 1-D and 3-D case, respectively. For 3-D Cauchy problem of system (1.3) with $\mu = \lambda = \kappa = 0$, the local existence of solutions was proved in [22]. Next, for a more simplified P_1-approximation model taking no account of energy balance law, the global existence of solutions was proved in [3], [47] for the multi-dimensional Cauchy problem. *Recently, the results were* *generalized into more physically relevant situation of polytropic flows by* [4]. For the P_1-approximation model (1.3), the nonrelativistic limit of local solutions was studied in [19] and in the case of $\mu = \lambda = \kappa = 0$, the nonequilibrium-diffusion limit at low Mach number was proved recently in [18]. Also, considerable progress has been obtained in the study of stability of basic wave patterns (such as shock wave, rarefaction wave and contact discontinuity) for the 1-D compressible fluid flow with radiation; we refer to [44], [13], [2] for system (1.1) and to [28], [23], [29], [30], [43], [27], [48], [37], [36], [45], [1] for system (1.1) with $\mu = \lambda = \kappa = 0$. Though, the multi-dimensional problem of system (1.1) was considered only in [46], where they showed the global existence of smooth solutions and decay rate for the 3-D Cauchy problem in the case of the ideal polytropic gas

$$P = R\varrho\theta, \quad e = c_v\theta, \tag{1.4}$$

where R, c_v are the positive constants. However, it does not seem to be suitable that the gas is ideal polytropic because the radiation effects will originate at high temperature. Thus, a natural question arises: can we show the similar result as in [46] for the more general gases including ideal polytropic gas? In this paper, we give the positive answer to this question for system (1.1) in 3-D with the initial condition and far field condition

$$(\varrho, \mathbf{u}, \theta)(\mathbf{x}, 0) = (\varrho_0, \mathbf{u}_0, \theta_0)(\mathbf{x}) \quad \text{and} \quad \lim_{|\mathbf{x}|\to\infty} (\varrho, \mathbf{u}, \theta, \mathbf{q})(\mathbf{x}, t) = (\bar{\varrho}, 0, \bar{\theta}, \mathbf{0}), \tag{1.5}$$

respectively, where $\bar{\varrho}$, $\bar{\theta}$ are the given positive constants, and with the gas law satisfying

$$P_\varrho(\varrho, \theta) > 0, \quad e_\theta(\varrho, \theta) > 0 \quad \text{for any } \varrho > 0,\ \theta > 0, \tag{1.6}$$

which is natural and more general than ideal polytropic gas (1.4).

Our main result is stated as follows:

Theorem 1.1. *Assume that* (1.6),

$$(\varrho_0 - \bar{\varrho}, \mathbf{u}_0, \theta_0 - \bar{\theta}) \in H^N(\mathbb{R}^3), \quad \inf_{\mathbf{x}\in\mathbb{R}^3} \varrho_0(\mathbf{x}) > 0, \quad \inf_{\mathbf{x}\in\mathbb{R}^3} \theta_0(\mathbf{x}) > 0 \tag{1.7}$$

for an integer $N \geqslant 3$. Then there exists a positive constant $\varepsilon_0 > 0$, dependent only on $\mu, \lambda, \kappa, \sigma_a, \widetilde{\sigma}, \bar{\varrho}, \bar{\theta}$, such that if

$$\|(\varrho_0 - \bar{\varrho}, \mathbf{u}_0, \theta_0 - \bar{\theta})\|_{H^3} \leqslant \varepsilon_0, \tag{1.8}$$

then the Cauchy problem (1.1), (1.5) *admits a unique solution* $(\varrho, \mathbf{u}, \theta, \mathbf{q})$ *on* $[0, \infty)$ *satisfying*

$$(\varrho-\bar{\varrho}, \mathbf{u}, \theta-\bar{\theta}) \in C([0,\infty), H^N(\mathbb{R}^3)), \quad (\mathbf{q}, \operatorname{div}\mathbf{q}) \in C([0,\infty), H^{N-1}(\mathbb{R}^3)),$$
$$\nabla\varrho \in L^2(0,\infty; H^{N-1}(\mathbb{R}^3)), \quad (\nabla\mathbf{u}, \nabla\theta, \mathbf{q}, \operatorname{div}\mathbf{q}) \in L^2(0,\infty; H^N(\mathbb{R}^3))$$

and

$$\sup_{t\in\mathbb{R}_+} (\|(\varrho-\bar{\varrho}, \mathbf{u}, \theta-\bar{\theta})(t)\|_{H^N}^2 + \|(\mathbf{q}, \operatorname{div}\mathbf{q})(t)\|_{H^{N-1}}^2) + \int_0^\infty \|\nabla\varrho\|_{H^{N-1}}^2 \,\mathrm{d}\tau + \int_0^\infty \|(\nabla\mathbf{u}, \nabla\theta, \mathbf{q}, \operatorname{div}\mathbf{q})\|_{H^N}^2 \,\mathrm{d}\tau \leqslant C\|(\varrho_0-\bar{\varrho}, \mathbf{u}_0, \theta_0-\bar{\theta})\|_{H^N}^2, \tag{1.9}$$

where C *is a positive constant independent of* $\mathbf{x}, t$ *and* ε_0.

Moreover, if $(\varrho_0-\bar{\varrho}, \mathbf{u}_0, \theta_0-\bar{\theta}) \in \dot{H}^{-s}(\mathbb{R}^3)$ *for some* $s \in [0, \frac{1}{2}]$*, then*

$$\|(\varrho-\bar{\varrho}, \mathbf{u}, \theta-\bar{\theta})(t)\|_{\dot{H}^{-s}}^2 \leqslant C_0, \tag{1.10}$$

and

$$\|(\varrho-\bar{\varrho}, \mathbf{u}, \theta-\bar{\theta})(t)\|_{H^N}^2 + \|(\mathbf{q}, \operatorname{div}\mathbf{q})(t)\|_{H^{N-1}}^2 \leqslant C_0(1+t)^{-s}, \tag{1.11}$$

where $\dot{H}^{-s}(\mathbb{R}^3)$ *denotes the homogeneous negative Sobolev space (see Definition* 1.1*).*

By Lemma 1.5, we obtain that for $p \in [\frac{3}{2}, 2)$, $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = 3(p^{-1} - \frac{1}{2}) \in (0, \frac{1}{2}]$. Then by Theorem 1.1, we have the following corollary of the usual L^p-L^2 type of decay result:

Corollary 1.1. *Under the assumptions of Theorem* 1.1 *except that we replace the* $\dot{H}^{-s}$ *assumption by that* $(\varrho_0-\bar{\varrho}, \mathbf{u}_0, \theta_0-\bar{\theta}) \in L^p(\mathbb{R}^3)$ *for some* $p \in [\frac{3}{2}, 2)$*, the following decay results hold:*

$$\|(\varrho-\bar{\varrho}, \mathbf{u}, \theta-\bar{\theta})(t)\|_{H^N}^2 + \|(\mathbf{q}, \operatorname{div}\mathbf{q})(t)\|_{H^{N-1}}^2 \lesssim (1+t)^{-3(1/p-1/2)}.$$

Remark 1.1.

(1) For the global existence of the solution, we only assume that the H^3-norm of initial data is small, while the higher-order Sobolev norms can be arbitrarily large (see (1.8)).

(2) It is not clear yet that the constraint $s \in [0, \frac{1}{2}]$ in the decay rate (1.11) of Theorem 1.1 can be extended to $s \in [0, \frac{3}{2})$.

R e m a r k 1.2. We briefly review key analytical technique. To prove Theorem 1.1, we will use the energy method originally developed by Guo-Wang [12], which relies essentially on the following two main steps: a priori estimates at lth level and negative Sobolev norm estimate. This idea was applied to the Cauchy problems for the compressible fluid flows in $\mathbb{R}^3$; we refer to [40] for Navier-Stokes-Poisson system, [39] for Navier-Stokes-Kortweg system, [11] for nematic liquid crystal flows and [34] for quantum Navier-Stokes system. Thus, the coupling between the Navier-Stokes equations and the radiative transport equation brings some trouble to get a priori estimates. To this end, we first obtain the basic energy estimate (2.10) for general gas satisfying (1.6). Next, we reformulate system (1.2) into (2.3) to get kth higher order estimates (see Lemma 2.2–Lemma 2.4). Last, we get estimate (3.1) in negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$ for $s \in (0, \frac{1}{2}]$ (see Lemma 3.1). Then, combining basic energy estimate, kth higher order estimates and negative Sobolev norm estimate, we will prove Theorem 1.1 in the last section.

Notation 1.1. *In this paper, $L^p(\mathbb{R}^3)$ and $W_p^k(\mathbb{R}^3)$ denote the usual Lebesgue and Sobolev spaces on $\mathbb{R}^3$, with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. When $p = 2$, we denote $W_p^k(\mathbb{R}^3)$ by $H^k(\mathbb{R}^3)$ with the norm $\|\cdot\|_{H^k}$ and $\|\cdot\|_{H^0} = \|\cdot\|$ will be used to denote the usual L^2-norm. The notation $\|(A_1, A_2, \ldots, A_l)\|_{H^k}$ means the summation of $\|A_i\|_{H^k}$ from $i = 1$ to $i = l$. For an integer m, the symbol ∇^m denotes the summation of all terms D^α with the multi-index α satisfying $|\alpha| = m$. We omit the spatial domain $\mathbb{R}^3$ in integrals for convenience. We also employ the notation $a \lesssim b$ to mean that $a \leqslant Cb$ for a universal constant $C > 0$ that only depends on the parameters coming from the problem.*

In order to establish the negative Sobolev estimates, we should review the following useful results. To this end, let us first introduce the following necessary definition.

Definition 1.1. For $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^3)$ is defined as the homogeneous Sobolev space of f, with the norm

$$\|f\|_{\dot{H}^s} = \|\Lambda^s f\|,$$

where Λ^s is defined by

$$(\Lambda^s f)(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i} \mathbf{x}\cdot\xi} \,\mathrm{d}\xi,$$

where $\hat{f}(\xi)$ is the Fourier transform of f, $\hat{f}(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^3} f(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{x}\cdot\xi} \,\mathrm{d}\mathbf{x}$.

Moreover, we need the following standard results which will be used extensively in our estimates.

Lemma 1.1 (Gagliardo-Nirenberg inequality, [31] or [40], Lemma A.1)**.** *Let l, s and k be any real numbers satisfying $0 \leqslant l,\ s < k$, and let $p, r, q \in [1, \infty]$ and $l/k \leqslant \theta \leqslant 1$ such that*

$$\frac{l}{3} - \frac{1}{p} = \Big(\frac{s}{3} - \frac{1}{r}\Big)(1-\theta) + \Big(\frac{k}{3} - \frac{1}{q}\Big)\theta.$$

Then for any $u \in W_q^k(\mathbb{R}^3)$ we have

$$\|\nabla^l u\|_{L^p} \lesssim \|\nabla^s u\|_{L^r}^{1-\theta} \|\nabla^k u\|_{L^q}^{\theta}. \tag{1.12}$$

Lemma 1.2 ([33], Lemma 2.5)**.** *Let $f(\varphi)$ and $f(\varphi, w)$ be smooth functions of φ and (φ, w), respectively, with bounded derivatives of any order, and $\|\varphi\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)} \lesssim 1$. Then for any integer $m \geqslant 1$ we have*

$$\begin{aligned} \|\nabla^m f(\varphi)\|_{L^p} &\leqslant C\|\nabla^m \varphi\|_{L^p}, \\ \|\nabla^m f(\varphi, w)\|_{L^p} &\leqslant C\|\nabla^m (\varphi, w)\|_{L^p} \end{aligned} \tag{1.13}$$

for any $1 \leqslant p \leqslant \infty$, where C may depend on f and m.

Lemma 1.3 ([33], Lemma 2.6)**.** *Let α be any multi-index with $|\alpha| = k$ and $1 < p < \infty$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} \|D^\alpha (fg)\|_{L^p} &\leqslant C(\|f\|_{L^{p_1}} \|\nabla^k g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}}), \\ \|[D^\alpha, f]g\|_{L^p} &\leqslant C(\|\nabla f\|_{L^{p_1}} \|\nabla^{k-1} g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}}), \end{aligned} \tag{1.14}$$

where $f, g \in S$ is the Schwarz class, $1 \leqslant p_2, p_3 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, and $[D^\alpha, f]g = D^\alpha(fg) - fD^\alpha g$.

By the Parseval theorem and Hölder's inequality, it is easy to check the following result (see [40]).

Lemma 1.4. *Let $s \geqslant 0$ and $l \geqslant 0$. Then we have*

$$\|\nabla^l f\| \lesssim \|\nabla^{l+1} f\|^{1-\theta} \|f\|_{\dot{H}^{-s}}^{\theta}, \quad \text{with } \theta = \frac{1}{l+s+1}. \tag{1.15}$$

If $s \in (0, 3)$, $\Lambda^{-s} g$ is the Riesz potential. Then we have the following L^p type inequality by the Hardy-Littlewood-Sobolev theorem (see [38], pp. 119, Theorem 1):

Lemma 1.5. *Let $0 < s < 3$, $1 < p < q < \infty$ and $1/q + s/3 = 1/p$. Then we have*

$$\|\Lambda^{-s} f\|_{L^q} \lesssim \|f\|_{L^p}. \tag{1.16}$$

Last, we need the following estimate on differential inequality:

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t}+c_0(f(t))^{1+1/(l+s)}\leqslant 0 \\ \Rightarrow f(t)\leqslant\Big(f(0)^{-1/(l+s)}+\frac{c_0t}{l+s}\Big)^{-(l+s)}\lesssim(1+t)^{-(l+s)}. \tag{1.17}$$

2. A priori estimate at kth level

In this section, we will derive the a priori estimates which play a crucial role in the proof of Theorem 1.1. By using the thermodynamical relation

$$-\varrho^2e_\varrho(\varrho,\theta)=\theta P_\theta(\varrho,\theta)-P(\varrho,\theta),\quad \hat{e}_S(\varrho,S)=\theta\quad\text{and}\quad \hat{e}_\varrho(\varrho,S)=\varrho^2P(\varrho,\theta), \tag{2.1}$$

where $\hat{e}(\varrho,S)=e(\varrho,\theta)$ and S is the entropy of gas, (see [14], (1.6) and (1.7)), we rewrite $(1.2)_3$ as

$$\begin{aligned} \theta_t+\mathbf{u}\cdot\nabla\theta+\frac{\theta P_\theta(\varrho,\theta)}{\varrho e_\theta(\varrho,\theta)}\operatorname{div}\mathbf{u}-\frac{\kappa}{\varrho e_\theta(\varrho,\theta)}\Delta\theta &\\ =\frac{2\mu\mathbf{D}:\mathbf{D}}{\varrho e_\theta(\varrho,\theta)}+\frac{\lambda(\operatorname{div}\mathbf{u})^2}{\varrho e_\theta(\varrho,\theta)}-\frac{\operatorname{div}\mathbf{q}}{\varrho e_\theta(\varrho,\theta)}&\\ S_t+\mathbf{u}\cdot\nabla S-\frac{\kappa}{\varrho\theta}\Delta\theta=\frac{2\mu\mathbf{D}:\mathbf{D}}{\varrho\theta}+\frac{\lambda(\operatorname{div}\mathbf{u})^2}{\varrho\theta}-\frac{\operatorname{div}\mathbf{q}}{\varrho\theta}&. \end{aligned} \tag{2.2}$$

Setting

$$\varphi=\varrho-\bar{\varrho}\quad\text{and}\quad\zeta=\theta-\bar{\theta},$$

and using $(2.2)_1$, we reformulate system (1.1) as

$$\begin{aligned} &\varphi_t+\bar{\varrho}\operatorname{div}\mathbf{u}=g_1,\\ &\mathbf{u}_t-\frac{\mu}{\bar{\varrho}}\Delta\mathbf{u}-\frac{\mu+\lambda}{\bar{\varrho}}\nabla\operatorname{div}\mathbf{u}+\frac{P_\varrho(\bar{\varrho},\bar{\theta})}{\bar{\varrho}}\nabla\varphi+\frac{P_\theta(\bar{\varrho},\bar{\theta})}{\bar{\varrho}}\nabla\zeta=\mathbf{g}_2,\\ &\zeta_t+\frac{\bar{\theta}P_\theta(\bar{\varrho},\bar{\theta})}{e_\theta(\bar{\varrho},\bar{\theta})}\operatorname{div}\mathbf{u}=\frac{\kappa}{\bar{\varrho}e_\theta(\bar{\varrho},\bar{\theta})}\Delta\theta-\frac{\operatorname{div}\mathbf{q}}{\bar{\varrho}e_\theta(\bar{\varrho},\bar{\theta})}+g_3,\\ &-\nabla\operatorname{div}\mathbf{q}+\sigma_a\mathbf{q}+4\widetilde{\sigma}\bar{\theta}\nabla\zeta=\mathbf{g}_4, \end{aligned} \tag{2.3}$$

where g_1, $\mathbf{g}_2$, g_3 and $\mathbf{g}_4$ are defined respectively by

$$\begin{aligned} g_1&=-\varphi\operatorname{div}\mathbf{u}-\mathbf{u}\cdot\nabla\varphi,\\ \mathbf{g}_2&=-(\mathbf{u},\nabla)\mathbf{u}+h_1(\varphi,\zeta)\nabla\varphi+h_2(\varphi,\zeta)\nabla\zeta-h_3(\varphi)(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}), \end{aligned} \tag{2.4}$$

$$\begin{aligned} h_1(\varphi,\zeta)&=\frac{P_\varrho(\bar{\varrho},\bar{\theta})}{\bar{\varrho}}-\frac{P_\varrho(\varrho,\theta)}{\varrho},\quad h_2(\varphi,\zeta)=\frac{P_\theta(\bar{\varrho},\bar{\theta})}{\bar{\varrho}}-\frac{P_\theta(\varrho,\theta)}{\varrho},\\ h_3(\varphi)&=\bar{\varrho}^{-1}-\varrho^{-1}, \end{aligned} \tag{2.5}$$

$$(2.6)\quad g_3 = -\mathbf{u}\cdot\nabla\zeta - \kappa h_4(\varphi,\zeta)\Delta\zeta + \frac{2\mu\mathbf{D}:\mathbf{D}}{\varrho e_\theta(\varrho,\theta)} + \frac{\lambda(\operatorname{div}\mathbf{u})^2}{\varrho e_\theta(\varrho,\theta)} + h_5(\varphi,\zeta)\operatorname{div}\mathbf{u} - h_4(\varphi,\zeta)\operatorname{div}\mathbf{q},$$

$$h_4(\varphi,\zeta) = \frac{1}{\bar\varrho e_\theta(\bar\varrho,\bar\theta)} - \frac{1}{\varrho e_\theta(\varrho,\theta)},\quad h_5(\varphi,\zeta) = \frac{\bar\theta P_\theta(\bar\varrho,\bar\theta)}{\bar\varrho e_\theta(\bar\varrho,\bar\theta)} - \frac{\theta P_\theta(\varrho,\theta)}{\varrho e_\theta(\varrho,\theta)}$$

and

$$(2.7)\qquad \mathbf{g}_4 = 4\widetilde{\sigma}(\zeta^3 + 3\bar\theta\zeta^2 + 3\bar\theta^2\zeta)\nabla\zeta.$$

We suppose that $(\varphi,\mathbf{u},\zeta,\mathbf{q}) \in X_N(0,T)$ is the solution to system (2.3) for any constant $T>0$ and an integer $N \geqslant 3$, where $X_N(I)$ is defined by

$$X_N(I) = \{(\varphi,\mathbf{u},\zeta,\mathbf{q}) \mid (\varphi,\mathbf{u},\zeta) \in C(I;H^N(\mathbb{R}^3)),\ (\mathbf{q},\operatorname{div}\mathbf{q}) \in C(I;H^{N-1}(\mathbb{R}^3)),\ \nabla\varphi \in L^2(I;H^{N-1}(\mathbb{R}^3)),\ (\nabla\mathbf{u},\nabla\zeta,\mathbf{q},\operatorname{div}\mathbf{q}) \in L^2(I;H^N(\mathbb{R}^3))\}$$

for any interval $I \subset [0,\infty)$. Also, we assume

$$(2.8)\qquad \sup_{0\leqslant t\leqslant T}(\|(\varphi,\mathbf{u},\zeta)(t)\|_{H^3} + \|\mathbf{q}(t)\|_{H^2}) \leqslant \varepsilon_1$$

for sufficiently small $\varepsilon_1>0$. By using (2.8), we can choose $\eta_1>0$ such that

$$(2.9)\qquad 0 < \frac{\bar\varrho}{2} \leqslant \varrho(x,t) \leqslant 2\bar\varrho \quad\text{and}\quad 0 < \frac{\bar\theta}{2} \leqslant \theta(x,t) \leqslant 2\bar\theta.$$

We first give the following basic energy estimate:

Lemma 2.1. *Under assumption* (2.8), *we have*

$$(2.10)\quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int \varrho\mathcal{E}\,\mathrm{d}x\Big) + \frac{\mu}{2}\|\nabla\mathbf{u}\|^2 + \frac{\kappa}{6\bar\theta}\|\nabla\zeta\|^2 + \frac{1}{32\widetilde{\sigma}\bar\theta^2}\|\operatorname{div}\mathbf{q}\|^2 + \frac{\sigma_a}{32\widetilde{\sigma}\bar\theta^2}\|\mathbf{q}\|^2 \leqslant 0,$$

where

$$(2.11)\qquad \mathcal{E} := (e-\bar e) + \frac{1}{2}\mathbf{u}^2 - \bar\theta(S-\overline{S}) + \overline{P}\Big(\frac{1}{\varrho} - \frac{1}{\bar\varrho}\Big),$$

where $\bar e = e(\bar\varrho,\bar\theta)$, $\overline{S} = S(\bar\varrho,\bar\theta)$ *and* $\overline{P} = P(\bar\varrho,\bar\theta)$.

Remark 2.1. It is easy to check (see [14]) that by (2.11) and (2.1), we have

$$(2.12)\qquad \int \varrho\mathcal{E}\,\mathrm{d}x \backsimeq \|(\varphi,\mathbf{u},\zeta)(t)\|^2.$$

P r o o f. By using (1.1) and (2.2), we obtain

(2.13)
$$
\begin{aligned}
(\varrho\mathcal{E})_t + \operatorname{div}(\varrho\mathbf{u}\mathcal{E}) &= \varrho\mathcal{E}_t + \varrho\mathbf{u}\cdot\nabla\mathcal{E}\\
&= \Big(1-\frac{\bar\theta}{\theta}\Big)(\kappa\Delta\theta + 2\mu\mathbf{D}:\mathbf{D} + \lambda(\operatorname{div}\mathbf{u})^2 - \operatorname{div}\mathbf{q}) - P\operatorname{div}\mathbf{u}\\
&\quad + \mu\Delta\mathbf{u}\cdot\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\cdot\mathbf{u} - \nabla P\cdot\mathbf{u} + \overline{P}\operatorname{div}\mathbf{u}.
\end{aligned}
$$

It is easy to check that

$$
\begin{gathered}
\int\Big(1-\frac{\bar\theta}{\theta}\Big)\Delta\theta\,\mathrm{d}x = -\int\frac{\bar\theta}{\theta^2}|\nabla\zeta|^2\,\mathrm{d}x,\\
-P\operatorname{div}\mathbf{u} - \mathbf{u}\cdot\nabla P + \overline{P}\operatorname{div}\mathbf{u} = -\operatorname{div}[(P-\overline{P})\mathbf{u}],\\
\int\Big(1-\frac{\bar\theta}{\theta}\Big)\operatorname{div}\mathbf{q}\,\mathrm{d}x = -\int\frac{\bar\theta}{\theta^2}\nabla\zeta\cdot\mathbf{q}\,\mathrm{d}x.
\end{gathered}
\tag{2.14}
$$

Multiplying $(2.3)_4$ by $\mathbf{q}/(4\widetilde{\sigma}\theta^2)$ and integrating the resulting equality for $x\in\mathbb{R}^3$, we get

$$
\begin{aligned}
\frac{1}{4\widetilde{\sigma}}\int\frac{1}{\theta^2}|\operatorname{div}\mathbf{q}|^2\,\mathrm{d}x + \frac{\sigma_a}{4\widetilde{\sigma}}\int\frac{1}{\theta^2}|\mathbf{q}|^2\,\mathrm{d}x + \int\frac{\bar\theta}{\theta^2}\nabla\zeta\cdot\mathbf{q}\,\mathrm{d}x&\\
= \frac{1}{4\widetilde{\sigma}}\int\frac{\mathbf{g}_4\cdot\mathbf{q}}{\theta^2}\,\mathrm{d}x + \frac{1}{2\widetilde{\sigma}}\int\frac{\operatorname{div}\mathbf{q}}{\theta^3}(\nabla\zeta\cdot\mathbf{q})\,\mathrm{d}x.&
\end{aligned}
\tag{2.15}
$$

Integrating (2.13) for $x\in\mathbb{R}^3$ and using (2.14), (2.15) and (2.7), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int\varrho\mathcal{E}\,\mathrm{d}x\Big) + \mu\|\nabla\mathbf{u}\|^2 + (\mu+\lambda)\|\operatorname{div}\mathbf{u}\|^2 + \kappa\int\frac{\bar\theta}{\theta^2}|\nabla\zeta|^2\,\mathrm{d}x&\\
+ \frac{1}{4\widetilde{\sigma}}\int\frac{1}{\theta^2}|\operatorname{div}\mathbf{q}|^2\,\mathrm{d}x + \frac{\sigma_a}{4\widetilde{\sigma}}\int\frac{1}{\theta^2}|\mathbf{q}|^2\,\mathrm{d}x = J_1 + J_2,&
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
J_1 &= \int\frac{\zeta}{\theta}(2\mu\mathbf{D}:\mathbf{D} + \lambda(\operatorname{div}\mathbf{u})^2)\,\mathrm{d}x,\\
J_2 &= \int\frac{1}{\theta^2}(\zeta^3 + 3\bar\theta\zeta^2 + 3\bar\theta^2\zeta)\nabla\zeta\cdot\mathbf{q}\,\mathrm{d}x + \frac{1}{2\widetilde{\sigma}}\int\frac{\operatorname{div}\mathbf{q}}{\theta^3}(\nabla\zeta\cdot\mathbf{q})\,\mathrm{d}x.
\end{aligned}
\tag{2.17}
$$

Using (2.8) and (2.9), we obtain from (2.17) that

$$
|J_1| \lesssim \varepsilon_1\|\nabla\mathbf{u}\|^2,\quad |J_2| \lesssim \varepsilon_1(\|\nabla\zeta\|^2 + \|\mathbf{q}\|^2 + \|\operatorname{div}\mathbf{q}\|^2).
\tag{2.18}
$$

Substituting (2.18) into (2.16) and using (2.9) and the smallness of $\varepsilon_1 > 0$, we get (2.10). The proof of Lemma 2.1 is completed. $\square$

Lemma 2.2. *Under assumption* (2.8), *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^k\mathbf{u}\|^2+\frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2}\|\nabla^k\varphi\|^2+\frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}\|\nabla^k\zeta\|^2\Big)+\frac{\mu}{\bar\varrho}\|\nabla^{k+1}\mathbf{u}\|^2 \\ +\frac{\kappa}{\bar\varrho\bar\theta}\|\nabla^{k+1}\zeta\|^2+\frac{1}{4\widetilde\sigma\bar\varrho\bar\theta^2}\|\nabla^k\operatorname{div}\mathbf{q}\|^2+\frac{\sigma_a}{4\widetilde\sigma\bar\varrho\bar\theta^2}\|\nabla^k\mathbf{q}\|^2 \\ \lesssim\varepsilon_1(\|\nabla^{k+1}\varphi\|^2+\|\nabla\varphi\|^2+\|\nabla\mathbf{u}\|^2+\|\nabla\zeta\|^2) \tag{2.19}$$

for $k=1,\dots,N-1$.

P r o o f. Applying ∇^k to (2.3) yields

$$\begin{aligned}&\nabla^k\varphi_t+\bar\varrho\operatorname{div}\nabla^k\mathbf{u}=\nabla^k g_1,\\ &\nabla^k\mathbf{u}_t-\frac{\mu}{\bar\varrho}\Delta\nabla^k\mathbf{u}-\frac{(\mu+\lambda)}{\bar\varrho}\nabla^{k+1}\operatorname{div}\mathbf{u}+\frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho}\nabla^{k+1}\varphi\\ &\qquad+\frac{P_\theta(\bar\varrho,\bar\theta)}{\bar\varrho}\nabla^{k+1}\zeta=\nabla^k\mathbf{g}_2,\\ &\nabla^k\zeta_t+\frac{\bar\theta P_\theta(\bar\varrho,\bar\theta)}{e_\theta(\bar\varrho,\bar\theta)}\operatorname{div}\nabla^k\mathbf{u}=\frac{\kappa}{\bar\varrho e_\theta(\bar\varrho,\bar\theta)}\Delta\nabla^k\zeta+\frac{\nabla^k\operatorname{div}\mathbf{q}}{\bar\varrho e_\theta(\bar\varrho,\bar\theta)}+\nabla^k g_3,\\ &-\nabla^{k+1}\operatorname{div}\mathbf{q}+\sigma_a\nabla^k\mathbf{q}+4\widetilde\sigma\bar\theta\nabla^{k+1}\zeta=\nabla^k\mathbf{g}_4.\end{aligned} \tag{2.20}$$

Multiplying equations $(2.20)_2$, $(2.20)_3$ and $(2.20)_4$ by $\nabla^k\mathbf{u}$, $(e_\theta(\bar\varrho,\bar\theta)/\bar\theta)\nabla^k\zeta$ and $(\sigma_a/(4\widetilde\sigma\bar\varrho\bar\theta^2))\nabla^k\mathbf{q}$, respectively, and using $(2.20)_1$ and (2.4)–(2.7), we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^k\mathbf{u}\|^2+\frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2}\|\nabla^k\varphi\|^2+\frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}\|\nabla^k\zeta\|^2\Big)+\frac{\kappa}{\bar\varrho\bar\theta}\int|\nabla^{k+1}\zeta|^2\,\mathrm{d}x\\ &\quad+\int\Big(\frac{\mu}{\bar\varrho}|\nabla^{k+1}\mathbf{u}|^2+\frac{(\mu+\lambda)}{\bar\varrho}|\operatorname{div}\nabla^k\mathbf{u}|^2\Big)\,\mathrm{d}x\\ &\quad+\frac{\sigma_a}{4\widetilde\sigma\bar\varrho\bar\theta^2}\int|\nabla^k\operatorname{div}\mathbf{q}|^2\,\mathrm{d}x+\frac{1}{4\widetilde\sigma\bar\varrho\bar\theta^2}\int|\nabla^k\mathbf{q}|^2\,\mathrm{d}x\\ &=\frac{P_\theta(\bar\varrho,\bar\theta)}{\bar\varrho^2}I_k^1+I_k^2+\frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}I_k^3+\frac{1}{\bar\varrho\bar\theta^2}I_k^4,\end{aligned} \tag{2.21}$$

where

$$I_k^1=-\int\nabla^k(\varphi\operatorname{div}\mathbf{u}+\mathbf{u}\nabla\varphi)\nabla^k\varphi\,\mathrm{d}x, \tag{2.22}$$

$$\begin{aligned}I_k^2=&-\int\nabla^k[(\mathbf{u},\nabla)\mathbf{u}]\cdot\nabla^k\mathbf{u}\,\mathrm{d}x\\ &+\int\nabla^k[h_1(\varphi,\zeta)\nabla\varphi+h_2(\varphi,\zeta)\nabla\zeta]\cdot\nabla^k\mathbf{u}\,\mathrm{d}x\\ &-\int\nabla^k[h_3(\varphi)(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u})]\cdot\nabla^k\mathbf{u}\,\mathrm{d}x,\\ =&\,I_k^{21}+I_k^{22}+I_k^{23},\end{aligned} \tag{2.23}$$

$$
\begin{aligned}
(2.24)\qquad I_k^3 =& -\int \nabla^k[\mathbf{u}\cdot\nabla\zeta - h_5(\varphi,\zeta)\operatorname{div}\mathbf{u} + h_4(\varphi,\zeta)\operatorname{div}\mathbf{q}]\nabla^k\zeta\,\mathrm{d}x\\
&-\kappa\int \nabla^k[h_4(\varphi,\zeta)\Delta\zeta]\nabla^k\zeta\,\mathrm{d}x\\
&+\int \nabla^k\Big[\frac{2\mu\mathbf{D}:\mathbf{D}+\lambda(\operatorname{div}\mathbf{u})^2}{\varrho e_\theta(\varrho,\theta)}\Big]\nabla^k\zeta\,\mathrm{d}x\\
=& I_k^{31}+I_k^{32}+I_k^{33},
\end{aligned}
$$

and

$$
(2.25)\qquad I_k^4 = -\int \nabla^k[(\zeta^3+3\bar\theta\zeta^2+3\bar\theta^2\zeta)\nabla\zeta]\cdot\nabla^k\mathbf{q}\,\mathrm{d}x.
$$

We estimate the right-hand side in (2.21). For I_k^1, using Hölder's inequality, (2.9) and (1.12), we obtain from (2.22) that

$$
\begin{aligned}
(2.26)\qquad I_k^1 \overset{(1.14)}{\lesssim}& (\|\varphi\|_{L^3}\|\nabla^{k+1}\mathbf{u}\| + \|\nabla\mathbf{u}\|\|\nabla^k\varphi\|_{L^3}\\
&+\|\mathbf{u}\|_{L^3}\|\nabla^{k+1}\varphi\| + \|\nabla\varphi\|\|\nabla^k\mathbf{u}\|_{L^3})\|\nabla^k\varphi\|_{L^6}\\
\overset{(1.12)}{\lesssim}& (\|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2}\|\nabla^{k+1}\mathbf{u}\| + \|\nabla\mathbf{u}\|\|\nabla\varphi\|^{1/(2k)}\|\nabla^{k+1}\varphi\|^{1-1/(2k)}\\
&+\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\|\nabla^{k+1}\varphi\|\\
&+\|\nabla\varphi\|\|\nabla\mathbf{u}\|^{1/(2k)}\|\nabla^{k+1}\mathbf{u}\|^{1-1/(2k)})\|\nabla^{k+1}\varphi\|\\
\overset{(2.8)}{\lesssim}& \varepsilon_1(\|\nabla^{k+1}\varphi\|^2+\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla\varphi\|^2+\|\nabla\mathbf{u}\|^2).
\end{aligned}
$$

For I_k^2, using Hölder's inequality, (2.9) and (1.12), we obtain from (2.23) that

$$
\begin{aligned}
(2.27)\qquad I_k^{21} \overset{(1.14)}{\lesssim}& (\|\mathbf{u}\|_{L^3}\|\nabla^{k+1}\mathbf{u}\| + \|\nabla\mathbf{u}\|\|\nabla^k\mathbf{u}\|_{L^3})\|\nabla^k\mathbf{u}\|_{L^6}\\
\overset{(1.12)}{\lesssim}& (\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\|\nabla^{k+1}\mathbf{u}\|\\
&+\|\nabla\mathbf{u}\|\|\nabla\mathbf{u}\|^{1/(2k)}\|\nabla^{k+1}\mathbf{u}\|^{1-1/(2k)})\|\nabla^{k+1}\mathbf{u}\|\\
\overset{(2.8)}{\lesssim}& \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla\mathbf{u}\|^2),\\
(2.28)\qquad I_k^{22} \overset{(1.14)}{\lesssim}& (\|h_1(\varphi,\zeta)\|_{L^3}\|\nabla^{k+1}\varphi\| + \|h_2(\varphi,\zeta)\|_{L^3}\|\nabla^{k+1}\zeta\|)\|\nabla^k\mathbf{u}\|_{L^6}\\
&+(\|\nabla\varphi\|\|\nabla^k h_1(\varphi,\zeta)\|_{L^6} + \|\nabla\zeta\|\|\nabla^k h_2(\varphi,\zeta)\|_{L^6})\|\nabla^k\mathbf{u}\|_{L^3}\\
\overset{(2.8),(1.13)}{\lesssim}& \varepsilon_1(\|\nabla^{k+1}\varphi\|+\|\nabla^{k+1}\zeta\|)\|\nabla^{k+1}\mathbf{u}\| + (\|\nabla\varphi\|+\|\nabla\zeta\|)\\
&\times(\|\nabla^{k+1}\varphi\|+\|\nabla\zeta\|^{k+1})\|\nabla\mathbf{u}\|^{1/(2k)}\|\nabla^{k+1}\mathbf{u}\|^{1-1/(2k)}\\
\lesssim& \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla^{k+1}\varphi\|^2+\|\nabla^{k+1}\zeta\|^2+\|\nabla\varphi\|^2+\|\nabla\zeta\|^2),
\end{aligned}
$$

$$
\begin{aligned}
(2.29)\quad I_k^{23} \quad &= \quad -\mu\int \nabla^{k-1}[\operatorname{div}(h_3(\varphi)\nabla\mathbf{u})-\nabla h_3(\varphi)\nabla\mathbf{u}]\cdot\nabla^{k+1}\mathbf{u}\,\mathrm{d}x\\
&\quad -(\lambda+\mu)\int \nabla^{k-1}[\nabla(h_3(\varphi)\operatorname{div}\mathbf{u})-\nabla h_3(\varphi)\operatorname{div}\mathbf{u}]\cdot\nabla^{k+1}\mathbf{u}\,\mathrm{d}x\\
&\overset{(1.14)}{\lesssim} (\|h_3(\varphi)\|_{L^\infty}\|\nabla^{k+1}\mathbf{u}\|+\|\nabla\mathbf{u}\|_{L^3}\|\nabla^k h_3(\varphi)\|_{L^6}\\
&\quad +\|\nabla h_3(\varphi)\|_{L^3}\|\nabla^k\mathbf{u}\|_{L^6})\|\nabla^{k+1}\mathbf{u}\|\\
&\overset{(2.8),(1.13)}{\lesssim} \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla^{k+1}\varphi\|^2).
\end{aligned}
$$

Therefore, we have

$$
(2.30)\quad I_k^2\lesssim\varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla^{k+1}\varphi\|^2+\|\nabla^{k+1}\zeta\|^2+\|\nabla\mathbf{u}\|^2+\|\nabla\varphi\|^2+\|\nabla\zeta\|^2).
$$

For I_k^3, using Hölder's inequality, (2.9) and (1.12), we obtain from (2.24) that

$$
\begin{aligned}
I_k^{31} &\overset{(1.14)}{\lesssim} (\|\mathbf{u}\|_{L^3}\|\nabla^{k+1}\zeta\|+\|\nabla\zeta\|_{L^{3/2}}\|\nabla^k\mathbf{u}\|_{L^6})\|\nabla^k\zeta\|_{L^6}\\
&\quad +(\|h_5(\varphi,\zeta)\|_{L^3}\|\nabla^{k+1}\mathbf{u}\|+\|\nabla\mathbf{u}\|_{L^{3/2}}\|\nabla^k h_5(\varphi,\zeta)\|_{L^6})\|\nabla^k\zeta\|_{L^6}\\
&\quad +(\|h_4(\varphi,\zeta)\|_{L^3}\|\nabla^k\operatorname{div}\mathbf{q}\|+\|\nabla\mathbf{q}\|_{L^{3/2}}\|\nabla^k h_4(\varphi,\zeta)\|_{L^6})\|\nabla^k\zeta\|_{L^6}\\
&\overset{(2.8),(1.13)}{\lesssim} \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla^{k+1}\varphi\|^2+\|\nabla^{k+1}\zeta\|^2+\|\nabla^k\operatorname{div}\mathbf{q}\|),\\
I_k^{32} &= \kappa\int\nabla^{k-1}[\operatorname{div}(h_4(\varphi,\zeta)\nabla\zeta)-\nabla h_4(\varphi,\zeta)\nabla\zeta]\cdot\nabla^{k+1}\zeta\,\mathrm{d}x\\
&\overset{(1.14)}{\lesssim} (\|h_4(\varphi,\zeta)\|_{L^\infty}\|\nabla^{k+1}\zeta\|+\|\nabla\zeta\|_{L^3}\|\nabla^k h_4(\varphi,\zeta)\|_{L^6}\\
&\quad +\|\nabla h_4(\varphi,\zeta)\|_{L^3}\|\nabla^k\zeta\|_{L^6})\|\nabla^{k+1}\zeta\|\\
&\overset{(2.8),(1.13)}{\lesssim} \varepsilon_1(\|\nabla^{k+1}\zeta\|^2+\|\nabla^{k+1}\varphi\|^2),\\
I_k^{33} &\overset{(1.14)}{\lesssim} \Big(\Big\|\frac{\mathbf{D}}{\varrho e_\theta(\varrho,\theta)}\Big\|_{L^3}\|\nabla^k\mathbf{D}\|+\|\mathbf{D}\|_{L^3}\Big\|\nabla^k\Big(\frac{\mathbf{D}}{\varrho e_\theta(\varrho,\theta)}\Big)\Big\|\Big)\|\nabla^k\zeta\|_{L^6}\\
&\quad +\Big(\Big\|\frac{\operatorname{div}\mathbf{u}}{\varrho e_\theta(\varrho,\theta)}\Big\|_{L^3}\|\nabla^k\operatorname{div}\mathbf{u}\|+\|\operatorname{div}\mathbf{u}\|_{L^3}\Big\|\nabla^k\Big(\frac{\operatorname{div}\mathbf{u}}{\varrho e_\theta(\varrho,\theta)}\Big)\Big\|\Big)\|\nabla^k\zeta\|_{L^6}\\
&\overset{(2.8)}{\lesssim} \varepsilon_1\Big(\|\nabla^{k+1}\mathbf{u}\|+\Big\|\nabla^k\Big(\frac{\mathbf{D}}{\varrho e_\theta(\varrho,\theta)}\Big)\Big\|+\Big\|\nabla^k\Big(\frac{\operatorname{div}\mathbf{u}}{\varrho e_\theta(\varrho,\theta)}\Big)\Big\|\Big)\|\nabla^{k+1}\zeta\|\\
&\overset{(2.8),(1.13)}{\lesssim} \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla^{k+1}\varphi\|^2+\|\nabla^{k+1}\zeta\|^2).
\end{aligned}
$$

Therefore, we have

$$
(2.31)\qquad I_k^3\lesssim\varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2+\|\nabla^{k+1}\varphi\|^2+\|\nabla^{k+1}\zeta\|^2+\|\nabla^k\operatorname{div}\mathbf{q}\|^2).
$$

For I_k^4, using Hölder's inequality, (2.9) and (1.12), we obtain from (2.25) that

$$
\begin{aligned}
I_k^4 &= \int \nabla^k[h_6(\zeta)\nabla\zeta]\cdot\nabla^k\mathbf{q}\,\mathrm{d}x \\
&\lesssim (\|h_6(\zeta)\|_{L^\infty}\|\nabla^{k+1}\zeta\| + \|\nabla\zeta\|_{L^3}\|\nabla h_6(\zeta)\|_{L^6})\|\nabla^k\mathbf{q}\| \\
&\lesssim \varepsilon_1(\|\nabla^{k+1}\zeta\|^2 + \|\nabla^k\mathbf{q}\|^2),
\end{aligned}
\tag{2.32}
$$

where $h_6(\zeta) = \zeta^3 + 3\bar\theta\zeta^2 + 3\bar\theta^2\zeta$.

Substituting (2.26), (2.30)–(2.32) into (2.21), and using the smallness of $\varepsilon_1 > 0$, we get (2.19). The proof of Lemma 2.2 is completed. □

Next, we have:

Lemma 2.3. *Under assumption* (2.8), *we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^k\mathbf{u}\|^2 + \frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2}\|\nabla^k\varphi\|^2 + \frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}\|\nabla^k\zeta\|^2\Big) + \frac{\mu}{\bar\varrho}\|\nabla^{k+1}\mathbf{u}\|^2 \\
&\qquad + \frac{\kappa}{\bar\varrho\bar\theta}\|\nabla^{k+1}\zeta\|^2 + \frac{1}{4\widetilde\sigma\bar\varrho\bar\theta^2}\|\nabla^k\operatorname{div}\mathbf{q}\|^2 + \frac{\sigma_a}{4\widetilde\sigma\bar\varrho\bar\theta^2}\|\nabla^k\mathbf{q}\|^2 \\
&\quad\lesssim \varepsilon_1(\|\nabla^k\varphi\|^2 + \|\nabla^k\mathbf{u}\|^2 + \|\nabla^k\zeta\|^2)
\end{aligned}
\tag{2.33}
$$

for $k = 1, 2, \dots, N$.

P r o o f. We use (2.21)–(2.25). Using $(1.13)_2$, along the same lines as in (2.26), we obtain from (2.22) that

$$
\begin{aligned}
I_k^1 &= \int\nabla^k(\varphi\operatorname{div}\mathbf{u})\nabla^k\varphi\,\mathrm{d}x - \int[\nabla^k,\mathbf{u}]\nabla\varphi\nabla^k\varphi\,\mathrm{d}x + \frac{1}{2}\int\operatorname{div}\mathbf{u}|\nabla^k\varphi|^2\,\mathrm{d}x \\
&\lesssim (\|\varphi\|_{L^\infty}\|\nabla^{k+1}\mathbf{u}\| + \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^k\varphi\|)\|\nabla^k\varphi\| \\
&\quad + (\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^k\varphi\| + \|\nabla\varphi\|_{L^3}\|\nabla^k\mathbf{u}\|_{L^6})\|\nabla^k\varphi\| + \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^k\varphi\|^2 \\
&\overset{(2.8)}{\lesssim} \varepsilon_1(\|\nabla^k\varphi\|^2 + \|\nabla^{k+1}\mathbf{u}\|^2).
\end{aligned}
\tag{2.34}
$$

Along the same lines as in (2.27)–(2.29), we have

$$
\begin{aligned}
I_k^{21} &= -\int\nabla^{k-1}[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\nabla^{k+1}\mathbf{u}\,\mathrm{d}x \\
&\overset{(1.14)}{\lesssim} (\|\mathbf{u}\|_{L^\infty}\|\nabla^k\mathbf{u}\| + \|\nabla\mathbf{u}\|_{L^3}\|\nabla^{k-1}\mathbf{u}\|_{L^6})\|\nabla^{k+1}\mathbf{u}\| \\
&\overset{(2.8)}{\lesssim} \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2 + \|\nabla^k\mathbf{u}\|^2),
\end{aligned}
$$

$$
\begin{aligned}
I_k^{22} &= -\int \nabla^{k-1}[h_1(\varphi,\zeta)\nabla\varphi + h_2(\varphi,\zeta)\nabla\zeta]\cdot\nabla^{k+1}\mathbf{u}\,\mathrm{d}x\\
&\overset{(1.14)}{\lesssim} (\|h_1(\varphi,\zeta)\|_{L^\infty}\|\nabla^k\varphi\| + \|\nabla\varphi\|_{L^3}\|\nabla^{k-1}h_1(\varphi,\zeta)\|_{L^6})\|\nabla^{k+1}\mathbf{u}\|\\
&\quad + (\|h_2(\varphi,\zeta)\|_{L^\infty}\|\nabla^k\zeta\| + \|\nabla\zeta\|_{L^3}\|\nabla^{k-1}h_2(\varphi,\zeta)\|_{L^6})\|\nabla^{k+1}\mathbf{u}\|\\
&\overset{(2.8),(1.13)}{\lesssim} \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2 + \|\nabla^k\varphi\|^2 + \|\nabla^k\zeta\|^2),\\
I_k^{23} &\overset{(1.14)}{\lesssim} (\|h_3(\varphi)\|_{L^\infty}\|\nabla^{k+1}\mathbf{u}\| + \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^k h_3(\varphi)\|\\
&\quad + \|\nabla h_3(\varphi)\|_{L^3}\|\nabla^k\mathbf{u}\|_{L^6})\|\nabla^{k+1}\mathbf{u}\|\\
&\overset{(2.8),(1.13)}{\lesssim} \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2 + \|\nabla^k\varphi\|^2).
\end{aligned}
$$

Therefore, we have

$$
I_k^2 \lesssim \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2 + \|\nabla^k\mathbf{u}\|^2 + \|\nabla^k\varphi\|^2 + \|\nabla^k\zeta\|^2). \tag{2.35}
$$

By similar arguments, it is easy to check that

$$
I_k^3 \lesssim \varepsilon_1(\|\nabla^{k+1}\mathbf{u}\|^2 + \|\nabla^{k+1}\zeta\|^2 + \|\nabla^k\varphi\|^2 + \|\nabla^k\zeta\|^2 + \|\nabla^k \operatorname{div}\mathbf{q}\|^2) \tag{2.36}
$$

and

$$
I_k^4 \lesssim \varepsilon_1(\|\nabla^k\zeta\|^2 + \|\nabla^k \operatorname{div}\mathbf{q}\|^2). \tag{2.37}
$$

Substituting (2.34), (2.35)–(2.37) into (2.21), and using the smallness of $\varepsilon_1 > 0$, we get (2.33). The proof of Lemma 2.3 is completed. □

Lemma 2.4. *Under assumption* (2.8), *we have*

(2.38)

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,\mathrm{d}x + \frac{P_\varrho(\bar\varrho,\bar\theta)}{2\bar\varrho}\|\nabla^{k+1}\varphi\|^2 \lesssim \|\nabla^{k+1}\mathbf{u}\|^2 + \|\nabla^{k+1}\zeta\|^2 + \|\nabla^{k+2}\mathbf{u}\|^2
$$

for $k = 0,\dots,N-1$.

Proof. Multiplying $(2.20)_2$ by $\nabla^{k+1}\varphi$ and using

$$
\int \nabla^k\mathbf{u}_t\cdot\nabla^{k+1}\varphi\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,\mathrm{d}x - \bar\varrho\int(\operatorname{div}\nabla^k\mathbf{u})^2\,\mathrm{d}x + \int\nabla^k g_1 \operatorname{div}\nabla^k\mathbf{u}\,\mathrm{d}x
$$

due to $(2.20)_1$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,\mathrm{d}x + \frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho}\|\nabla^{k+1}\varphi\|^2 - \bar\varrho\int(\operatorname{div}\nabla^k\mathbf{u})^2\,\mathrm{d}x = \sum_{j=5}^{7} I_k^j, \tag{2.39}
$$

where

$$I_k^5 = \int \nabla^k \mathrm{div}(\varphi \mathbf{u}) \mathrm{div} \nabla^k \mathbf{u} \,\mathrm{d}x - \int \nabla^k [(\mathbf{u}, \nabla)\mathbf{u}] \cdot \nabla^{k+1} \varphi \,\mathrm{d}x + \int \nabla^k [h_1(\varphi, \zeta)\nabla \varphi + h_2(\varphi, \zeta)\nabla \zeta] \cdot \nabla^{k+1} \varphi \,\mathrm{d}x, \tag{2.40}$$

$$I_k^6 = -\int \nabla^k [h_3(\varphi)(\mu \Delta \mathbf{u} + (\mu + \lambda)\nabla \mathrm{div} \mathbf{u})] \cdot \nabla^{k+1} \varphi \,\mathrm{d}x, \tag{2.41}$$

$$I_k^7 = -\frac{\mu}{\bar\varrho} \int \nabla^k \Delta \mathbf{u} \nabla^{k+1} \varphi \,\mathrm{d}x - \frac{\mu + \lambda}{\bar\varrho} \int \nabla^{k+1} \operatorname{div} \mathbf{u} \nabla^{k+1} \varphi \,\mathrm{d}x - \frac{P_\theta(\bar\varrho, \bar\theta)}{\bar\varrho} \int \nabla^{k+1} \zeta \nabla^{k+1} \varphi \,\mathrm{d}x. \tag{2.42}$$

We estimate the right-hand side in (2.39). For I_k^5, along the same lines as in (2.26)–(2.28), we obtain from (2.40) that

$$I_k^5 \lesssim \varepsilon_1 (\|\nabla^{k+1} \mathbf{u}\|^2 + \|\nabla^{k+1} \varphi\|^2 + \|\nabla^{k+1} \zeta\|^2). \tag{2.43}$$

For I_k^6 we obtain from (2.41) that

$$\begin{aligned} I_k^6 &\overset{(1.14)}{\lesssim} (\|h_3(\varphi)\|_{L^\infty} \|\nabla^{k+2} \mathbf{u}\| + \|\nabla^2 \mathbf{u}\|_{L^3} \|\nabla^k h_3(\varphi)\|_{L^6}) \|\nabla^{k+1} \varphi\| \\ &\overset{(2.8),(1.13)}{\lesssim} \varepsilon_1 (\|\nabla^{k+2} \mathbf{u}\|^2 + \|\nabla^{k+1} \varphi\|^2). \end{aligned} \tag{2.44}$$

For I_k^7 we obtain from (2.42) that

$$\begin{aligned} I_k^7 &\lesssim \|\nabla^{k+2} \mathbf{u}\| \|\nabla^{k+1} \varphi\| + \|\nabla^{k+1} \zeta\| \|\nabla^{k+1} \varphi\| \\ &\lesssim \eta \|\nabla^{k+1} \varphi\|^2 + \eta^{-1} (\|\nabla^{k+2} \mathbf{u}\|^2 + \|\nabla^{k+1} \zeta\|^2) \end{aligned} \tag{2.45}$$

for any $\eta > 0$.

Substituting (2.43)–(2.45) into (2.39), and using the smallness of $\varepsilon_1 > 0$ and $\eta > 0$, we get (2.38). The proof of Lemma 2.4 is completed. □

3. Negative Sobolev estimate

In this section, we will derive the evolution of the negative Sobolev norms of solutions to system (2.3).

Lemma 3.1. *Under assumption* (2.8), *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\frac{P_\varrho(\bar\varrho, \bar\theta)}{\bar\varrho^2} \|\Lambda^{-s} \varphi\|^2 + \|\Lambda^{-s} \mathbf{u}\|^2 + \frac{e_\theta(\bar\varrho, \bar\theta)}{\bar\theta} \|\Lambda^{-s} \zeta\|^2 \Big) \lesssim (\|\nabla(\varphi, \mathbf{u}, \zeta)\|_{H^2}^2 + \|\operatorname{div} \mathbf{q}\|^2) \|\Lambda^{-s}(\varphi, \mathbf{u}, \zeta)\| + \varepsilon_1 \|\nabla \zeta\|_{H^1}^2 \tag{3.1}$$

for $s \in (0, \frac{1}{2}]$.

P r o o f. Applying Λ^{-s} to (2.3) yields

$$
\begin{aligned}
&\Lambda^{-s}\varphi_t + \bar\varrho \operatorname{div}\Lambda^{-s}\mathbf{u} = \Lambda^{-s} g_1, \\
&\Lambda^{-s}\mathbf{u}_t - \frac{\mu}{\bar\varrho}\Delta\Lambda^{-s}\mathbf{u} - \frac{(\mu+\lambda)}{\bar\varrho}\nabla\operatorname{div}\Lambda^{-s}\mathbf{u} + \frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho}\nabla\Lambda^{-s}\varphi \\
&\qquad\qquad + \frac{P_\theta(\bar\varrho,\bar\theta)}{\bar\varrho}\nabla\Lambda^{-s}\zeta = \Lambda^{-s}\mathbf{g}_2, \\
&\Lambda^{-s}\zeta_t + \frac{\bar\theta P_\theta(\bar\varrho,\bar\theta)}{e_\theta(\bar\varrho,\bar\theta)}\operatorname{div}\Lambda^{-s}\mathbf{u} = \frac{\kappa}{\bar\varrho e_\theta(\bar\varrho,\bar\theta)}\Delta\Lambda^{-s}\zeta - \frac{\Lambda^{-s}\operatorname{div}\mathbf{q}}{\bar\varrho e_\theta(\bar\varrho,\bar\theta)} + \Lambda^{-s} g_3, \\
&-\nabla\Lambda^{-s}\operatorname{div}\mathbf{q} + \sigma_a\Lambda^{-s}\mathbf{q} + 4\widetilde\sigma\bar\theta\nabla\Lambda^{-s}\zeta = \Lambda^{-s}\mathbf{g}_4.
\end{aligned}
\tag{3.2}
$$

Multiplying equations $(3.2)_2$, $(3.2)_3$ and $(3.2)_4$ by $\Lambda^{-s}\mathbf{u}$, $(e_\theta(\bar\varrho,\bar\theta)/\bar\theta)\Lambda^{-s}\zeta$ and $1/(4\widetilde\sigma\bar\varrho\bar\theta^2)\Lambda^{-s}\mathbf{q}$, respectively, and using $(3.2)_1$ and (2.4)–(2.7), we get

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\Lambda^{-s}\mathbf{u}\|^2 + \frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2}\|\Lambda^{-s}\varphi\|^2 + \frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}\|\Lambda^{-s}\zeta\|^2\Big) \\
&\qquad + \int\Big(\frac{\mu}{\bar\varrho}|\nabla\Lambda^{-s}\mathbf{u}|^2 + \frac{(\mu+\lambda)}{\bar\varrho}|\operatorname{div}\Lambda^{-s}\mathbf{u}|^2\Big)\,\mathrm{d}x \\
&\qquad + \frac{\kappa}{\bar\varrho\bar\theta}\int|\nabla\Lambda^{-s}\zeta|^2\,\mathrm{d}x + \frac{1}{4\widetilde\sigma\bar\varrho\bar\theta^2}\int|\Lambda^{-s}\operatorname{div}\mathbf{q}|^2\,\mathrm{d}x \\
&\qquad + \frac{\sigma_a}{4\widetilde\sigma\bar\varrho\bar\theta^2}\int|\Lambda^{-s}\mathbf{q}|^2\,\mathrm{d}x \\
&\quad = \frac{P_\theta(\bar\varrho,\bar\theta)}{\bar\varrho^2}I^8 + I^9 + \frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}I^{10} + \frac{1}{\bar\varrho\bar\theta^2}I^{11},
\end{aligned}
\tag{3.3}
$$

where

$$
I^8 = \int\Lambda^{-s}(\varphi\operatorname{div}\mathbf{u} + \mathbf{u}\nabla\varphi)\Lambda^{-s}\varphi\,\mathrm{d}x,
\tag{3.4}
$$

$$
\begin{aligned}
I^9 = &-\int\Lambda^{-s}[(\mathbf{u},\nabla)\mathbf{u}]\cdot\Lambda^{-s}\mathbf{u}\,\mathrm{d}x \\
&+\int\Lambda^{-s}[h_1(\varphi,\zeta)\nabla\varphi + h_2(\varphi,\zeta)\nabla\zeta]\cdot\Lambda^{-s}\mathbf{u}\,\mathrm{d}x \\
&-\int\Lambda^{-s}[h_3(\varphi)(\mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\mathrm{div}\mathbf{u})]\cdot\Lambda^{-s}\mathbf{u}\,\mathrm{d}x,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
I^{10} = &-\int\Lambda^{-s}[\mathbf{u}\cdot\nabla\zeta - h_5(\varphi,\zeta)\operatorname{div}\mathbf{u} + h_4(\varphi,\zeta)\operatorname{div}\mathbf{q}]\Lambda^{-s}\zeta\,\mathrm{d}x \\
&-\kappa\int\Lambda^{-s}[h_4(\varphi,\zeta)\Delta\zeta]\Lambda^{-s}\zeta\,\mathrm{d}x \\
&+\int\Lambda^{-s}\Big[\frac{2\mu\mathbf{D}:\mathbf{D} + \lambda(\operatorname{div}\mathbf{u})^2}{\varrho e_\theta(\varrho,\theta)}\Big]\Lambda^{-s}\zeta\,\mathrm{d}x \\
= {}& I^{10}_1 + I^{10}_2 + I^{10}_3,
\end{aligned}
\tag{3.6}
$$

and

$$I^{11} = -\int \Lambda^{-s}[h_6(\zeta)\nabla\zeta]\cdot\Lambda^{-s}\mathbf{q}\,\mathrm{d}x, \tag{3.7}$$

where $h_6(\zeta) = (\zeta^3 + 3\bar{\theta}\zeta^2 + 3\bar{\theta}^2\zeta)$.

In order to estimate the right-hand side of (3.3), we shall use estimate (1.16). This forces us to require that $s \in (0, \frac{3}{2})$. If $s \in (0, \frac{1}{2}]$, then $\frac{1}{2} + \frac{1}{3}s < 1$ and $3s^{-1} \geqslant 6$. Then, along the same arguments as in [40], Section 3, we obtain from (3.4) and (3.5) respectively that

$$I^8 \lesssim (\|\nabla\mathbf{u}\|_{H^1}^2 + \|\nabla\varphi\|_{H^1}^2)\|\Lambda^{-s}\varphi\|, \quad I^9 \lesssim (\|\nabla\mathbf{u}\|_{H^1}^2 + \|\nabla\varphi\|_{H^1}^2 + \|\nabla\zeta\|_{H^1}^2)\|\Lambda^{-s}\mathbf{u}\|. \tag{3.8}$$

For I^{10}, using Hölder's inequality and (1.16), we obtain from (3.6) that

$$\begin{aligned}
I_1^{10} &\lesssim (\|\mathbf{u}\cdot\nabla\zeta\|_{L^{1/(1/2+s/3)}} + \|h_5(\varphi,\zeta)\operatorname{div}\mathbf{u}\|_{L^{1/(1/2+s/3)}} \\
&\quad + \|h_4(\varphi,\zeta)\operatorname{div}\mathbf{q}\|_{L^{1/(1/2+s/3)}})\|\Lambda^{-s}\zeta\| \\
&\lesssim (\|\mathbf{u}\|_{L^{3/s}}\|\nabla\zeta\| + \|h_5(\varphi,\zeta)\|_{L^{3/s}}\|\nabla\mathbf{u}\| + \|h_4(\varphi,\zeta)\|_{L^{3/s}}\|\mathrm{div}|,\mathbf{q}\|)\|\Lambda^{-s}\zeta\| \\
&\overset{(1.12)}{\lesssim} (\|\nabla\mathbf{u}\|^{1/2+s}\|\nabla^2\mathbf{u}\|^{1/2-s} + \|\nabla\varphi\|^{1/2+s}\|\nabla^2\varphi\|^{1/2-s} \\
&\quad + \|\nabla\zeta\|^{1/2+s}\|\nabla^2\zeta\|^{1/2-s})(\|\nabla\zeta\| + \|\nabla\mathbf{u}\| + \|\operatorname{div}\mathbf{q}\|)\|\Lambda^{-s}\zeta\| \\
&\lesssim (\|\nabla\mathbf{u}\|_{H^1}^2 + \|\nabla\varphi\|_{H^1}^2 + \|\nabla\zeta\|_{H^1}^2)\|\Lambda^{-s}\zeta\|, \\
I_2^{10} &\lesssim \|h_4(\varphi,\zeta)\Delta\zeta\|_{L^{1/(1/2+s/3)}}\|\Lambda^{-s}\zeta\| \lesssim \|h_4(\varphi,\zeta)\|_{L^{3/s}}\|\Delta\zeta\|\|\Lambda^{-s}\zeta\| \\
&\lesssim (\|\varphi\|_{L^{3/s}} + \|\zeta\|_{L^{3/s}})\|\Delta\zeta\|\|\Lambda^{-s}\zeta\| \\
&\overset{(1.12)}{\lesssim} (\|\nabla\varphi\|^{1/2+s}\|\nabla^2\varphi\|^{1/2-s} + \|\nabla\zeta\|^{1/2+s}\|\nabla^2\zeta\|^{1/2-s})\|\nabla^2\zeta\|\|\Lambda^{-s}\zeta\| \\
&\lesssim (\|\nabla\varphi\|_{H^1}^2 + \|\nabla\zeta\|_{H^1}^2)\|\Lambda^{-s}\zeta\|, \\
I_3^{10} &\lesssim \Big(\Big\|\frac{\mathbf{D}:\mathbf{D}}{\varrho e_\theta(\varrho,\theta)}\Big\|_{L^{1/(1/2+s/3)}} + \Big\|\frac{(\operatorname{div}\mathbf{u})^2}{\varrho e_\theta(\varrho,\theta)}\Big\|_{L^{1/(1/2+s/3)}}\Big)\|\Lambda^{-s}\zeta\| \\
&\lesssim \Big(\Big\|\frac{\mathbf{D}}{\varrho e_\theta(\varrho,\theta)}\Big\|_{L^{3/s}} + \Big\|\frac{\operatorname{div}\mathbf{u}}{\varrho e_\theta(\varrho,\theta)}\Big\|_{L^{3/s}}\Big)\|\|\nabla\mathbf{u}\|\Lambda^{-s}\zeta\| \lesssim \|\nabla\mathbf{u}\|_{L^{3/s}}\|\nabla\mathbf{u}\|\Lambda^{-s}\zeta\| \\
&\overset{(1.12)}{\lesssim} \|\nabla^2\mathbf{u}\|^{1/2+s}\|\nabla^3\mathbf{u}\|^{1/2-s}\|\nabla\mathbf{u}\|\Lambda^{-s}\zeta\| \lesssim \|\nabla\mathbf{u}\|_{H^2}^2\|\Lambda^{-s}\zeta\|.
\end{aligned}$$

Therefore, we have

$$I^{10} \lesssim (\|\nabla(\varphi,\zeta)\|_{H^1}^2 + \|\nabla\mathbf{u}\|_{H^2}^2 + \|\operatorname{div}\mathbf{q}\|^2)\|\Lambda^{-s}\zeta\|. \tag{3.9}$$

For I^{11}, using Hölder's inequality and (1.12), we obtain from (3.7) that

$$
(3.10)\qquad I^{11} \lesssim \|h_6(\zeta)\nabla\zeta\|_{L^{1/(1/2+s/3)}}\|\Lambda^{-s}\mathbf{q}\| \lesssim \|h_6(\zeta)\|_{L^{3/s}}\|\nabla\zeta\|\|\Lambda^{-s}\mathbf{q}\| \underset{(2.8),(1.12)}{\lesssim} \varepsilon_1(\|\nabla\zeta\|_{H^1}^2 + \|\Lambda^{-s}\mathbf{q}\|^2).
$$

Substituting (3.8)–(3.10) into (3.3), we get (3.1). The proof of Lemma 3.1 is completed. $\square$

4. The proof of Theorem 1.1

In this section, we shall combine all the estimates that we have derived in the previous two sections and the Sobolev interpolation to prove Theorem 1.1.

4.1. The existence of global solutions. By Lemma 2.1, we have

$$
(4.1)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int \varrho\mathcal{E}\,\mathrm{d}x\Big) + \|\nabla(\mathbf{u},\zeta)\|^2 + \|(\mathbf{q},\operatorname{div}\mathbf{q})\|^2 \leqslant 0.
$$

We close the estimates at each kth level to prove (1.9). For $N \geqslant 3$, let $3 \leqslant m \leqslant N$. Summing up estimate (2.19) in Lemma 2.2 from $k = 1$ to $m-1$, we get

$$
(4.2)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leqslant k\leqslant m-1}\Big(\|\nabla^k\mathbf{u}\|^2 + \frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2}\|\nabla^k\varphi\|^2 + \frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}\|\nabla^k\zeta\|^2\Big) + \sum_{2\leqslant k\leqslant m}\|\nabla^k(\mathbf{u},\zeta)\|^2 + \sum_{1\leqslant k\leqslant m-1}\|\nabla^k(\mathbf{q},\operatorname{div}\mathbf{q})\|^2 \leqslant C_1\varepsilon_1\Big(\sum_{2\leqslant k\leqslant m}\|\nabla^k\varphi\|^2 + \|\nabla(\varphi,\mathbf{u},\zeta)\|^2\Big).
$$

By estimate (2.33) of Lemma 2.3 with $k = m$, we have

$$
(4.3)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^m\mathbf{u}\|^2 + \frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2}\|\nabla^m\varphi\|^2 + \frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}\|\nabla^m\zeta\|^2\Big) + \|\nabla^{m+1}(\mathbf{u},\zeta)\|^2 + \|\nabla^k(\mathbf{q},\operatorname{div}\mathbf{q})\|^2 \leqslant C_2\varepsilon_1\|\nabla^m(\varphi,\mathbf{u},\zeta)\|^2.
$$

Summing up estimate (2.38) of Lemma 2.4 from $k = 0$ to $m-1$, we get

$$
(4.4)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\sum_{0\leqslant k\leqslant m-1}\int \nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,\mathrm{d}x + \sum_{1\leqslant k\leqslant m}\|\nabla^k\varphi\|^2 \leqslant C_3\sum_{1\leqslant k\leqslant m+1}\|\nabla^k(\mathbf{u},\zeta)\|^2.
$$

Let $\varepsilon \in (0,1]$ be suitably small. Then, summing (4.1)–(4.3) and $\varepsilon \times$ (4.4), we find that

$$
(4.5)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0^m(t) + \Lambda_0^m(t) + \sum_{0\leqslant k\leqslant m}\|\nabla^k(\mathbf{q},\operatorname{div}\mathbf{q})\|^2 \leqslant 0,
$$

where

$$\mathcal{E}_0^m(t) := \sum_{1\leqslant k\leqslant m} \|\nabla^k \mathbf{u}\|^2 + \varepsilon \sum_{1\leqslant k\leqslant m-1} \int \nabla^k \mathbf{u} \cdot \nabla^{k+1}\varphi \, \mathrm{d}x + \frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2} \sum_{1\leqslant k\leqslant m} \|\nabla^k \varphi\|^2 + \frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta} \sum_{1\leqslant k\leqslant m} \|\nabla^k \zeta\|^2 + \int \varrho \mathcal{E} \, \mathrm{d}x \tag{4.6}$$

and

$$\Lambda_0^m(t) := (1-\varepsilon C_3) \sum_{1\leqslant k\leqslant m+1} \|\nabla^k (\mathbf{u},\zeta)\|^2 + (\varepsilon - C_1\varepsilon_1 - C_2\varepsilon_1) \sum_{1\leqslant k\leqslant m} \|\nabla^k \varphi\|^2. \tag{4.7}$$

Since $\varepsilon \in (0,1]$ and $\varepsilon_1 > 0$ are suitably small, using (2.12), we obtain from (4.6) and (4.7), respectively, that

$$\mathcal{E}_0^m(t) \backsimeq \|(\varphi,\mathbf{u},\zeta)(t)\|_{H^m}^2 \tag{4.8}$$

and

$$\Lambda_0^m(t) \backsimeq \|\nabla\varphi(t)\|_{H^{m-1}}^2 + \|\nabla(\mathbf{u},\zeta)(t)\|_{H^m}^2 \tag{4.9}$$

uniformly for all $t \geqslant 0$.

Integrating (4.5) over $(0,t)$, and using (4.8) and (4.9), we have

$$\begin{aligned} \|(\varphi,\mathbf{u},\zeta)(t)\|_{H^m}^2 &+ \int_0^t \|\nabla\varphi\|_{H^{m-1}}^2 \, \mathrm{d}\tau + \int_0^t \|\nabla(\mathbf{u},\zeta)\|_{H^m}^2 \, \mathrm{d}\tau \\ &+ \int_0^t \|(\mathbf{q}, \operatorname{div}\mathbf{q})\|_{H^N}^2 \, \mathrm{d}\tau \\ &\leqslant C\|(\varphi,\mathbf{u},\zeta)(0)\|_{H^N}^2. \end{aligned} \tag{4.10}$$

Applying ∇^k to $(1.1)_4$, multiplying it by $\nabla^k \mathbf{q}$ and summing up the resulting equalities for $k = 0,1,\ldots,m-1$, we get

$$\begin{aligned} \sum_{0\leqslant k\leqslant m-1} \|\nabla^k \operatorname{div}\mathbf{q}\|^2 + \sigma_a \sum_{0\leqslant k\leqslant m-1} \|\nabla^k \mathbf{q}\|^2 &= 4\widetilde{\sigma} \sum_{0\leqslant k\leqslant m-1} \int \nabla^k(\theta^3\nabla\zeta) \cdot \nabla^k \mathbf{q} \, \mathrm{d}x \lesssim \|\nabla\zeta\|_{H^{m-1}} \|\mathbf{q}\|_{H^{m-1}} \\ &\leqslant \frac{\sigma_a}{2}\|\mathbf{q}\|_{H^{m-1}}^2 + C\|\zeta\|_{H^m}^2, \end{aligned}$$

which implies

$$\|(\operatorname{div}\mathbf{q},\mathbf{q})\|_{H^{m-1}}^2 \lesssim \|\zeta\|_{H^m}^2. \tag{4.11}$$

Now, taking $m=3$ in (4.10) and (4.11), we arrive at

$$\|(\varphi,\mathbf{u},\zeta)(t)\|_{H^3}^2+\|(\operatorname{div}\mathbf{q},\mathbf{q})\|_{H^2}^2\lesssim\|(\varphi,\mathbf{u},\zeta)(0)\|_{H^3}^2. \tag{4.12}$$

Using (1.8) and (4.12), by a standard continuity argument, we can close the a priori estimate (2.8). This in turn allows us to take $m=N$ in (4.10) and (4.11), which then implies (1.9).

Now, the proof of existence and uniqueness of global solution is standard. So, we omit the details for brevity.

4.2. Decay rate. We prove (1.10) for $s\in(0,\frac{1}{2}]$ because it is obvious to check (1.10) with $s=0$. Setting

$$\mathcal{E}_{-s}(t):=\frac{P_\varrho(\bar\varrho,\bar\theta)}{\bar\varrho^2}\|\Lambda^{-s}\varphi\|^2+\|\Lambda^{-s}\mathbf{u}\|^2+\frac{e_\theta(\bar\varrho,\bar\theta)}{\bar\theta}\|\Lambda^{-s}\zeta\|^2,$$

and using (1.6), we get

$$\mathcal{E}_{-s}(t)\backsimeq\|\Lambda^{-s}(\varphi,\mathbf{u},\zeta)(t)\|^2. \tag{4.13}$$

Then, integrating in time (3.1), and using (4.13) and (4.10), we obtain that for $s\in(0,\frac{1}{2}]$,

$$\begin{aligned}\mathcal{E}_{-s}(t)&\lesssim\mathcal{E}_{-s}(0)+\varepsilon_1\int_0^t\|\nabla\zeta\|_{H^1}^2\,\mathrm{d}\tau+\int_0^t(\|\nabla(\varphi,\mathbf{u},\zeta)\|_{H^2}^2+\|\operatorname{div}\mathbf{q}\|^2)\sqrt{\mathcal{E}_{-s}(\tau)}\,\mathrm{d}\tau\\&\lesssim1+\sup_{0\leqslant\tau\leqslant t}\sqrt{\mathcal{E}_{-s}(\tau)},\end{aligned}$$

which implies (1.10).

Next, we will continue the proof of (1.11) for $s\in[0,\frac{1}{2}]$. We may use (1.15) to find that

$$\|\nabla f\|\gtrsim\|\Lambda^{-s}f\|^{-1/s}\|f\|^{1+1/s}. \tag{4.14}$$

By (4.14) and (1.10), we get

$$\|\nabla(\varphi,\mathbf{u},\zeta)\|\gtrsim\|(\varphi,\mathbf{u},\zeta)\|^{1+1/s}. \tag{4.15}$$

Also, using $\|\varphi(t)\|_{H^N}\lesssim1$ due to (1.9), we have

$$\|\nabla^N\varphi\|\gtrsim\|\nabla^N\varphi\|^{1+1/s}. \tag{4.16}$$

By (4.15) and (4.16), we obtain from (4.8) and (4.9) that

$$\Lambda_0^N(t) \gtrsim (\mathcal{E}_0^N(t))^{1+1/s}. \tag{4.17}$$

Using (4.17), we deduce from (4.5) with $m = N$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0^N(t) + C_0(\mathcal{E}_0^N(t))^{1+1/s} \leqslant 0.$$

Solving this inequality directly gives (see (1.17))

$$\mathcal{E}_0^N(t) \lesssim (1+t)^{-s},$$

that is,

$$\|(\varphi, \mathbf{u}, \zeta)(t)\|_{H^N}^2 \lesssim (1+t)^{-s} \tag{4.18}$$

due to (4.8). By (4.18) and (4.11) with $m = N$, we obtain (1.11). The proof of (1.10)–(1.11) is completed.

References

[1] *Z. Chen, X. Chai, W. Wang*: Convergence rate of solutions to strong contact discontinuity for the one-dimensional compressible radiation hydrodynamics model. Acta Math. Sci., Ser. B, Engl. Ed. *36* (2016), 265–282. zbl MR doi

[2] *K.-I. Choe, H. Hong, J. Kim*: Non-degenerate stationary solution for outflow problem on the 1-D viscous heat-conducting gas with radiation. Commun. Math. Sci. *18* (2020), 1661–1684. zbl MR doi

[3] *R. Danchin, B. Ducomet*: On a simplified model for radiating flows. J. Evol. Equ. *14* (2014), 155–195. zbl MR doi

[4] *R. Danchin, B. Ducomet*: Existence of strong solutions with critical regularity to a polytropic model for radiating flows. Ann. Mat. Pura Appl. (4) *196* (2017), 107–153. zbl MR doi

[5] *B. Ducomet, E. Feireisl, Š. Nečasová*: On a model in radiation hydrodynamics. Ann. Inst. Henri Poincaré, Anal. Non Linéaire *28* (2011), 797–812. zbl MR doi

[6] *B. Ducomet, Š. Nečasová*: Global existence of solutions for the one-dimensional motions of a compressible viscous gas with radiation: An “infrarelativistic model”. Nonlinear Anal., Theory Methods Appl., Ser. A *72* (2010), 3258–3274. zbl MR doi

[7] *B. Ducomet, Š. Nečasová*: Global weak solutions to the 1D compressible Navier-Stokes equations with radiation. Commun. Math. Anal. *8* (2010), 23–65. zbl MR

[8] *B. Ducomet, Š. Nečasová*: Large-time behavior of the motion of a viscous heat-conducting one dimensional gas coupled to radiation. Ann. Mat. Pura Appl. (4) *191* (2012), 219–260. zbl MR doi

[9] *B. Ducomet, Š. Nečasová*: Asymptotic behavior of the motion of a viscous heat-conducting one-dimensional gas with radiation: The pure scattering case. Anal. Appl., Singap. *11* (2013), Article ID 1350003, 29 pages. zbl MR doi

[10] *B. Ducomet, Š. Nečasová*: Global existence for weak solutions of the Cauchy problem in a model of radiation hydrodynamics. J. Math. Anal. Appl. *420* (2014), 464–482. zbl MR doi

[11] *J. Gao, Q. Tao, Z.-A. Yao*: Long-time behavior of solution for the compressible nematic liquid crystal flows in $\mathbb{R}^3$. J. Differ. Equations *261* (2016), 2334–2383. zbl MR doi
[12] *Y. Guo, Y. Wang*: Decay of dissipative equations and negative Sobolev spaces. Commun. Partial Differ. Equations *37* (2012), 2165–2208. zbl MR doi
[13] *H. Hong*: Asymptotic behavior toward the combination of contact discontinuity with rarefaction waves for 1-D compressible viscous gas with radiation. Nonlinear Anal., Real World Appl. *35* (2017), 175–199. zbl MR doi
[14] *H. Hong, T. Wang*: Stability of stationary solutions to the inflow problem for full compressible Navier-Stokes equations with a large initial perturbation. SIAM J. Math. Anal. *49* (2017), 2138–2166. zbl MR doi
[15] *P. Jiang*: Unique global solution of an initial-boundary value problem to a diffusion approximation model in radiation hydrodynamics. Discrete Contin. Dyn. Syst. *35* (2015), 3015–3037. zbl MR doi
[16] *P. Jiang*: Global well-posedness and large time behavior of classical solutions to the diffusion approximation model in radiation hydrodynamics. Discrete Contin. Dyn. Syst. *37* (2017), 2045–2063. zbl MR doi
[17] *P. Jiang*: Unique global solution to the thermally radiative magnetohydrodynamics equations. Z. Angew. Math. Phys. *69* (2018), Article ID 96, 29 pages. zbl MR doi
[18] *S. Jiang, Q. Ju, Y. Liao*: Nonequilibrium-diffusion limit of the compressible Euler-P_1 approximation radiation model at low Mach number. SIAM J. Math. Anal. *53* (2021), 2491–2522. zbl MR doi
[19] *S. Jiang, F. Li, F. Xie*: Nonrelativistic limit of the compressible Navier-Stokes-Fourier-P_1 approximation model arising in radiation hydrodynamics. SIAM J. Math. Anal. *47* (2015), 3726–3746. zbl MR doi
[20] *S. Jiang, F. Xie, J. W. Zhang*: A global existence result in radiation hydrodynamics. Industrial and Applied Mathematics in China. Series in Contemporary Applied Mathematics 10. World Scientific, Hackensack, 2009, pp. 25–48. zbl MR
[21] *P. Jiang, F. Yu*: Global well posedness for the thermally radiative magnetohydrodynamic equations in 3D. J. Funct. Spaces *2020* (2020), Article ID 4748101, 11 pages. zbl MR doi
[22] *P. Jiang, Y. Zhou*: Smooth solutions to diffusion approximation radiation hydrodynamics equations. J. Math. Anal. Appl. *466* (2018), 324–337. zbl MR doi
[23] *C. Lattanzio, C. Mascia, D. Serre*: Shock waves for radiative hyperbolic-elliptic systems. Indiana Univ. Math. J. *56* (2007), 2601–2640. zbl MR doi
[24] *Y. Li, S. Zhu*: Formation of singularities in solutions to the compressible radiation hydrodynamics equations with vacuum. J. Differ. Equations *256* (2014), 3943–3980. zbl MR doi
[25] *Y. Li, S. Zhu*: Existence results for compressible radiation hydrodynamic equations with vacuum. Commun. Pure Appl. Anal. *14* (2015), 1023–1052. zbl MR doi
[26] *Y. Li, S. Zhu*: Existence results and blow-up criterion of compressible radiation hydrodynamic equations. J. Dyn. Differ. Equations *29* (2017), 549–595. zbl MR doi
[27] *C. Lin*: Asymptotic stability of rarefaction waves in radiative hydrodynamics. Commun. Math. Sci. *9* (2011), 207–223. zbl MR doi
[28] *C. Lin, J.-F. Coulombel, T. Goudon*: Shock profiles for non-equilibrium radiating gases. Physica D *218* (2006), 83–94. zbl MR doi
[29] *C. Lin, J.-F. Coulombel, T. Goudon*: Asymptotic stability of shock profiles in radiative hydrodynamics. C. R., Math., Acad. Sci. Paris *345* (2007), 625–628. zbl MR doi
[30] *T. Nguyen, R. G. Plaza, K. Zumbrun*: Stability of radiative shock profiles for hyperbolic-elliptic coupled systems. Physica D *239* (2010), 428–453. zbl MR doi
[31] *L. Nirenberg*: On elliptic partial differential equations. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. *13* (1959), 115–162. zbl MR

[32] *G. C. Pomraning*: The Equations of Radiation Hydrodynamics. Pergamon Press, Elmsford, 1973.
[33] *X. Pu, B. Guo*: Global existence and semiclassical limit for quantum hydrodynamic equations with viscosity and heat conduction. Kinet. Relat. Models *9* (2016), 165–191. zbl MR doi
[34] *X. Pu, X. Xu*: Asymptotic behaviors of the full quantum hydrodynamic equations. J. Math. Anal. Appl. *454* (2017), 219–245. zbl MR doi
[35] *Y. Qin, B. Feng, M. Zhang*: Large-time behavior of solutions for the one-dimensional infrarelativistic model of a compressible viscous gas with radiation. J. Differ. Equations *252* (2012), 6175–6213. zbl MR doi
[36] *C. Rohde, W. Wang, F. Xie*: Hyperbolic-hyperbolic relaxation limit for a 1D compressible radiation hydrodynamics model: Superposition of rarefaction and contact waves. Commun. Pure Appl. Anal. *12* (2013), 2145–2171. zbl MR doi
[37] *C. Rohde, F. Xie*: Decay rates to viscous contact waves for a 1D compressible radiation hydrodynamics model. Math. Models Methods Appl. Sci. *23* (2013), 441–469. zbl MR doi
[38] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series 30. Princeton University Press, Princeton, 1970. zbl MR doi
[39] *Z. Tan, R. Zhang*: Optimal decay rates of the compressible fluid models of Korteweg type. Z. Angew. Math. Phys. *65* (2014), 279–300. zbl MR doi
[40] *Y. Wang*: Decay of the Navier-Stokes-Poisson equations. J. Differ. Equations *253* (2012), 273–297. zbl MR doi
[41] *Z. Wang*: Vanishing viscosity limit of the radiation hydrodynamic equations with far field vacuum. J. Math. Anal. Appl. *452* (2017), 747–779. zbl MR doi
[42] *Z. Wang*: Existence results for the radiation hydrodynamic equations with degenerate viscosity coefficients and vacuum. J. Differ. Equations *265* (2018), 354–388. zbl MR doi
[43] *J. Wang, F. Xie*: Asymptotic stability of viscous contact wave for the 1D radiation hydrodynamics system. J. Differ. Equations *251* (2011), 1030–1055. zbl MR doi
[44] *J. Wang, F. Xie*: Asymptotic stability of viscous contact wave for the one-dimensional compressible viscous gas with radiation. Nonlinear Anal., Theory Methods Appl., Ser. A *74* (2011), 4138–4151. zbl MR doi
[45] *J. Wang, F. Xie*: Singular limit to strong contact discontinuity for a 1D compressible radiation hydrodynamics model. SIAM J. Math. Anal. *43* (2011), 1189–1204. zbl MR doi
[46] *W. Wang, F. Xie*: The initial value problem for a multi-dimensional radiation hydrodynamics model with viscosity. Math. Methods Appl. Sci. *34* (2011), 776–791. zbl MR doi
[47] *W. Wang, F. Xie, X. Yang*: Decay rates of solutions to a P1-approximation model arising from radiation hydrodynamics. J. Differ. Equations *264* (2018), 2936–2969. zbl MR doi
[48] *F. Xie*: Nonlinear stability of combination of viscous contact wave with rarefaction waves for a 1D radiation hydrodynamics model. Discrete Contin. Dyn. Syst., Ser. B *17* (2012), 1075–1100. zbl MR doi

Authors' addresses: *Hyejong O*, Graduate School, Kim Chol Ju University of Education, Daesong District, 355 Pyongyang, Democratic People's Republic of Korea; *Hakho Hong* (corresponding author), Institute of Mathematics, State Academy of Sciences, Unjong District, 355 Pyongyang, Democratic People's Republic of Korea, e-mail: `hhhong@star.co-net.kp`; *Jongsung Kim*, School of Mathematics, University of Mechanical Engineering Pyongyang, Sunguo District, 355 Pyongyang, Democratic People's Republic of Korea, e-mail: `pmu@star-co.net.kp`.