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A SURVEY OF SOME RECENT RESULTS ON CLIFFORD ALGEBRAS IN $\mathbb{R}^4$

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To the memory of Prof. Gerhard Opfer, University of Hamburg.

Abstract. We will study applications of numerical methods in Clifford algebras in $\mathbb{R}^4$, in particular in the skew field of quaternions, in the algebra of coquaternions and in the other nondivision algebras in $\mathbb{R}^4$. In order to gain insight into the multidimensional case, we first consider linear equations in quaternions and coquaternions. Then we will search for zeros of one-sided (simple) quaternion polynomials. Three different classes of zeros can be distinguished. In general, the quaternionic coefficients can be placed on both sides of the powers. Then there are even five different classes of zeros. All results can be extended to other noncommutative algebras in $\mathbb{R}^4$. In the paper by R. Lauterbach and G. Opfer (2014), the authors constructed an exact Jacobi matrix for functions defined in noncommutative algebraic systems without the use of any partial derivative. We applied this technique to find the eigenvalues of the companion matrix as zeros of the companion polynomial by Newton's method.

Keywords: linear equations in quaternions and coquaternions; polynomials over $\mathbb{R}^4$ algebras; the algebraic eigenvalue problem over noncommutative algebras; Newton's method; companion matrix and companion polynomial; Niven's algorithm

MSC 2020: 15A06, 15A18, 15A66

1. Brief (pre)historical overview

Clifford algebras find their use in many areas of mathematics. From the thirties of the last century up to today, mathematicians have been attracted by [24], [19], [30], [22], [2], [25], [23], [7].

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2. Basic notation

2.1. Quaternions. $\mathbb{H} := \mathbb{R}^4$—the skew field of quaternions. Let $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{H}$.

- ▷ $x_1 := \Re x$—real part of x, $x_2 := \Im x$—imaginary part of x,
- ▷ $x_v := (0, x_2, x_3, x_4)$—vector part of x,
- ▷ $\overline{x} = (x_1, -x_2, -x_3, -x_4)$—conjugate of x,
- ▷ $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$—absolute value of x,
- ▷ $x^{-1} = \overline{x}/|x|^2$ for $x \in \mathbb{H} \setminus \{0\}$—inverse quaternion,
- ▷ unit vectors: $\mathbf{1} = (1,0,0,0)$, $\mathbf{i} = (0,1,0,0)$, $\mathbf{j} = (0,0,1,0)$, $\mathbf{k} = (0,0,0,1)$,
- ▷ $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$,
- ▷ $xy = (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4,\ x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3,\ x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2,\ x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1)$,
- ▷ $\Re(xy) = \Re(yx)$, $\alpha x = x\alpha$ for $x, y \in \mathbb{H}$, $\alpha \in \mathbb{R}$.

2.1.1. Matrices isomorphic to quaternions. The field $\mathbb{H}$ is isomorphic to a certain class $\mathbb{H}_{\mathbb{C}}$ of $\mathbb{C}^{2\times 2}$ matrices: Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$ and let $w = a_1 + a_2\mathrm{i}$, $z = a_3 + a_4\mathrm{i}$. Then $\mathbb{H}_{\mathbb{C}}$ is the space of matrices of the form

$$\mathbf{H} = \begin{pmatrix} w & z \\ -\overline{z} & \overline{w} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}} \text{ with ordinary matrix addition and multiplication.} \tag{2.1}$$

The field $\mathbb{H}$ is also isomorphic to a certain class $\mathbb{H}_{\mathbb{R}}$ of $\mathbb{R}^{4\times 4}$ matrices:

Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$. We introduce the linear mapping $\omega_1\colon\ \mathbb{H} \to \mathbb{H}_{\mathbb{R}}$ by

$$\omega_1(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \in \mathbb{H}_{\mathbb{R}}. \tag{2.2}$$

This mapping represents the isomorphic image of the quaternion a in the matrix space $\mathbb{R}^{4\times 4}$, see [29].

Let us introduce also the mapping $\omega_2\colon\ \mathbb{H} \to \mathbb{R}^{4\times 4}$,

$$\omega_2(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4\times 4}. \tag{2.3}$$

The two matrices $\omega_1(a)$, $\omega_2(b)$ coincide if and only if $a = b \in \mathbb{R}$.

The second mapping ω_2 reverses the multiplication order [1]:

$$\omega_2(ab) = \omega_2(b)\omega_2(a).$$

Let us put

$$\omega_3(a,b) := \omega_1(a)\omega_2(b) \in \mathbb{R}^{4\times 4}, \quad a, b \in \mathbb{H}. \tag{2.4}$$

2.2. Coquaternions. $\mathbb{H}_{\mathrm{coq}} := \mathbb{R}^4$—the field of coquaternions. Let $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{H}_{\mathrm{coq}}$.

- ▷ $x_1 := \Re x$—real part of x, $x_2 := \Im x$—imaginary part of x,
- ▷ $x_v := (0, x_2, x_3, x_4)$—vector part of x,
- ▷ $\overline{x} = (x_1, -x_2, -x_3, -x_4)$—conjugate of x,
- ▷ $\mathrm{abs}_2(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$—modulus of x; it may be also negative, $\mathrm{abs}_2(x)$ *is not* the square of a norm,
- ▷ unit vectors: $\mathbf{1} = (1,0,0,0)$, $\mathbf{i} = (0,1,0,0)$, $\mathbf{j} = (0,0,1,0)$, $\mathbf{k} = (0,0,0,1)$,
- ▷ $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$,
- ▷ $xy = (x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4, x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3, x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2, x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1)$.

Coquaternions were introduced in 1849 by Sir James Cockle, see [4] and [5], as complex matrices of the form

$$\mathbf{C} := \begin{pmatrix} w & z \\ \overline{z} & \overline{w} \end{pmatrix}. \tag{2.5}$$

The decisive difference between $\mathbf{H}$ and $\mathbf{C}$ is the inverse:

$$\mathbf{H}^{-1} = \frac{1}{|w|^2 + |z|^2}\begin{pmatrix} \overline{w} & -z \\ \overline{z} & w \end{pmatrix}, \quad \mathbf{C}^{-1} = \frac{1}{|w|^2 - |z|^2}\begin{pmatrix} \overline{w} & -z \\ -\overline{z} & w \end{pmatrix}.$$

A quaternion $\mathbf{H}$ has an inverse as long as $\mathbf{H} \neq \mathbf{0}$. The algebra of quaternions is free of zero divisors, it is a field, though not commutative.

The inverse of $\mathbf{C}$ exists if and only if the denominator $|w|^2 - |z|^2 \neq 0$. Thus, the algebra of coquaternions has zero divisors, does not form a field, see [29].

2.2.1. Matrices isomorphic to coquaternions. Let $a = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k} \in \mathbb{H}_{\mathrm{coq}}$ and define the matrix

$$\mathbf{C}_4(a) := \begin{pmatrix} a_1 & -a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix}. \tag{2.6}$$

Then the set of all matrices of the type $\mathbf{C}_4$ forms an algebra, and this algebra is isomorphic to the algebra $\mathbb{H}_{\mathrm{coq}}$ of coquaternions, see [10] and [13].

Let us note that the two matrices $\omega_1(a)$ in (2.2) and $\mathbf{C}_4(a)$ in (2.6) differ only by the signs of the elements of the 2×2 submatrix in the upper right corner.

The algebra of coquaternions is also isomorphic to the algebra of all real 2×2 matrices, see [20],

$$\mathbf{C}_2(a) := a_1 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{E}} + a_2 \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathbf{I}} + a_3 \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{J}} + a_4 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathbf{K}}$$
$$= \begin{pmatrix} a_1 + a_4 & a_2 + a_3 \\ -a_2 + a_3 & a_1 - a_4 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{2.7}$$

Given the four matrix elements $c_{11}, c_{12}, c_{21}, c_{22} \in \mathbb{R}$, the four components of a can be recovered by

$$a_1 = \frac{1}{2}(c_{11} + c_{22}), \quad a_2 = \frac{1}{2}(c_{12} - c_{21}), \quad a_3 = \frac{1}{2}(c_{12} + c_{21}), \quad a_4 = \frac{1}{2}(c_{11} - c_{22}).$$

If we denote the four basis elements in the order of the equation (2.7) by $\mathbf{E}$, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, then they obey the same multiplication rules as $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively.

An algebra of this type is also called a split algebra, in the current case the algebra of split quaternions, see [20].

Coquaternions are used, e.g., in physics; it is shown by Brody and Graefe [3]. That paper also contains an overview over relevant properties of coquaternions. There is another, very subtle, investigation mainly on the analysis of coquaternions with application to physics by Frenkel and Libine [8].

2.3. Extension to other nondivision algebras in $\mathbb{R}^4$. Let us go from the algebra of coquaternions to other algebras in $\mathbb{R}^4$, in particular, let us consider tessarines, cotessarines, nectarines, conectarines, tangerines, and cotangerines. We may observe many similarities with the quaternionic and coquaternionic case. The following table is obtained by allowing all eight combinations of signs ± 1 for the squares $\mathbf{i}^2$, $\mathbf{j}^2$, $\mathbf{k}^2$ and keeping the product $\mathbf{ij} = \mathbf{k}$ the same for all algebras.

No	Name of algebra	in short	$\mathbf{i}^2$	$\mathbf{j}^2$	$\mathbf{k}^2$	$\mathbf{ij}$	$\mathbf{jk}$	$\mathbf{ki}$
1.	Quaternions	$\mathbb{H}$	-1	-1	-1	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{j}$
2.	Coquaternions	$\mathbb{H}_{\text{coq}}$	-1	1	1	$\mathbf{k}$	$-\mathbf{i}$	$\mathbf{j}$
3.	Tessarines	$\mathbb{H}_{\text{tes}}$	-1	1	-1	$\mathbf{k}$	$\mathbf{i}$	$-\mathbf{j}$
4.	Cotessarines	$\mathbb{H}_{\text{cot}}$	1	1	1	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{j}$
5.	Nectarines	$\mathbb{H}_{\text{nec}}$	1	-1	1	$\mathbf{k}$	$\mathbf{i}$	$-\mathbf{j}$
6.	Conectarines	$\mathbb{H}_{\text{con}}$	1	1	-1	$\mathbf{k}$	$-\mathbf{i}$	$-\mathbf{j}$
7.	Tangerines	$\mathbb{H}_{\text{tan}}$	1	-1	-1	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{j}$
8.	Cotangerines	$\mathbb{H}_{\text{cotan}}$	-1	-1	1	$\mathbf{k}$	$-\mathbf{i}$	$-\mathbf{j}$

The eight algebras separate into four noncommutative ones, in particular No. 1, 2, 5, 6, and into four commutative ones, those with numbers 3, 4, 7, 8. We note that the real numbers commute with all members of all algebras.

The names of No. 2–4 were introduced by James Cockle in 1849 in a series of articles in Philosophical Magazine, see [6], [4], [27]. The names nectarines, conectarines, tangerines, cotangerines were introduced very recently by Bernd Schmeikal (Vienna), see [28].

2.4. Inversion formula.

Lemma 2.1. *Let $\mathcal{A}$ be one of the four noncommutative algebras (number 1, 2, 5, or 6) and $a \in \mathcal{A}$. Then the product $a\overline{a} = \overline{a}a$ is real and*

$$a^{-1} = \frac{\overline{a}}{\overline{a}a} \quad \text{if } \overline{a}a \neq 0. \tag{2.8}$$

Proof. Because $\operatorname{abs}_2(a) = \overline{a}a$, $a = (a_1, a_2, a_3, a_4)$, we apply the multiplication rules and obtain

$$\operatorname{abs}_2(a) := a\overline{a} = \begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 & \text{for algebra } \mathbb{H}, \\ a_1^2 + a_2^2 - a_3^2 - a_4^2 & \text{for algebra } \mathbb{H}_{\text{coq}}, \\ a_1^2 - a_2^2 + a_3^2 - a_4^2 & \text{for algebra } \mathbb{H}_{\text{nec}}, \\ a_1^2 - a_2^2 - a_3^2 + a_4^2 & \text{for algebra } \mathbb{H}_{\text{con}}. \end{cases}$$

If $\operatorname{abs}_2(a) \neq 0$, the inversion formula

$$a^{-1} = \frac{\overline{a}}{\operatorname{abs}_2(a)}$$

is valid in all four noncommutative algebras. □

2.5. Classes of equivalence.

2.5.1. Classes of equivalence in quaternions. Two quaternions x and y are equivalent, denoted by $x \sim y$, if $y = h^{-1}xh$ for some $h \in \mathbb{H} \setminus \{0\}$. For fixed $x \in \mathbb{H}$ the set

$$[x] = \{y \in \mathbb{H}\colon\ y = h^{-1}xh \text{ for } h \in \mathbb{H} \setminus \{0\}\} \tag{2.9}$$

is called the equivalence class of x. If x is real, then obviously $[x] = \{x\}$, i.e., the equivalence class $[x]$ consists of x alone.

Lemma 2.2. *Two quaternions x and y are equivalent if and only if*

$$\Re x = \Re y \quad \text{and} \quad |x| = |y|. \tag{2.10}$$

Let $x = (x_1, x_2, x_3, x_4)$. It follows that $y = (x_1, \sqrt{x_2^2 + x_3^2 + x_4^2}, 0, 0)$ is equivalent to x and $y \in [x]$ is the only complex element in $[x]$ with nonnegative imaginary part, see [12].

2.5.2. Classes of equivalence in coquaternions. In (2.7), we have already seen that the algebra $\mathbb{H}_{\mathrm{coq}}$ is isomorphic to $\mathbb{R}^{2\times 2}$, where this isomorphism is for $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}_{\mathrm{coq}}$ defined by

$$\mathbf{C}_2(a) := \begin{bmatrix} a_1 + a_4 & a_2 + a_3 \\ -a_2 + a_3 & a_1 - a_4 \end{bmatrix}.$$

Definition 2.1. Two coquaternions a, b will be called similar, denoted by $a \sim b$, if the corresponding matrices $C_2(a), C_2(b)$ are similar, or in other words, if there is a nonsingular (= invertible) coquaternion h such that $a = h^{-1}bh$. By

$$[a] := \{b\colon\ b = h^{-1}ah \text{ for all invertible } h \in \mathbb{H}_{\mathrm{coq}}\}$$

we denote the equivalence class of all coquaternions which are similar to a.

Remark 2.1. The equivalence class $[a]$ consists of one single element if and only if a is real.

Lemma 2.3. *Let $a \sim b$, $a, b \in \mathbb{H}_{\mathrm{coq}}$. Then*

$$\Re(a) = \Re(b), \quad \mathrm{abs}_2(a) = \mathrm{abs}_2(b). \tag{2.11}$$

Example 2.1. Let $a = (\alpha, 0, 0, 0)$, $\alpha \in \mathbb{R}$, and $b = (\alpha, 5, 4, 3)$ be two coquaternions. Because a is real, then $[a]$ consists of one single element only, i.e., a, b are not similar. However, $\Re(a) = \Re(b) = \alpha$ and $\mathrm{abs}_2(a) = \mathrm{abs}_2(b) = \alpha^2$, i.e., in contrast to the quaternionic case, conditions (2.11) are not sufficient for similarity of coquaternions.

Definition 2.2. Two coquaternions a, b are said to be quasi-similar, written as $a \sim^q b$, if (2.11) is valid.

Lemma 2.4. *The relation $\sim^q$ is an equivalence relation.*

Proof. The three properties of equivalence relations, $a \sim^q a$ (reflexivity), $a \sim^q b \Leftrightarrow b \sim^q a$ (symmetry), and $a \sim^q b$, $b \sim^q c \Rightarrow a \sim^q c$ (transitivity) are easily verified. $\square$

The corresponding quasisimilarity classes are denoted by

$$[a]_q = \{b \in \mathbb{H}_{\mathrm{coq}}\colon\ b \sim^q a\}.$$

Let us note that

$$a \sim b \Rightarrow a \sim^q b \quad \text{and} \quad [a] \subset [a]_q.$$

Due to the first condition in (2.11), distinct real numbers are in different equivalence classes with respect to both $\sim$ and $\sim^q$.

3. Linear mappings over $\mathbb{R}$ in general

3.1. Column operator. Let $L\colon\ \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping over $\mathbb{R}$. It can be represented by a real matrix $\mathbf{M}$ of size $(m \times n)$, see [11]. In order to find this matrix, we define a column operator col by

$$\operatorname{col}(x) := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $x_1, x_2, \ldots, x_n$ are the components of x. If it happens that x is a matrix, we put the columns of that matrix from the left to the right into one column in order to define col for that matrix. By evaluating L at $x \in \mathbb{R}^n$ and applying the col operator we obtain

$$\operatorname{col}(L(x)) = \mathbf{M}\operatorname{col}(x), \quad \mathbf{M} \in \mathbb{R}^{m\times n}, \quad \operatorname{col}(\cdot) \in \mathbb{R}^n, \tag{3.1}$$

where $\mathbf{M}$ is unknown so far.

Let e_j be the standard unit vectors in $\mathbb{R}^n$, $j = 1, 2, \ldots, n$. Putting $x := e_j$, we obtain

$$\operatorname{col}(L(e_j)) = \mathbf{M}\operatorname{col}(e_j) = \begin{bmatrix} \mu_{1j} \\ \mu_{2j} \\ \vdots \\ \mu_{mj} \end{bmatrix}, \quad j = 1, \ldots, n, \tag{3.2}$$

where μ_{ij} are the elements of the matrix $\mathbf{M}$ and the right-hand side of (3.2) represents the jth column of $\mathbf{M}$. Hence, the matrix $\mathbf{M}$ is completely known by the n values $L(e_j)$. We note that $\mathbf{M}$ will be an integer if the values of L are integers, see [16].

A typical nontrivial example is the mapping $L\colon\ \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$L(\mathbf{X}) = \mathbf{AXB}, \quad \mathbf{A} \in \mathbb{R}^{p\times q}, \quad \mathbf{X} \in \mathbb{R}^{q\times r}, \quad \mathbf{B} \in \mathbb{R}^{r\times s}, \tag{3.3}$$

where in this case we have $m = ps$, $n = qr$.

3.2. Quaternionic linear mappings in one variable. Let us study quaternionic linear mappings $\Lambda\colon\ \mathbb{H}\to\mathbb{H}$ over $\mathbb{R}$ which are defined by the property

$$\Lambda(\gamma x+\delta y)=\gamma\Lambda(x)+\delta\Lambda(y)\quad \forall\, x,y\in\mathbb{H}\ \forall\,\gamma,\delta\in\mathbb{R}. \tag{3.4}$$

All linear mappings have the property $\Lambda(0)=0$. This follows from (3.4) by putting $x=y$, $\gamma=1$, $\delta=-1$.

Definition 3.1. A quaternionic linear mapping $\Lambda\colon\ \mathbb{H}\to\mathbb{H}$ will be called nonsingular if $\Lambda(x)=e$ has a unique solution for all choices of $e\in\mathbb{H}$. The mapping Λ will be called singular if it is not nonsingular.

Lemma 3.1. *The linear mapping Λ is singular if and only if the homogeneous equation $\Lambda(x)=0$ has nontrivial solutions.*

Proof. For the proof see [19]. $\square$

3.2.1. Null space of Λ. The null space (or kernel) of Λ is defined by

$$\mathcal{N}:=\{x\in\mathbb{H}\colon\ \Lambda(x)=0\}. \tag{3.5}$$

It is a linear subspace of $\mathbb{H}$, regarded as a subspace over $\mathbb{R}^4$, i.e., the dimension of $\mathcal{N}$ may vary from zero to four.

The linear mapping $\Lambda\colon\ \mathbb{H}\to\mathbb{H}$ also defines a linear mapping $\widetilde{\Lambda}\colon\ \mathbb{R}^4\to\mathbb{R}^4$ which can be defined by a 4×4 real matrix.

Theorem 3.1. *Let $\Lambda\colon\ \mathbb{H}\to\mathbb{H}$ be a quaternionic linear mapping. Then there exists a matrix $\mathbf{M}\in\mathbb{R}^{4\times 4}$ such that $\Lambda(x)=\mathbf{M}x$, where the quaternions x, $\Lambda(x)$ have to be identified with the column vectors $x,\mathbf{M}x\in\mathbb{R}^4$, respectively, and where*

$$\mathbf{M}:=(\,\Lambda(1),\Lambda(\mathbf{i}),\Lambda(\mathbf{j}),\Lambda(\mathbf{k})\,)\,. \tag{3.6}$$

The entries $\Lambda(\mathbf{j})$, $j=1,2,3,4$, must be read as column vectors in $\mathbb{R}^4$.

This theorem allows us to reduce the given linear equation to a linear matrix equation in four variables for which solution techniques are known.

3.2.2. Quaternionic linear mappings of type $\Lambda(x):=\sum\limits_{j=1}^{m}b_jxc_j$. Let us consider a quaternionic mapping

$$\Lambda_m(x):=\sum_{j=1}^{m}b_jxc_j \tag{3.7}$$

defined for any fixed positive integer m and for $2m$ quaternionic constants b_j, $c_j\in\mathbb{H}$, $j=1,2,\dots,m$.

The set of all such linear mappings Λ_m separates into two classes. One class consists of all mappings Λ_m with $m \leqslant 2$ for which a complete answer to all reasonable questions can be found. All other Λ_m, namely those with $m \geqslant 3$, belong to a class for which we can gather only incomplete information. We introduce a formal simplification: Let $\mu(x) := b_m^{-1} x c_1^{-1}$. This linear mapping is nonsingular and thus (put $\tilde{b}_j = b_m^{-1} b_j$, $\tilde{c}_j = c_j c_1^{-1}$, $j = 2, 3, \ldots, m-1$),

$$(\mu \circ \lambda_m)(x) = b_m^{-1} \Big(\sum_{j=1}^{m} b_j x c_j \Big) c_1^{-1} = \begin{cases} b_1 x + \displaystyle\sum_{j=2}^{m-1} \tilde{b}_j x \tilde{c}_j + x c_m & \text{for } m \geqslant 2, \\ x & \text{for } m = 1 \end{cases}$$

will be singular if and only if λ_m is singular. After a formal simplification, it turns out that without loss of generality we can investigate

$$\Lambda_m(x) := ax + \sum_{j=1}^{m-2} b_j x c_j + xd, \quad m \geqslant 2, \tag{3.8}$$

for example

$$\Lambda_2(x) := ax + xd \quad \text{for } m = 2,$$

and for $m = 3$ we write simply

$$\Lambda_3(x) := ax + bxc + xd.$$

3.3. Linear equations in coquaternions. Linear equations in coquaternions are formally similar to linear equations in quaternions. A linear system will always be a linear system over $\mathbb{R}$. Linearity with respect to $\mathbb{C}$ or to $\mathbb{H}_{\mathrm{coq}}$ is in general not granted.

A linear system in n coquaternions x_k, $k = 1, 2, \ldots, n$, and m equations is defined as follows: Let

$$l_j(u) := \sum_{k=1}^{K_j} a_k^{(j)} u b_k^{(j)}, \quad j = 1, 2, \ldots, mn \tag{3.9}$$

be an arbitrary set of mn linear, coquaternionic mappings in one coquaternionic variable u, where $a_k^{(j)}$, $b_k^{(j)}$, $k = 1, 2, \ldots, K_j$, are given coquaternions, and K_j are given positive integers, $j = 1, 2, \ldots, mn$.

A linear system $L(x)$ in n coquaternionic variables x_k, $k = 1, 2, \dots, n$, and m equations will then be defined by

$$L_1(x) := \sum_{j=1}^{n} l_j(x_j), \quad L_2(x) := \sum_{j=1}^{n} l_{j+n}(x_j), \dots,$$

$$L_m(x) := \sum_{j=1}^{n} l_{j+(m-1)n}(x_j) \Rightarrow L(x) := \begin{bmatrix} L_1(x) \\ L_2(x) \\ \vdots \\ L_m(x) \end{bmatrix},$$

where $x \in \mathbb{R}^{4n}$ consists of one column composed out of $x_1, x_2, \dots, x_n$.

Note that L is not a linear mapping over $\mathbb{C}$ or $\mathbb{H}_{\mathrm{coq}}$. However, L is a linear mapping over $\mathbb{R}$, because real coquaternions commute with arbitrary coquaternions. Hence, a matrix $\mathbf{M}$ as in (3.1) and (3.2) exists.

4. One-sided (simple) quaternionic polynomials

Let p_n be a given quaternionic polynomial of degree n:

$$p_n(z) = \sum_{j=0}^{n} a_j z^j, \quad z,\, a_j \in \mathbb{H},\ j = 0, 1, 2, \dots, n, \quad a_0 a_n \neq 0. \tag{4.1}$$

Polynomial p_n is called one-sided (or simple) quaternionic polynomial.

Let us note that we suppose that $a_0 \neq 0$, i.e., the origin is never a zero of p_n, $a_n \neq 0$, i.e., the polynomial degree is not less than n.

Definition 4.1. Let z_0 be a zero of p_n. If z_0 is not real and it has the property

$$p_n(z) = 0 \quad \forall\, z \in [z_0],$$

we say that z_0 generates a spherical zero or z_0 is a spherical zero.

If z_0 is real or does not generate a spherical zero, it is called an isolated zero.

4.1. All powers of a quaternion. To get rid of the powers z^j and to simplify formula (4.1) we apply the following two term recursion: Iteration process (two term recursion):

$$z^j = \alpha_j z + \beta_j, \quad \alpha_j, \beta_j \in \mathbb{R}, \quad j = 0, 1, \dots, \tag{4.2}$$

where

$$\alpha_0 = 0, \quad \beta_0 = 1, \quad \alpha_{j+1} = 2\Re(z)\alpha_j + \beta_j, \quad \beta_{j+1} = -|z|^2\alpha_j, \quad j = 0, 1, \dots$$

In order to compute all powers of $z \in \mathbb{H}$ up to degree n, one needs $n-1$ quaternionic multiplications (one quaternionic multiplication = 28 flops), whereas the above recursion needs only $3n$ flops.

Lemma 4.1. *Let $z_1 \sim z_2$. Then α_j, β_j, $j = 0, 1, \ldots, n$, defined in (4.2) are the same for z_1, z_2.*

Proof. For the proof see [26]. $\square$

By means of recursion (4.2), the polynomial p_n can be written as

$$p_n(z) = \sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} a_j(\alpha_j + \beta_j z) = A(z) + B(z)z, \tag{4.3}$$

where

$$A(z) = \sum_{j=0}^{n} \alpha_j a_j, \quad B(z) = \sum_{j=0}^{n} \beta_j a_j. \tag{4.4}$$

Theorem 4.1. *Let $z_0 \in \mathbb{H}$ be fixed. Then both $A(z)$ and $B(z)$ are constant for all $z \in [z_0]$. Let z_0 be a zero of p_n, i.e., $p_n(z_0) = A(z_0) + B(z_0)z_0 = 0$. Then*

$$p_n(z) = A(z) + B(z)z = 0 \quad \forall\, z \in [z_0]. \tag{4.5}$$

The quantities $A(z)$, $B(z)$ in (4.5) can vanish only simultaneously.

If $A(z_0) = 0$ and z_0 is not real, then all $z \in [z_0]$ are zeros of p_n, thus, z_0 generates a spherical zero of p_n.

If $A(z_0) \neq 0$, then z_0 is an isolated zero of p_n.

Theorem 4.2. *Let $z_0, z_1 \in \mathbb{H}$ be two different zeros of p_n with $z_0 \in [z_1]$. Then $p_n(z) = 0$ for all $z \in [z_1]$, z_0 generates a spherical zero of p_n, and $A(z) = B(z) = 0$ for all $z \in [z_0]$. In particular, $z_0 \notin \mathbb{R}$ is a spherical zero of p_n if and only if $A(z_0) = 0$.*

Finally, let us summarize: Let z_0 be a zero of the one sided (simple) quaternionic polynomial p_n. Then:

1. If z_0 is real, then z_0 is isolated.
2. If z_0 is not real and $A(z_0) = 0$, then all $z \in [z_0]$ are zeros of p_n, z_0 is spherical.
3. If z_0 is not real and $A(z_0) \neq 0$, then z_0 is isolated.

The computation of all zeros of p_n, including their types, can be reduced to the computation of all zeros of a real polynomial of degree $2n$, see [14].

5. Two-sided type quaternionic polynomials

We consider so-called two-sided polynomials

$$p(z) := \sum_{j=0}^{n} a_j z^j b_j, \quad z, a_j, b_j \in \mathbb{H}, \quad j = 0, 1, \ldots, n \in \mathbb{N}, \tag{5.1}$$
$$a_0 b_0 \neq 0, \quad a_n b_n \neq 0.$$

By means of (4.2) the polynomial p can be written as

$$\begin{aligned} p(z) &:= \sum_{j=0}^{n} a_j z^j b_j = \sum_{j=0}^{n} a_j(\alpha_j + \beta_j z) b_j \\ &= \sum_{j=0}^{n} \alpha_j a_j\, b_j + \sum_{j=0}^{n} \beta_j a_j z b_j = A(z) + \sum_{j=0}^{n} \beta_j a_j z b_j, \end{aligned} \tag{5.2}$$

where

$$A(z) := \sum_{j=0}^{n} \alpha_j a_j b_j. \tag{5.3}$$

Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}$. We introduce a column operator $\mathrm{col}\colon \mathbb{H} \to \mathbb{R}^4$,

$$\mathrm{col}(a) := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

The purpose of this column operator is to regard a quaternion as a matrix with one column and four rows.

Lemma 5.1. *The column operator is linear over $\mathbb{R}$, i.e.,*

$$\mathrm{col}(\alpha a + \beta b) = \alpha\, \mathrm{col}(a) + \beta\, \mathrm{col}(b), \quad a, b \in \mathbb{H}, \alpha, \beta \in \mathbb{R}.$$

Lemma 5.2. *For arbitrary quaternions a, b, c we have*

$$\mathrm{col}(ab) = \omega_1(a)\mathrm{col}(b) = \omega_2(b)\mathrm{col}(a), \tag{5.4}$$

$$\mathrm{col}(abc) = \omega_1(a)\omega_2(c)\mathrm{col}(b). \tag{5.5}$$

We apply the mapping ω_3 to the last term in (5.2), in particular,

$$\mathrm{col}(abc) = \omega_3(a, c)\mathrm{col}(b), \quad a, b, c, \in \mathbb{H},$$

to the equation

$$p(z) = A(z) + \sum_{j=0}^{n} \beta_j a_j z b_j$$

and obtain

$$\mathrm{col}(p(z)) = \mathrm{col}(A(z)) + \sum_{j=0}^{n} \beta_j \omega_3(a_j, b_j)\mathrm{col}(z) = \mathrm{col}(A(z)) + \mathbf{B}(z)\mathrm{col}(z), \tag{5.6}$$

where

$$\mathbf{B}(z) = \sum_{j=0}^{n} \beta_j \omega_3(a_j, b_j) \in \mathbb{R}^{4\times 4}. \tag{5.7}$$

The matrix $\mathbf{B}$ plays an important role in the classification of zeros of two-sided quaternionic polynomials.

5.1. Classification of zeros of two-sided quaternionic polynomials.

Definition 5.1. Let z be a nonreal zero of p, defined in (5.1), and let z_0 be the complex representative of $[z]$. Then we classify the zeros z of p with respect to the rank of $\mathbf{B}(z_0)$, see [15].

The zero z is called zero of type k if

$$\mathrm{rank}(\mathbf{B}(z_0)) = 4 - k, \quad 0 \leqslant k \leqslant 4\,.$$

A zero of type 4 ($\mathrm{rank}(\mathbf{B}(z_0)) = 0$) will be called a spherical zero. It has the property that all $z \in [z_0]$ are zeros.

A zero of type 0 ($\mathbf{B}(z_0)$ is nonsingular) will be called an isolated zero. In this case

$$\mathrm{col}(z) = -(B(z_0))^{-1}\mathrm{col}(A(z_0))$$

and this z is the only zero in $[z_0]$.

A real zero will also be called an isolated zero.

6. Zeros of one-sided polynomials over $\mathbb{R}^4$ algebras

Let $\mathcal{A}$ be one of eight algebras in $\mathbb{R}^4$. We have already divided them into noncommutative algebras: quaternions, coquaternions, nectarines, conectarines and commutative ones: tessarins, cotessarins, tangerines, cotangerines.

Now we will treat unilateral polynomials p of the form

$$p(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + a_n z^n,\ z, a_j \in \mathcal{A}, \qquad j = 0, 1, \ldots, n,\ a_0, a_n \text{ invertible in } \mathcal{A}. \tag{6.1}$$

Let us consider three noncommutative algebras $\mathbb{H}_{\text{coq}}$, $\mathbb{H}_{\text{nec}}$ and $\mathbb{H}_{\text{con}}$.

Let $a = (a_1, a_2, a_3, a_4) \in \mathcal{A}$. We will deal with the following problem, see [9]: Given a polynomial p over $\mathcal{A}$ and a similarity class $[z] \subset \mathcal{A}$, which is known to contain a zero $z_0 \in [z]$ of p, how to find the zero?

The main idea is to write the polynomial p in a formally linear form. Similarly as for quaternions, we use the two-term recursion (4.2) and obtain

$$z^2 = -\text{abs}_2(z) + 2\Re(z) z, \tag{6.2}$$

which is valid in $\mathcal{A}$.

We define the conjugate of a, denoted either by $\overline{a}$ or by $\text{conj}(a)$, by

$$\overline{a} = \text{conj}(a) = (a_1, -a_2, -a_3, -a_4). \tag{6.3}$$

For the product $a\overline{a}$ we use again the notation

$$\text{abs}_2(a) = a\overline{a}. \tag{6.4}$$

Let us restrict our attention to one quasi-similarity class $[z]_q$. Then the coefficients α_k, β_k for $k \geqslant 0$ are constant on this class. Applying (4.2) to all powers in the polynomial p, we obtain

$$p(z) = \sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n} a_k(\alpha_k + \beta_k z) = \sum_{k=0}^{n} \alpha_k a_k + \left(\sum_{k=0}^{n} \beta_k a_k\right) z := A + Bz \tag{6.5}$$

and A, B are constant on the quasi similarity class $[z]_q$.

Theorem 6.1. *Let B be invertible on the given class $[z]_q$ and let $[z]_q$ contain a zero z_0 of p. Then*

$$z_0 = -B^{-1} A \tag{6.6}$$

is the only zero in $[z]_q$. If $A = B = 0$, then all elements in $[z]_q$ are zeros of p and there may exist another undetected zero quasi similar to r_1.

P r o o f. From (6.5) it follows that $p(z_0) = 0$. Let there be two distinct zeros, $z_0, z_1 \in [z]_q$. Then $p(z_0) = A + Bz_0 = 0$ and $p(z_1) = A + Bz_1 = 0$, which implies $B(z_0 - z_1) = 0$. If B is invertible, then $z_0 = z_1$ would follow, a contradiction. Thus, B is noninvertible if there exist two distinct zeros $z_1, z_2 \in [z]_q$. The last part is obvious. $\square$

The algorithm to find all zeros of polynomials over $\mathcal{A}$ can be found in [17].

Remark 6.1. Number of zeros: Let p be a polynomial of degree n as defined in (6.1) over $\mathcal{A}$. Then p may have $n(2n-1)$ zeros.

Finally, we would like to mention that in this part we used two essential ideas of other authors, namely:

(1) The idea by Anatoliy Pogoruy and Michael Shapiro was to write the powers z^j in the form $\alpha z + \beta$ with real α, β, which reduces the two-sided polynomials to a sum of terms of the form azb, see [26]. This idea also gave birth to the introduction of equivalence classes of zeros in $\mathbb{H}$.
(2) The idea by Ludmilla Aramanovitch was the introduction of the matrix ω_2 (formula (2.3)) which permitted to pull out the variable z from azb (formula (5.5)). Both ingredients allowed the development of the important formula (6.2), see [1].

7. The algebraic eigenvalue problem over noncommutative algebras

7.1. Eigenvalues of matrices with general algebraic entries.

Definition 7.1. Let $\mathbf{A} \in \mathcal{A}^{n\times n}$ for an algebra $\mathcal{A}$. If there exists an element $\lambda \in \mathcal{A}$ and a column vector $\mathbf{x} \in \mathcal{A}^{n\times 1}$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{x}\lambda, \quad \mathbf{x} \text{ contains an invertible component}, \tag{7.1}$$

then λ is called the eigenvalue of $\mathbf{A}$ with respect to the eigenvector $\mathbf{x}$. The pair $(\lambda, \mathbf{x})$ is called the eigenpair.

The set of all eigenvalues of $\mathbf{A}$ is denoted by $\sigma(\mathbf{A})$.

Lemma 7.1. *Let $\mathcal{A}$ be a noncommutative algebra and λ an eigenvalue of $\mathbf{A}$ with respect to an eigenvector $\mathbf{x}$. Then for all invertible $h \in \mathcal{A}$, the set $\Lambda := \{h^{-1}\lambda h\}$ consists of eigenvalues of $\mathbf{A}$ with respect to $\mathbf{x}h$.*

Lemma 7.2. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}^{n\times n}$ be two similar matrices. Then $\sigma(\mathbf{A}) = \sigma(\mathbf{B})$.*

7.2. Some facts about eigenvalues. In order to find all eigenvalues of a given matrix $\mathbf{A} \in \mathcal{A}^{n\times n}$, it is sufficient to find one representative in each similarity class of eigenvalues. The number of eigenvalues of $\mathbf{A}$ will be, correspondingly, defined as the number of distinct similarity classes of $\sigma(\mathbf{A})$.

Theorem 7.1. *Let $\mathcal{A}$ be either a division algebra or a commutative algebra and let $\mathbf{A} \in \mathcal{A}^{n\times n}$ be an upper triangular matrix. Then the diagonal elements of $\mathbf{A}$ are the eigenvalues of $\mathbf{A}$. The same is true for lower triangular matrices.*

Theorem 7.2. *Let $\mathbf{A} \in \mathcal{A}^{n\times n}$ be a given upper or lower triangular matrix with matrix entries a_{jk}, $j,k = 1,2,\dots,n$. The matrix $\mathbf{A}$ is singular if and only if one of the diagonal elements a_{jj}, $j = 1,2,\dots,n$, is singular.*

The eigenvalue problem over an arbitrary algebra can be expressed in the real form

$$(\mathbf{M} - \Lambda)\mathrm{col}(\mathbf{x}) = \mathbf{0}, \tag{7.2}$$

and there will be a solution $\mathbf{x} \neq \mathbf{0}$ if and only if there is a matrix Λ such that $\mathbf{M} - \Lambda$ is singular. The matrix $\mathbf{M} - \Lambda$ is a triangular block matrix.

8. Computation of eigenvalues by Newton's technique

For a general square matrix $\mathbf{A} \in \mathcal{A}^{n\times n}$ we consider the eigenvalue problem (7.1) in the form

$$G_1(\mathbf{x}, \lambda) := \mathbf{x}\lambda - \mathbf{A}\mathbf{x}, \tag{8.1}$$

$$G_2(\mathbf{x}) := \|\mathrm{col}(\mathbf{x})\|^2 - 1 \tag{8.2}$$

and solve

$$G(\mathbf{x}, \lambda) := \begin{bmatrix} G_1(\mathbf{x}, \lambda) \\ G_2(\mathbf{x}) \end{bmatrix} = \mathbf{0} \tag{8.3}$$

by Newton's method. The quantity $\|\cdot\|^2$ is the square of the standard Euclidean norm in $\mathbb{R}^{nN}$. The condition $G_2(\mathbf{x}) = 0$ is a normalization condition for the eigenvectors. It is independent of the algebra $\mathcal{A}$ under investigation. However, it does not imply uniqueness of $\mathbf{x}$. In all algebras the eigenvectors $\mathbf{x}$ and $-\mathbf{x}$ are simultaneous eigenvectors or not.

If $z \in \mathcal{A}$ commutes with an eigenvalue λ and $\|\mathrm{col}(z)\| = 1$ (Euclidean norm), then $\mathbf{x}$ and $\mathbf{x}z$ are both eigenvectors for the same λ.

Applying the techniques developed in [21], we obtain the derivative of G in the form

$$G'(\mathbf{x}, \lambda)(\mathbf{h}, h_1) = \begin{bmatrix} \mathbf{h}\lambda + \mathbf{x}h_1 - \mathbf{A}\mathbf{h} \\ 2\mathrm{col}(\mathbf{x})^\top \mathrm{col}(\mathbf{h}) \end{bmatrix}, \quad \mathbf{x}, \mathbf{h} \in \mathcal{A}^{n\times 1},\ h_1 \in \mathcal{A}^{1\times 1}. \tag{8.4}$$

The Newton's technique consists of solving the linear system

$$G'(\mathbf{x}_k, \lambda_k)(\mathbf{h}, h_1) = -G(\mathbf{x}_k, \lambda_k), \quad k = 0,1,\dots, \tag{8.5}$$

for $(\mathbf{h}, h_1)$, where the start values $\mathbf{x}_0$, λ_0 are given, in principle, arbitrarily.

We observe that the number of real unknowns $(\mathbf{x}, \lambda)$ in equation (8.3) is $(n+1)N$, whereas the number of real equations is $nN+1$. In the linear Newton equation (8.5) we have, thus, $(n+1)N$ real unknowns $(\mathbf{h}, h_1)$ and $nN+1$ real equations. Thus, the system is underdetermined.

Finally, the left-hand side of (8.5) is linear in $(\mathbf{h}, h_1)$ and thus, can be expressed by a matrix the form of which is given in (8.4). The result is

$$G'(\mathbf{x}, \lambda)(\mathbf{h}, h_1) = \mathbf{M} \begin{bmatrix} \operatorname{col}(\mathbf{h}) \\ \operatorname{col}(h_1) \end{bmatrix}, \quad \text{where } \mathbf{M} \in \mathbb{R}^{(nN+1)\times(n+1)N}.$$

9. Calculation of eigenvalues of a quaternion matrix using the companion matrix and companion polynomial

Given the polynomial coefficients $a_0, a_1, \ldots, a_n$, $a_n = 1$, the companion matrix is defined as

$$\mathbf{C} := \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & -a_{n-2} \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix} \in \mathbb{H}^{n\times n}. \tag{9.1}$$

The essence of the method is to find the set of eigenvalues of $\mathbf{C}$. How to find these eigenvalues?

Let us recall that we identify complex numbers $d_1 + d_2\mathrm{i}$ with quaternions $d = (d_1, d_2, 0, 0)$.

Theorem 9.1. *Two complex numbers with nonnegative imaginary part are similar if and only if they are identical.*

In order to find the eigenvalues of $\mathbf{A}$ we use an isomorphism

$$\iota\colon\ \mathbb{H} \to \mathbf{C}^{2\times 2}$$

defined by

$$\iota(a) = \begin{bmatrix} w & z \\ -\overline{z} & \overline{w} \end{bmatrix} = \begin{bmatrix} a_1 + a_2\mathrm{i} & a_3 + a_4\mathrm{i} \\ -a_3 + a_4\mathrm{i} & a_1 - a_2\mathrm{i} \end{bmatrix} \tag{9.2}$$

and apply this isomorphism to the whole matrix $\mathbf{A}$.

Theorem 9.2. *Eigenvalues of the matrix $\iota(\mathbf{A})$:*

(i) *The $2n$ eigenvalues of the complex matrix $\iota(\mathbf{A})$ are either real or pairs of complex conjugate numbers.*
(ii) *Real eigenvalues of $\iota(\mathbf{A})$ appear in duplicate.*
(iii) *The eigenvalues of $\iota(\mathbf{A})$ are similar to the eigenvalues of $\mathbf{A}$.*
(iv) *The similarity classes of the eigenvalues of $\iota(\mathbf{A})$ are identical with the eigenvalues of $\mathbf{A}$.*

Let
$$p_n(z) = \sum_{j=0}^{n} a_j z^j, \quad z, a_j \in \mathbb{H}, \quad j = 0, 1, 2, \ldots, n, \quad a_n = 1.$$
The main information is the following.

Theorem 9.3. *The zeros of the polynomial p are located in the similarity classes of the eigenvalues of the complex matrix $\iota(\mathbf{C})$.*

How to find the zeros of p if the similarity classes of the eigenvalues of $\iota(\mathbf{C})$ are known?

9.1. Niven's algorithm. Niven's algorithm of 1941 is based on the following representation of p.

Let r be a quadratic polynomial of the form $r(z; u, v) = z^2 - 2uz + v$, $u, v \in \mathbb{R}$. The central property of this polynomial r is that it vanishes if we choose $u = \Re(z)$, $v = \mathrm{abs}_2(z)$. Due to (2.10), r will even vanish if we replace z by all elements of $[z]$ but keep u, v unchanged, i.e., r will vanish for the whole class $[z]$. Dividing of p by r yields

$$p(z) = q(z)r(z; u, v) + R_0(u, v)z + R_1(u, v), \tag{9.3}$$

where q is a polynomial of degree $n-2$ and $R_0(u, v), R_1(u, v)$ follow from a (simple) recursion based on (9.3). Now $R_0(u, v)z + R_1(u, v)$ may be regarded as the remainder of the division. The polynomial q is of no importance.

We insert one of the eigenvalues z of $\iota(\mathbf{C})$ including $u = \Re(z)$, $v = \mathrm{abs}_2(z)$ into Niven's equation (9.3). Then $r(z; u, v) = 0$ and the equation

$$p(z) = R_0(u, v)z + R_1(u, v) = 0 \tag{9.4}$$

remains to be solved. If $R_0(u, v)$ is invertible, then

$$z = -(R_0(u, v))^{-1} R_1(u, v).$$

This is an isolated zero. If $R_0(u, v) = 0$, then also $R_1(u, v) = 0$ and all elements of $[z]$ solve (9.4). This zero of p is spherical.

Now it is natural to ask how many zeros the polynomial p has. The number of eigenvalues of $\iota(\mathbf{C})$ is $2n = 2n_1 + 2n_2$, where $2n_1$ is the number of real eigenvalues and $2n_2$ is the number of complex eigenvalues.

Altogether, there are at most n_1 distinct real eigenvalues and at most n_2 distinct complex eigenvalues with positive real part. Thus, the number of zeros of p is at most $n_1 + n_2 = n$. This result was already known by Gordon and Motzkin, 1965.

9.2. The introduction of the companion polynomial. Again based on the coefficients $a_0, a_1, \ldots, a_n = 1$ of the given polynomial p, the following real polynomial c is defined by

$$c(z) = \sum_{j,k=0}^{n} a_j \overline{a_k} z^{j+k} = \sum_{\ell=0}^{2n} b_\ell z^\ell, \tag{9.5}$$

$$b_\ell = \sum_{m=\max(0,\ell-n)}^{\min(\ell,n)} \overline{a_m} a_{\ell-m} \in \mathbb{R}, \tag{9.6}$$

and c is called the companion polynomial of p. This name with this definition was for the first time used by Janovská and Opfer, [14].

We will call the solutions of $c(z) = 0$ the roots of c, in contrast to zeros, which are the solutions of $p(z) = 0$.

Since the polynomial c is real and of degree $2n$, it has an even number (zero allowed) of real roots and an even number (zero allowed) of complex roots, where the complex roots always appear in pairs of complex conjugate roots.

Theorem 9.4. *The zeros of the polynomial p are located in the similarity classes of the roots of the real companion polynomial c.*

Proof. For the proof see [18]. □

In order to find the zeros of p we do exactly the same steps as before with the eigenvalues of $\iota(\mathbf{C})$. The zeros of p are located in the similarity classes of the eigenvalues of $\iota(\mathbf{C})$ and simultaneously in the similarity classes of the roots of c. Since these quantities are real or complex, they must be identical, which implies:

Theorem 9.5. *Let $\mathbf{C}$ be the companion matrix defined by the quaternionic polynomial coefficients of p and let c be the companion polynomial of p. Then the companion polynomial c is the characteristic polynomial of $\iota(\mathbf{C})$.*

Proof. For the proof see [18]. □

Let us see a simple example. Let $p(z) = z + a$ and let $a = (a_1, a_2, a_3, a_4)$. The companion polynomial of p is

$$c(z) = z^2 + 2\Re(a)z + \mathrm{abs}_2(a).$$

The companion matrix is $\mathbf{C} = [-a]$ and (see equation (9.2))

$$\iota(\mathbf{C}) = \begin{bmatrix} -(a_1 + a_2\mathrm{i}) & -(a_3 + a_4\mathrm{i}) \\ a_3 - a_4\mathrm{i} & -a_1 + a_2\mathrm{i} \end{bmatrix}$$

and

$$\det(\mathbf{I}_2\lambda - \iota(\mathbf{C})) = \lambda^2 + 2a_1\lambda + a_1^2 + a_2^2 + a_3^2 + a_4^4,$$

which coincides with the above companion polynomial c.

In the meantime it was shown (various papers by Janovská and Opfer) that last Theorem 9.5 is also valid in the remaining three noncommutative $\mathbb{R}^4$ algebras:

▷ $\mathbb{H}_{\mathrm{coq}}$, the algebra of coquaternions,
▷ $\mathbb{H}_{\mathrm{nec}}$, the algebra of nectarines,
▷ $\mathbb{H}_{\mathrm{con}}$, the algebra of conectarines.

There is one difference: The equation

$$p(z) = R_0(u, v)z + R_1(u, v) = 0$$

has more possibilities for solving than in the quaternionic case and the number of zeros is limited to the larger number

$$\binom{2n}{2} = n(2n - 1).$$

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