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EXACT L_1 PENALTY FUNCTION FOR NONSMOOTH MULTIOBJECTIVE INTERVAL-VALUED PROBLEMS

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Our objective in this article is to explore the idea of an unconstrained problem using the exact l_1 penalty function for the nonsmooth multiobjective interval-valued problem (MIVP) having inequality and equality constraints. First of all, we figure out the KKT-type optimality conditions for the problem (MIVP). Next, we establish the equivalence between the set of weak LU-efficient solutions to the problem (MIVP) and the penalized problem (MIVP_ρ) with the exact l_1 penalty function. The utility of this transformation lies in the fact that it converts constrained problems to unconstrained ones. To accurately predict the applicability of the results presented in the paper, meticulously crafted examples are provided.

Keywords: interval-valued problem, multiobjective programming, exact l_1 penalty function, LU-efficient solution

Classification: 49J52, 49M30, 90C29, 90C46

1. INTRODUCTION

Multiobjective programming problems came into existence in order to maximize several objective functions simultaneously. Such problems are also known as vector programming problems or multi-criteria programming problems. The multiobjective problems are solved with deterministic coefficient values to get precise results. In real situations, many economic and engineering problems do not satisfy the deterministic assumptions since real-life problems are full of uncertainty. Uncertainty can be addressed using the concepts of fuzzy numbers, stochastic processes, and interval-valued problems. These techniques make the models closer to real-world applications. In interval-valued problems, coefficients appearing in the objective function as well as constraints vary over some closed intervals.

Moore [20] was the first who started the work on interval analysis. Some initial approaches and applications of interval analysis can be seen in the paper of Moore [21]. Later on, Wu ([26, 27]) derived the KKT optimality conditions and duality theorem for the Wolfe dual for the differentiable interval-valued programming problem. Jayswal et al. [16] extracted the sufficiency for functions satisfying generalized invexity and derived the duality results. Zhang [29] proposed the necessary and sufficient conditions for interval-valued problems along with duality under the extended concept of invexity and

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preinvexity for interval problems. Zhou and Wang [30] worked on a mixed-type dual problem that connects the Mond Weir as well as Wolfe-type duals and rooted the sufficiency. Consequently, Jayswal and Banerjee [17] introduced an exact l_1 penalty function for interval-valued programming problems and established the equivalence between an *LU*-optimal solution of the considered problem as well as a penalized problem using convexity.

The basic framework of invexity was first conceptualized by Hanson [15] in 1981 but coined later on by Craven [11]. These functions became a more enjoyable class of functions in optimization theory, as all stationary points for these functions are found to be global minimizers. Invexity was also found suitable for relaxing necessary optimality conditions because one can weaken the constraint qualifications expressed in terms of convexity with the help of invexity. The same has been designed by Ben-Israel and Mond [9], where they have used modified or generalized Slater constraint qualifications. Following the research of Hanson and Craven, many authors have utilized invexity in various generalized forms for establishing many important results in the field of optimization, like Antczak [1], Martin [19], Weir and Jeyakumar [25], and many others. Recently, Khatri and Prasad [24] used the idea of invexity to derive duality results for smooth fractional variational problems. Antczak and Farajzadeh [6] studied a class of nonsmooth semi-infinite programming problems with multiple intervals and formulated the Fritz-John and KKT-type optimality conditions using invexity. Moreover, they implemented an exact l_1 penalty technique for solving the considered nonsmooth semi-infinite problems and established the equivalence between the considered problem and its corresponding penalized problem via the exact l_1 penalty function under invexity.

Nonlinear optimization problems with an exact penalty approach have been the center of a number of decent works on optimization problems. Zangwill [28] and Pietrzykowski [22] were the first to work on exact nondifferentiable penalty functions. An exact penalty method converts a constrained problem into an unconstrained problem. In this approach, the objective functions merge with the constraints through the penalty parameter, and it is necessary to select both the penalty parameter as well as the penalty function in such a way that the optimal solution to the penalized problem as well as the original problem are equal. An exact l_1 penalty function or the absolute value penalty function is usually nondifferentiable. The nondifferentiable exact l_1 penalty method has been studied by many researchers like Bazaraa et al. [8], Bertsekas and Koksal-Ozdaglar [10], Fletcher [13], Mangasarian [18], etc. Recently, Antczak [2] has introduced some pathbreaking results based on the penalty approach for nonconvex differentiable programming problems that consists of both inequality as well as equality constraints. Later on, Antczak [4] extended this work to nonsmooth, convex interval-valued problems using an exact l_1 penalty function approach and established the equivalence between the problem and its penalized problem based on LU-optimal solution. Further, Antczak and Studniarski [7] gave some properties based on the exactness of the penalization problem for the exact l_1 penalty approach for the nonsmooth nonconvex multiobjective programming problems.

The present paper is structured as follows: Section 2 recalls some notations and definitions that we use in the sequel of the paper. In Section 3, we establish the equivalence between the set of weak LU-efficient solutions to the considered problem and its corresponding penalization problem. Moreover, we demonstrate the equivalence between the weak LU-efficient solutions to the problem and the corresponding penalized solutions using the Lagrange function under invexity. Also, an example of a nonsmooth multiobjective interval-valued problem with the exact l_1 penalty function method is forcasted. Finally, Section 4 summarizes the accomplished work in the form of conclusions.

2. PRELIMINARIES

This section begins with the following convention for inequalities and equalities, which is utilized in the sequel of the article. For any $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ where $\mathbb{R}^n$ stands for n-dimensional Euclidean space, we have

(i) $u = v \Leftrightarrow u_i = v_i \ \forall i = 1, 2, \ldots .n$;

(ii) $u > v \Leftrightarrow u_i > v_i \ \forall i = 1, 2, \ldots .n$;

(iii) $u \geqq v \Leftrightarrow u_i \geqq v_i \ \forall i = 1, 2, \ldots .n$;

(iv) $u \geq v \Leftrightarrow u \geqq v,\ u \neq v$.

Let the set of all bounded and closed intervals of $\mathbb{R}$ be represented by $\mathcal{I}$. If $A = [a^L, a^U],\ B = [b^L, b^U]\ \in \mathcal{I}$, we define

(i) $A + B = [a^L + b^L, a^U + b^U]$,

(ii) $-A = [-a^U, -a^L]$,

(iii) $A - B = \{A + (-B)\} = [a^L - b^U, a^U - b^L]$,

(iv) $\kappa + A = [\kappa + a^L, \kappa + a^U]$,

(v) $\kappa A = \begin{cases} [\kappa a^L, \kappa a^U],\ \kappa > 0, \\ [\kappa a^U, \kappa a^L],\ \kappa \leqq 0, \end{cases}$

where κ is any real number. If we take $a^L = a^U = a$, then the interval A reduces to a real number. If $\widehat{F}$ is an interval-valued function, then it can be represented more appropriately by $\widehat{F}(\pi) = [F^L(\pi), F^U(\pi)]$, where $F^L(\pi) : \mathbb{R}^n \to \mathbb{R}$ and $F^U(\pi) : \mathbb{R}^n \to \mathbb{R}$ have the components satisfying conditions $F^L(\pi) \leqq F^U(\pi),\ \forall\ \pi \in \mathbb{R}^n$. In short, we write $[F(\pi)]^L$ and $[F(\pi)]^U$ in place of $F^L(\pi)$ and $F^U(\pi)$, respectively.

Symbolically, we use the symbol $A \leqq_{\mathrm{LU}} B$ to denote $a^L \leqq b^L$ and $a^U \leqq b^U$. Similarly, $A <_{\mathrm{LU}} B \Leftrightarrow A \leqq_{\mathrm{LU}} B,\ A \neq B$. That is, $A <_{\mathrm{LU}} B$ means any one of the following three conditions:

$$a^L < b^L,\ a^U < b^U,$$

or,

$$a^L \leqq b^L,\ a^U < b^U,$$

or,

$$a^L < b^L,\ a^U \leqq b^U.$$

Throughout the entire article, $\mathbb{X}$ signify an open subset (nonempty) of $\mathbb{R}^n$.

Definition 2.1. (Antczak and Studniarski [7]) A function Ξ defined from $\mathbb{R}^n$ to $\mathbb{R}$ is known as locally Lipschitz at a point $\pi \in \mathbb{R}^n$ if there exist positive scalars C and ε satisfying

$$|\Xi(y) - \Xi(z)| \leqq C\|y - z\|$$

for every y and z belong to $\pi + \varepsilon B$, B being an open unit ball in $\mathbb{R}^n$.

Definition 2.2. (Clarke [12]) The Clarke generalized directional derivative of a locally Lipschitz function $\Xi : \mathbb{X} \to \mathbb{R}$ at a point $\pi \in \mathbb{X}$ along $d \in \mathbb{R}^n$ is expressed as

$$\Xi^0(\pi; d) := \limsup_{\sigma\to\pi,\gamma\downarrow 0} \frac{\Xi(\sigma + \gamma d) - \Xi(\sigma)}{\gamma}.$$

Definition 2.3. (Clarke [12]) The Clarke generalized subdifferential of a locally Lipschitz function $\Xi : \mathbb{X} \to \mathbb{R}$ at a point $\pi \in \mathbb{X}$ is expressed mathematically as

$$\partial\Xi(\pi) := \left\{ \upsilon \in \mathbb{R}^n : \Xi^0(\pi; d) \geqq \langle \upsilon, d\rangle,\ \forall\ d \in \mathbb{R}^n \right\}.$$

Lemma 2.4. (Clarke [12]) Let $\pi \in \mathbb{X}$ be an arbitrary point, and the locally Lipschitz function Ξ is defined from $\mathbb{X}$ to $\mathbb{R}$. Then for any scalar $\lambda \in \mathbb{R}$, one can get

$$\partial\big(\lambda\Xi\big)(\pi) \subseteq \lambda\partial\Xi(\pi).$$

Proposition 2.5. (Clarke [12]) Let $\pi \in \mathbb{X}$ be an arbitrary point and the locally Lipschitz functions $\Xi_k,\ k = 1, \ldots, s$, defined from $\mathbb{X}$ to $\mathbb{R}$. Then

$$\partial\left(\sum_{k=1}^{s} \Xi_k\right)(\pi) \subseteq \sum_{k=1}^{s} \partial\Xi_k(\pi).$$

In the above relation, equality holds if all but at most one of the functions Ξ_k is strictly differentiable at a point π.

Corollary 2.6. (Clarke [12]) Let $\pi \in \mathbb{X}$ be an arbitrary point and the locally Lipschitz functions $\Xi_k,\ k = 1, \ldots, s$, defined from $\mathbb{X}$ to $\mathbb{R}$. Then for any scalar value $\lambda_k \in \mathbb{R},\ k = 1, \ldots, s$, one has

$$\partial\left(\sum_{k=1}^{s} \lambda_k\Xi_k\right)(\pi) \subseteq \sum_{k=1}^{s} \lambda_k\partial\Xi_k(\pi).$$

In the above relation, equality holds if all but at most one of the functions Ξ_k are strictly differentiable at a point π.

Theorem 2.7. (Clarke [12]) If the locally Lipschitz function Ξ is defined from $\mathbb{R}^n$ to $\mathbb{R}$ and attains its (local) minimum at a point $\bar{\pi} \in \mathbb{R}^n$, then

$$0 \in \partial\Xi(\bar{\pi}). \tag{1}$$

Proposition 2.8. (Clarke [12]) Let the functions $\Xi_k,\ k = 1,\dots,s$, defined from $\mathbb{R}^n$ to $\mathbb{R}$ be locally Lipschitz at point $\bar{\pi} \in \mathbb{R}^n$. Then $\Xi : \mathbb{R}^n \to \mathbb{R}$ defined by $\Xi(\pi) := \max_{k=1,\dots,s}\Xi_k(\pi)$ is also locally Lipschitz at a point $\bar{\pi}$. Moreover,

$$\partial\Xi(\bar{\pi}) \subset \operatorname{conv}\big\{\partial\Xi_k(\bar{\pi}) : k \in \mathfrak{K}(\bar{\pi})\big\},$$

where $\mathfrak{K}(\bar{\pi}) := \big\{k \in \mathfrak{K} : \Xi(\bar{\pi}) = \Xi_k(\bar{\pi})\big\}$.

Definition 2.9. (Antczak and Studniarski [7]) The functions $\Xi_k,\ k = 1,\dots,s$ defined from $\mathbb{X}$ to $\mathbb{R}$ is known as invex in connection with η at $\bar{\pi} \in \mathbb{X}$, if each component is locally Lipschitz at $\bar{\pi} \in \mathbb{X}$, and for each $\pi \in \mathbb{X}$, there exists a function $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$ satisfying

$$\Xi_k(\pi) - \Xi_k(\bar{\pi}) \geqq \langle \upsilon_k, \eta(\pi, \bar{\pi})\rangle,\ \forall \upsilon_k \in \partial\Xi_k(\bar{\pi}). \tag{2}$$

In the above definition, if we take $\eta(\pi, \bar{\pi}) = (\pi - \bar{\pi})$, then the above definition will reduce to the definition of convexity. Therefore, invexity incorporates a larger class of functions compared to the class of convex functions. The term $(\pi - \bar{\pi})$ appearing in the definition of convexity plays no role in the proof of the sufficiency of Kuhn–Tucker conditions. This observation motivated Hanson [15] to prove that Kuhn–Tucker necessary conditions are also sufficient if the objective function and constraints are invex in connection with the same function η. The centeral attraction of an invex function is that each stationary point is a global minimizer.

Note:

1. If we consider the scalar case instead of the vectorial case, i. e., $\Xi : \mathbb{X} \to \mathbb{R}$, then Definition 2.9 reduces to the definition of an invex function, given by Antzak [3].
2. If we consider Ξ is differentiable, then Definition 2.9 reduces to the definition of a differentiable invex function, which was given by Reiland [23] and also given in [2].

Remark 2.10. The interval-valued functions $\aleph_k,\ k = 1,\dots,s$, defined from $\mathbb{X}$ to $\mathcal{I}$, are known as invex in connection with η (where $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$) at a point $\bar{\pi} \in \mathbb{X}$ if both the functions $\aleph_k^L$ and $\aleph_k^U$ are invex in connection with η. That is, if it satisfy

$$\begin{aligned}\aleph_k^L(\pi) - \aleph_k^L(\bar{\pi}) \geqq \langle \upsilon_k^L, \eta(\pi, \bar{\pi})\rangle,\ \forall \upsilon_k^L \in \partial\aleph_k^L(\bar{\pi}),\\ \aleph_k^U(\pi) - \aleph_k^U(\bar{\pi}) \geqq \langle \upsilon_k^U, \eta(\pi, \bar{\pi})\rangle,\ \forall \upsilon_k^U \in \partial\aleph_k^U(\bar{\pi}).\end{aligned} \tag{3}$$

Definition 2.11. (Antczak and Studniarski [7]) The functions $\Xi_k,\ k = 1,\dots,s$ defined from $\mathbb{X}$ to $\mathbb{R}$ are known as strictly invex in connection with η at $\bar{\pi} \in \mathbb{X}$, if each component is locally Lipschitz at $\bar{\pi} \in \mathbb{X}$, and for each $\pi \in \mathbb{X}$, there exists a function $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$ satisfying

$$\Xi_k(\pi) - \Xi_k(\bar{\pi}) > \langle \upsilon_k, \eta(\pi, \bar{\pi})\rangle,\ \forall \upsilon_k \in \partial\Xi_k(\bar{\pi}). \tag{4}$$

Remark 2.12. The interval-valued functions $\aleph_k,\ k = 1,\dots,s$, defined from $\mathbb{X}$ to $\mathcal{I}$ are known as strictly invex in connection with η (defined by $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$) at a point $\bar{\pi} \in \mathbb{X}$ if at least one or both the functions $\aleph_k^L$ and $\aleph_k^U$ are strictly invex in connection with η. That is, if it satisfy

$$\aleph_k(\pi) - \aleph_k(\bar{\pi}) >_{\mathrm{LU}} \langle \upsilon_k, \eta(\pi, \bar{\pi})\rangle,\ \forall \upsilon_k \in \partial\aleph_k(\bar{\pi}). \tag{5}$$

Note: For (strictly) incave functions, the direction of the inequalities used in (strictly) invex functions will reverse its direction.

Example 2.13. Let us consider a nonsmooth multiobjective interval-valued functions $\aleph_k$ $(k = 1, 2) : \mathbb{X} \to \mathcal{I}$ where $\aleph_k = [\aleph_k^L, \aleph_k^U]$ and $\aleph_k^L, \aleph_k^U : \mathbb{X} \to \mathbb{R}$ $(k = 1, 2)$ be given by $\aleph_1^L = |\pi|,\ \aleph_1^U = |\pi| + 1,\ \aleph_2^L = |\pi| + 2,\ \aleph_2^U = |\pi| + 3.$
Let us assume $\bar{\pi} = 0$, and the function $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be defined by

$$\eta(\pi, \bar{\pi}) = |\pi| - |\bar{\pi}|.$$

Using the definition of invex function, it can be shown that $\aleph_k$ $(k = 1, 2)$ are invex functions with respect to η at a point $\bar{\pi}$.

Note: It should be noted that the function η defined in Example 2.13 need not be unique. Indeed, if we define η by

$$\eta(\pi, \bar{\pi}) = \frac{|\pi| + |\bar{\pi}|}{2},$$

then $\aleph_k$ $(k = 1, 2)$ are invex with respect to η at $\bar{\pi}$.

Proposition 2.14. (Antczak and Studniarski [7]) Let $\bar{\pi} \in \mathbb{X}$ be an arbitrary point, and the locally Lipschitz functions $\phi_j,\ j = 1, \ldots, m$ be defined from $\mathbb{X}$ to $\mathbb{R}$. Moreover, let the functions $\phi_j^+ : \mathbb{X} \to \mathbb{R}$ be defined by $\phi_j^+(\pi) := \max\{0, \phi_j(\pi)\}$. If the functions ϕ_j are invex in connection with η (defined by $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$) at a point $\bar{\pi} \in \mathbb{X}$, then ϕ_j^+ are locally Lipschitz invex at a point $\bar{\pi} \in \mathbb{X}$ in connection with η.

Theorem 2.15. (Ha and Luu [14]) Let $\Xi_k,\ k = 1, \ldots, s$ defined from $\mathbb{X}$ to $\mathbb{R}$ be continuous at $\bar{\pi} \in \mathbb{X}$, the vector d that maps to $\Xi_k^0(\bar{\pi}; d),\ \forall d \in \mathbb{R}^n$ be upper semicontinuous at $\bar{\pi}$ and the mapping $\pi \to \partial \Xi_k(\pi)$ be upper semicontinuous at a point $\bar{\pi}$. Moreover, suppose the functions Ξ_j are invex in connection with η (where $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$) at a point $\bar{\pi} \in \mathbb{X}$. Then $\Xi : \mathbb{R}^n \to \mathbb{R}$ defined by $\Xi(\pi) := \max_{k=1,\ldots,s} \Xi_k(\pi)$ is also invex at a point $\bar{\pi} \in \mathbb{X}$ in connection with η.

Generally, the unconstrained multiobjective interval-valued problem is expressed as

$$\begin{aligned} \text{(UMIVP)} \quad \text{minimize} \quad & \aleph(\pi) = \big(\aleph_1(\pi), \ldots, \aleph_s(\pi)\big) \\ & \quad = \big([\aleph_1^L(\pi), \aleph_1^U(\pi)], \ldots, [\aleph_s^L(\pi), \aleph_s^U(\pi)]\big) \\ \text{subject to} \quad & \pi \in \mathbb{X}, \end{aligned}$$

where $\aleph_k : \mathbb{X} \to \mathcal{I},\ k \in \mathfrak{K} = \{1, \ldots, s\}$ are interval-valued functions and $\aleph_k^L,\ \aleph_k^U, : \mathbb{X} \to \mathbb{R}, \forall\ k \in \mathfrak{K}$ are locally Lipschitz functions on $\mathbb{X} \subseteq \mathbb{R}^n$.
Let us examine the following nonsmooth multiobjective interval-valued problem:

$$\begin{aligned} \text{(MIVP)} \quad \text{minimize} \quad & \aleph(\pi) = \big(\aleph_1(\pi), \ldots, \aleph_s(\pi)\big) \\ & \quad = \big([\aleph_1^L(\pi), \aleph_1^U(\pi)], \ldots, [\aleph_s^L(\pi), \aleph_s^U(\pi)]\big) \\ \text{subject to} \quad & \phi_j(\pi) \leqq 0;\ j \in \mathfrak{J} = \{1, \ldots, m\}, \end{aligned}$$

$$\psi_l(\pi) = 0;\ l \in \mathfrak{L} = \{1, \ldots, q\};\ \pi \in \mathbb{X}.$$

where $\aleph_k : \mathbb{X} \to \mathcal{I},\ k \in \mathfrak{K} = \{1, \ldots, s\}$, are interval-valued functions whereas $\aleph_k^L,\ \aleph_k^U$, ϕ_j, and $\psi_l : \mathbb{X} \to \mathbb{R}, \forall\, k \in \mathfrak{K},\ \forall\, j \in \mathfrak{J},\ \forall\, l \in \mathfrak{L}$, are locally Lipschitz functions on $\mathbb{X} \subseteq \mathbb{R}^n$.

For convenience, we use $\aleph^L := (\aleph_1^L, \ldots, \aleph_s^L) : \mathbb{X} \to \mathbb{R}^s$, $\aleph^U := (\aleph_1^U, \ldots, \aleph_s^U) : \mathbb{X} \to \mathbb{R}^s$, $\phi := (\phi_1, \ldots, \phi_m) : \mathbb{X} \to \mathbb{R}^m$, and $\psi := (\psi_1, \ldots, \psi_q) : \mathbb{X} \to \mathbb{R}^q$ to simplify our representation. Assume that $\Gamma := \{\pi \in \mathbb{X} : \phi_j(\pi) \leqq 0,\ j \in \mathfrak{J},\ \psi_l(\pi) = 0,\ l \in \mathfrak{L}\}$ represent the set of feasible solutions to the problem (MIVP). Moreover, the set of active constraint indices is defined by $\mathfrak{J}(\bar{\pi}) := \{j \in \mathfrak{J} : \phi_j(\bar{\pi}) = 0\}$ at a point $\bar{\pi}$.

Definition 2.16. (Antczak [5]) A feasible point $\bar{\pi} \in \Gamma$ is known as a LU-efficient solution (LU-Pareto solution) to (MIVP) if there does not exist any point $\pi \in \Gamma$ satisfying

$$\aleph_k(\pi) \leqq_{\text{LU}} \aleph_k(\bar{\pi}),\ \text{for each } k \in \{1, \ldots, s\}$$

and

$$\aleph_k(\pi) <_{\text{LU}} \aleph_k(\bar{\pi}),\ \text{for at least one } k \in \{1, \ldots, s\}.$$

Definition 2.17. (Antczak [5]) A feasible point $\bar{\pi} \in \Gamma$ is known as a weak LU-efficient solution (weak LU-Pareto solution) to (MIVP) if there does not exist any point $\pi \in \Gamma$ satisfying

$$\aleph_k(\pi) <_{\text{LU}} \aleph_k(\bar{\pi}),\ \text{for each } k \in \{1, \ldots, s\}.$$

Theorem 2.18. (KKT-type necessary optimality conditions) Let the feasible point $\bar{\pi} \in \Gamma$ be a weak LU-efficient solution to the problem (MIVP), and it satisfies the suitable constraint qualification. Then there exist Lagrange multipliers $\mu^L,\ \mu^U \in \mathbb{R}^s,\ \xi \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^q$ satisfying

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi}) + \sum_{j=1}^{m} \xi_j \partial \phi_j(\bar{\pi}) + \sum_{l=1}^{q} \zeta_l \partial \psi_l(\bar{\pi}), \tag{6}$$

$$\xi_j \partial \phi_j(\bar{\pi}) = 0,\ j \in \mathfrak{J}, \tag{7}$$

$$\mu^L,\ \mu^U \geq 0, \sum_{k=1}^{s} (\mu_k^L + \mu_k^U) = 1,\ \xi \geqq 0. \tag{8}$$

Definition 2.19. The feasible point $\bar{\pi} \in \Gamma$ is known as the KKT point of the proposed nonsmooth multiobjective interval-valued problem (MIVP) if the KKT conditions (6)–(8) are satisfied at a feasible point $\bar{\pi}$.

Now, let us formulate the necessary optimality criteria for an unconstrained multiobjective interval-valued problem (UMIVP).

Theorem 2.20. The necessary criteria for the feasible point $\bar{\pi}$ to be a weak LU-efficient solution to the unconstrained multiobjective interval-valued problem (UMIVP) is that the Lagrange multipliers $\mu^L \in \mathbb{R}^s$ and $\mu^U \in \mathbb{R}^s$ exist and satisfy

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi}),$$

$$\mu^L,\ \mu^U \geq 0,\ \sum_{k=1}^{s} (\mu_k^L + \mu_k^U) = 1.$$

Definition 2.21. The Lagrange function or the Lagrangian $\mathcal{L}$ for the proposed nonsmooth multiobjective interval-valued problem (MIVP) is defined as

$$\mathcal{L}\big(\pi, \mu^L, \mu^U, \xi,\ \zeta\big) := \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\pi) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\pi) + \sum_{j=1}^{m} \xi_j \phi_j(\pi) + \sum_{l=1}^{q} \zeta_l \psi_l(\pi) \tag{9}$$

3. EXACT L_1 PENALTY METHOD

In this section, we analyze the given nonsmooth multiobjective interval-valued problem (MIVP) using the penalty function. An exact l_1 penalty function transforms the constrained problem to an unconstrained problem. The fundamental concept behind an exact penalty approach is to solve the given extremization problem by choosing a positive penalty parameter ρ and a penalty function p with the property that there exists a nonnegative lower bound $\bar{\rho}$ satisfying $\rho > \bar{\rho}$ so that the optimal solution π is the same for both the penalized problem and the original extremization problem.

The usual scalar optimization problem with an exact l_1 penalty function is given by

$$\text{minimize} \quad P^{\circ}(\pi, \rho) := \Xi(\pi) + \rho \left[\sum_{j=1}^{m} \phi_j^+(\pi) + \sum_{l=1}^{q} |\psi_l(\pi)| \right],$$

where $P^{\circ}(\pi, \rho) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$. The functions $\Xi,\ \phi_j (j \in \mathfrak{J})$ and $\psi_l (l \in \mathfrak{L})$ are real valued functions defined on $\mathbb{R}^n$. The inequality constraint function $\phi_j (j \in \mathfrak{J})$ in exact l_1 penalty function is defined such that the function $\phi_j^+ : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\phi_j^+(\pi) = \begin{cases} 0, & \text{if } \phi_j(\pi) \leqq 0, \\ \phi_j(\pi), & \text{if } \phi_j(\pi) > 0. \end{cases} \tag{10}$$

Thus, we observe that ϕ_j^+ is zero for the feasible points and positive for the infeasible points. Next, we focus on the definition of an exact penalty function for a multiobjective interval-valued problem.

Definition 3.1. If a nonnegative threshold value $\bar{\rho}$ exists for each $\rho > \bar{\rho}$ satisfying

$$\text{arg weak LU-efficient solution of } \{P(\pi, \rho) : \pi \in \mathbb{R}^n\}$$

$$= \text{arg weak LU-efficient solution of } \{\aleph(\pi) : \pi \in \Gamma\},$$

then the function $P(\pi, \rho)$ is known as multiobjective interval-valued exact penalty function.

Next, let us explore the penalized problem approach for the multiobjective interval-valued problem (MIVP) with exact l_1 penalty functions:

$$\begin{aligned}(\text{MIVP}_\rho) \quad \text{minimize} \quad P(\pi,\rho) &= \Big(P_1(\pi,\rho),\dots,P_s(\pi,\rho)\Big) \\ &= \Big(\big[P_1^L(\pi,\rho),P_1^U(\pi,\rho)\big],\dots,\big[P_s^L(\pi,\rho),P_s^U(\pi,\rho)\big]\Big),\end{aligned}$$

where $P_k^L(.,\rho),\ P_k^U(.,\rho),\ k \in \mathfrak{K} = \{1,\dots,s\}$, are the endpoint functions of the multiple interval-valued exact l_1 penalty function $P_k(.,\rho)$ defined by

$$P_k^L(\pi,\rho) = \aleph_k^L(\pi) + \rho\left[\sum_{j=1}^{m}\phi_j^+(\pi) + \sum_{l=1}^{q}|\psi_l(\pi)|\right],\ \forall\ k \in \mathfrak{K},$$

and

$$P_k^U(\pi,\rho) = \aleph_k^U(\pi) + \rho\left[\sum_{j=1}^{m}\phi_j^+(\pi) + \sum_{l=1}^{q}|\psi_l(\pi)|\right],\ \forall\ k \in \mathfrak{K}.$$

Now, we establish the equivalence between the weak LU-efficient solution of the problem (MIVP) and the penalty problem (MIVP_ρ). First of all, we prove that the KKT point for the constructed problem (MIVP) is a weak LU-efficient solution corresponding to the penalized problem (MIVP_ρ).

Theorem 3.2. Let $\bar{\pi}$ be a feasible point to the considered multiobjective interval-valued problem (MIVP). Moreover, suppose it satisfies the following assumptions:

(i) $\bar{\pi}$ satisfies the KKT-type necessary conditions given by (6) – (8) with the Lagrange multipliers $\mu_k^L,\ \mu_k^U (k \in \mathfrak{K});\ \xi_j (j \in \mathfrak{J})$ and $\zeta_l (l \in \mathfrak{L})$;

(ii) The functions $\aleph_k^L$ and $\aleph_k^U$ $(k \in \mathfrak{K})$ that appear in the objective function are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η;

(iii) The constraints $\phi_j (j \in \mathfrak{J}),\ \psi_l (l \in \mathfrak{L}^+(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l > 0\})$ and $-\psi_l (l \in \mathfrak{L}^-(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l < 0\})$ are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η.

If we take enough large penalty parameter $\rho \geqq \max\{\xi_j,\ \forall\ j \in \mathfrak{J},\ |\zeta_l|,\ \forall\ l \in \mathfrak{L}\}$, then the feasible point $\bar{\pi}$ is weak LU-efficient solution to the multiobjective interval-valued penalized problem (MIVP_ρ) with exact l_1 penalty function.

Proof. Assume that $\bar{\pi}$ satisfies the KKT-type necessary conditions given by (6) – (8) with the Lagrange multipliers $\mu_k^L,\ \mu_k^U (k \in \mathfrak{K});\ \xi_j (j \in \mathfrak{J})$ and $\zeta_l (l \in \mathfrak{L})$. Suppose that the point $\bar{\pi}$ is not weak LU-efficient solution to the multiobjective interval-valued penalized problem (MIVP$_\rho$) with exact l_1 penalty functions, then there exist $\hat{\pi} \in \mathbb{X}$ satisfying

$$P_k(\hat{\pi}, \rho) <_{\mathrm{LU}} P_k(\bar{\pi}, \rho), \ \forall k \in \mathfrak{K},$$

that is,

$$\begin{matrix} P_k^L(\hat{\pi}, \rho) < P_k^L(\bar{\pi}, \rho) & \text{or,} & P_k^L(\hat{\pi}, \rho) \leqq P_k^L(\bar{\pi}, \rho) & \text{or,} & P_k^L(\hat{\pi}, \rho) < P_k^L(\bar{\pi}, \rho) \\ \\ P_k^U(\hat{\pi}, \rho) < P_k^U(\bar{\pi}, \rho) & & P_k^U(\hat{\pi}, \rho) < P_k^U(\bar{\pi}, \rho) & & P_k^U(\hat{\pi}, \rho) \leqq P_k^U(\bar{\pi}, \rho). \end{matrix}$$

The above inequalities can be rewritten as

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \end{cases}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^L(\bar{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \end{cases} \tag{11}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^U(\bar{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \end{cases}$$

for each $k \in \mathfrak{K}$. As given, $\bar{\pi}$ is a feasible point to the constructed multiobjective interval-valued programming problem (MIVP). Therefore, using the feasibility of $\bar{\pi}$ and (10), the above inequalities reduce to

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}), \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^m \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^q |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}), \end{cases}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^{q} |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^L(\bar{\pi}), \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}), \end{cases} \tag{12}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}), \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^{q} |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^U(\bar{\pi}), \end{cases}$$

for each $k \in \mathfrak{K}$. Multiplying the lower bounded inequalities of (12) by $\mu^L \geq 0$, the upper bounded inequalities by $\mu^U \geq 0$, and then, summing up from $k = \{1, \ldots, s\}$, we get

$$\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\hat{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\hat{\pi}) + \rho \sum_{k=1}^{s} \left(\mu_k^L + \mu_k^U\right) \left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right]$$

$$< \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}).$$

Using (8) in the above inequality, we obtain

$$\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\hat{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\hat{\pi}) + \rho \left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right]$$

$$< \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}). \tag{13}$$

Since the penalty parameter $\rho \geqq \max\{\xi_j,\ \forall\ j \in \mathfrak{J},\ |\zeta_l|,\ \forall\ l \in \mathfrak{L}\}$, therefore inequality (13) can be written as

$$\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\hat{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\hat{\pi}) + \sum_{j=1}^{m} \xi_j \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\zeta_l||\psi_l(\hat{\pi})|$$

$$< \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}).$$

Using condition (10), the above inequality gives

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$<\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi}).$$

With the help of equation (7), it can be rewritten as

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$<\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi}). \tag{14}$$

Due to invexity of $\aleph_k^L$ and $\aleph_k^U$ at a feasible point $\bar{\pi}\in\mathbb{X}$ in connection with η, we see that

$$\aleph_k^L(\pi)-\aleph_k^L(\bar{\pi})\geqq\langle\alpha_k^L,\eta(\pi,\bar{\pi})\rangle,\ \forall\alpha_k^L\in\partial\aleph_k^L(\bar{\pi}),\ \forall k\in\mathfrak{K},$$

$$\aleph_k^U(\pi)-\aleph_k^U(\bar{\pi})\geqq\langle\alpha_k^U,\eta(\pi,\bar{\pi})\rangle,\ \forall\alpha_k^U\in\partial\aleph_k^U(\bar{\pi}),\ \forall k\in\mathfrak{K},$$

hold for each $\pi\in\mathbb{X}$. Moreover, the constraints $\phi_j(j\in\mathfrak{J})$, $\psi_l(l\in\mathfrak{L}^+(\bar{\pi}))$ and $-\psi_l(l\in\mathfrak{L}^-(\bar{\pi}))$ are invex at a feasible point $\bar{\pi}\in\mathbb{X}$ in connection with η. So,

$$\phi_j(\pi)-\phi_j(\bar{\pi})\geqq\langle\beta_j,\eta(\pi,\bar{\pi})\rangle,\ \forall\beta_j\in\partial\phi_j(\bar{\pi}),\ \forall j\in\mathfrak{J},$$

$$\psi_l(\pi)-\psi_l(\bar{\pi})\geqq\langle\delta_l,\eta(\pi,\bar{\pi})\rangle,\ \forall\delta_l\in\partial\psi_l(\bar{\pi}),\ \forall l\in\mathfrak{L}^+(\bar{\pi}),$$

$$-\psi_l(\pi)+\psi_l(\bar{\pi})\geqq\langle-\delta_l,\eta(\pi,\bar{\pi})\rangle,\ \forall\delta_l\in\partial\psi_l(\bar{\pi}),\ \forall l\in\mathfrak{L}^-(\bar{\pi}),$$

hold for all $\pi\in\mathbb{X}$. Taking $\pi=\hat{\pi}$ in the above inequalities and using (8), we obtain

$$\mu_k^L\aleph_k^L(\hat{\pi})-\mu_k^L\aleph_k^L(\bar{\pi})\geqq\langle\mu_k^L\alpha_k^L,\eta(\hat{\pi},\bar{\pi})\rangle,\ \forall\alpha_k^L\in\partial\aleph_k^L(\bar{\pi}),\ \forall k\in\mathfrak{K}, \tag{15}$$

$$\mu_k^U\aleph_k^U(\hat{\pi})-\mu_k^U\aleph_k^U(\bar{\pi})\geqq\langle\mu_k^U\alpha_k^U,\eta(\hat{\pi},\bar{\pi})\rangle,\ \forall\alpha_k^U\in\partial\aleph_k^U(\bar{\pi}),\ \forall k\in\mathfrak{K}, \tag{16}$$

$$\xi_j\phi_j(\hat{\pi})-\xi_j\phi_j(\bar{\pi})\geqq\langle\xi_j\beta_j,\eta(\hat{\pi},\bar{\pi})\rangle,\ \forall\beta_j\in\partial\phi_j(\bar{\pi}),\ \forall j\in\mathfrak{J}, \tag{17}$$

$$\zeta_l\psi_l(\hat{\pi})-\zeta_l\psi_l(\bar{\pi})\geqq\langle\zeta_l\delta_l,\eta(\hat{\pi},\bar{\pi})\rangle,\ \forall\delta_l\in\partial\psi_l(\bar{\pi}),\ \forall l\in\mathfrak{L}^+(\bar{\pi})\cup\mathfrak{L}^-(\bar{\pi}). \tag{18}$$

On summing up the inequalities (15) − (18), we have

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$-\left(\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi})\right)$$

$$\geqq\left\langle\sum_{k=1}^{s}\mu_k^L\alpha_k^L+\sum_{k=1}^{s}\mu_k^U\alpha_k^U+\sum_{j=1}^{m}\xi_j\beta_j+\sum_{l=1}^{q}\zeta_l\delta_l,\eta(\hat{\pi},\bar{\pi})\right\rangle,$$

for all $\alpha_k^L\in\partial\aleph_k^L(\bar{\pi}),\ \alpha_k^U\in\partial\aleph_k^U(\bar{\pi})(k\in\mathfrak{K});\ \beta_j\in\partial\phi_j(\bar{\pi})(j\in\mathfrak{J})$ and $\delta_l\in\partial\psi_l(\bar{\pi})(l\in\mathfrak{L})$. Using the KKT condition (6), we get

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$\geqq\left(\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi})\right)$$

which contradicts the inequality (14). Hence, the proof is complete. □

Next, we will derive the Theorem 3.2 under the Lagrange type assumptions.

Theorem 3.3. Let $\bar{\pi}$ be a feasible point to the multiobjective interval-valued programming problem (MIVP) and the stationary condition

$$0\in\mathcal{L}\big(\bar{\pi},\mu^L,\mu^U,\xi,\ \zeta\big)\tag{19}$$

be satisfied at a feasible point $\bar{\pi}$ with the Lagrange multipliers $\mu^L,\ \mu^U,\ \xi$ and ζ. Moreover, suppose that the Lagrange function $\mathcal{L}\big(*,\mu^L,\mu^U,\xi,\ \zeta\big)$ is invex at a point $\bar{\pi}\in\mathbb{X}$. If we take enough large penalty parameter $\rho\geqq\max\{\xi_j,\ \forall\ j\in\mathfrak{J},\ |\zeta_l|,\ \forall\ l\in\mathfrak{L}\}$, then the feasible point $\bar{\pi}$ is weak LU-efficient solution to the multiobjective interval-valued penalized problem (MIVP$_\rho$) with exact l_1 penalty function.

P r o o f. Let the point $\bar{\pi}$ be not weak LU-efficient solution to the multiobjective interval-valued penalized problem (MIVP$_\rho$) with exact l_1 penalty function. Then, there exists $\hat{\pi}\in\mathbb{X}$ satisfying

$$P_k(\hat{\pi},\rho)<_{\mathrm{LU}}P_k(\bar{\pi},\rho),\ \forall k\in\mathfrak{K}$$

that is,

$$P_k^L(\hat{\pi},\rho)<\ P_k^L(\bar{\pi},\rho)\quad\text{or,}\ \ P_k^L(\hat{\pi},\rho)\leqq\ P_k^L(\bar{\pi},\rho)\quad\text{or,}\ \ P_k^L(\hat{\pi},\rho)<\ P_k^L(\bar{\pi},\rho)$$

$$P_k^U(\hat{\pi},\rho) < P_k^U(\bar{\pi},\rho) \qquad P_k^U(\hat{\pi},\rho) < P_k^U(\bar{\pi},\rho) \qquad P_k^U(\hat{\pi},\rho) \leqq P_k^U(\bar{\pi},\rho).$$

The above inequalities can be rewritten as

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \end{cases}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^L(\bar{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \end{cases} \tag{20}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^U(\bar{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \end{cases}$$

for each $k \in \mathfrak{K}$. As given, $\bar{\pi}$ is a feasible point to the multiobjective interval-valued problem (MIVP). Therefore, using (10) together with the feasibility of $\bar{\pi}$, the above inequalities reduce to

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}), \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}), \end{cases}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^L(\bar{\pi}), \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^U(\bar{\pi}), \end{cases} \tag{21}$$

or,

$$\begin{cases} \aleph_k^L(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^{q} |\psi_l(\hat{\pi})|\right] < \aleph_k^L(\bar{\pi}), \\ \\ \aleph_k^U(\hat{\pi}) + \rho\left[\sum\limits_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum\limits_{l=1}^{q} |\psi_l(\hat{\pi})|\right] \leqq \aleph_k^U(\bar{\pi}), \end{cases}$$

for each $k \in \mathfrak{K}$. Multiplying the lower bounbed inequalities of (12) by $\mu^L \geq 0$, the upper bounbed inequalities by $\mu^U \geq 0$, and then, summing up from $k = \{1, \dots, s\}$, we get

$$\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\hat{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\hat{\pi}) + \rho \sum_{k=1}^{s} \left(\mu_k^L + \mu_k^U\right)\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right]$$

$$< \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}).$$

Using (8) in the above inequality, we obtain

$$\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\hat{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\hat{\pi}) + \rho\left[\sum_{j=1}^{m} \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\psi_l(\hat{\pi})|\right]$$

$$< \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}). \tag{22}$$

Since the penalty parameter $\rho \geqq \max\{\xi_j,\ \forall\ j \in \mathfrak{J},\ |\zeta_l|,\ \forall\ l \in \mathfrak{L}\}$, therefore inequality (22) can be written as

$$\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\hat{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\hat{\pi}) + \sum_{j=1}^{m} \xi_j \phi_j^+(\hat{\pi}) + \sum_{l=1}^{q} |\zeta_l||\psi_l(\hat{\pi})|$$

$$< \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}).$$

Using condition (10), the above inequality gives

$$\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\hat{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\hat{\pi}) + \sum_{j=1}^{m} \xi_j \phi_j(\hat{\pi}) + \sum_{l=1}^{q} \zeta_l \psi_l(\hat{\pi})$$

$$< \sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}).$$

With the help of equation (7), it can be written as

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$<\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi}). \tag{23}$$

As the Lagrange function $\mathcal{L}\big(*,\mu^L,\mu^U,\xi,\ \zeta\big)$ is invex at a point $\bar{\pi}\in\mathbb{X}$ on $\mathbb{X}$, therefore,

$$\mathcal{L}\big(\hat{\pi},\mu^L,\mu^U,\xi,\ \zeta\big)-\mathcal{L}\big(\bar{\pi},\mu^L,\mu^U,\xi,\ \zeta\big)\geqq\langle\varsigma,\eta(\hat{\pi},\bar{\pi})\rangle,\ \forall\varsigma\in\partial\mathcal{L}\big(\bar{\pi},\mu^L,\mu^U,\xi,\ \zeta\big).$$

Using the properties of the Lagrange function $\mathcal{L}\big(*,\mu^L,\mu^U,\xi,\ \zeta\big)$, we obtain

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$-\left(\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi})\right)$$

$$\geqq\langle\varsigma,\eta(\hat{\pi},\bar{\pi})\rangle,$$

for all $\varsigma\in\partial\left(\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi})\right)$. Moreover, with the help of (19), we get

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$-\left(\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi})\right)$$

$$\geqq\langle 0,\eta(\hat{\pi},\bar{\pi})\rangle,$$

that can be written as

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\hat{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\hat{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\hat{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\hat{\pi})$$

$$\geqq\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi})+\sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi})+\sum_{j=1}^{m}\xi_j\phi_j(\bar{\pi})+\sum_{l=1}^{q}\zeta_l\psi_l(\bar{\pi}),$$

which contradicts the inequality (23). Hence, the proof is complete. □

Corollary 3.4. Let the feasible point $\bar{\pi}$ be a weak LU-efficient solution to the multiobjective interval-valued problem (MIVP) and satisfies all the assumptions of Theorem 3.2. If we take enough large penalty parameter $\rho \geqq \max\{\xi_j,\ \forall\, j \in \mathfrak{J},\ |\zeta_l|,\ \forall\, l \in \mathfrak{L}\}$, then the feasible point $\bar{\pi}$ is weak LU-efficient solution to the multiobjective interval-valued penalized problem (MIVP$_\rho$) with exact l_1 penalty function.

Next, we demonstrate the relation between the KKT point to the considered multiobjective interval-valued problem (MIVP) and an LU-efficient solution to the penalized problem (MIVP$_\rho$) with an exact l_1 penalty function.

Theorem 3.5. Let the feasible point $\bar{\pi}$ be a solution to the considered multiobjective interval-valued programming problem (MIVP). Moreover, suppose it satisfies the following assumptions:

(i) $\bar{\pi}$ satisfies the KKT-type necessary conditions given by (6) – (8) with the Lagrange multipliers $\mu_k^L,\ \mu_k^U (k \in \mathfrak{K});\ \xi_j (j \in \mathfrak{J})$ and $\zeta_l (l\ \in \mathfrak{L})$;

(ii) The Lagrange multipliers $\mu_k^L, \mu_k^U (k \in \mathfrak{K})$ corresponding with the objective functions $\aleph_k^L$ and $\aleph_k^U$ are considered as a positive real value;

(iii) The objective functions $\aleph_k^L$ and $\aleph_k^U (k \in \mathfrak{K})$ are strictly invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η;

(iv) The constraints $\phi_j (j \in \mathfrak{J})$, $\psi_l (l \in \mathfrak{L}^+(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l > 0\})$ and $-\psi_l (l \in \mathfrak{L}^-(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l < 0\})$ are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η.

If we take enough large penalty parameter $\rho \geqq \max\{\xi_j,\ \forall\, j \in \mathfrak{J},\ |\zeta_l|,\ \forall\, l \in \mathfrak{L}\}$, then the feasible point $\bar{\pi}$ is LU-efficient solution to the multiobjective interval-valued penalized problem (MIVP$_\rho$) with exact l_1 penalty function.

Proof. It's proof is similar to that of Theorem 3.2. Hence, the proof is omitted. □

Next, using the properties of the Lagrange functions, we demonstrate the result stated in Theorem 3.5.

Theorem 3.6. Let the feasible point $\bar{\pi}$ be a solution to the problem (MIVP) and satisfy the stationary condition (19) at point $\bar{\pi}$ with the Lagrange multipliers $\mu^L,\ \mu^U,\ \xi$ and ζ. Moreover, suppose that the Lagrange function $\mathcal{L}(*, \mu^L, \mu^U, \xi,\ \zeta)$ is strictly invex at $\bar{\pi} \in \mathbb{X}$. If we take enough large penalty parameter $\rho \geqq \max\{\xi_j,\ \forall\, j \in \mathfrak{J},\ |\zeta_l|,\ \forall\, l \in \mathfrak{L}\}$, then the feasible point $\bar{\pi}$ is LU-efficient solution to the multiobjective interval-valued penalized problem (MIVP$_\rho$) with exact l_1 penalty function.

Proof. It's proof is similar to that Theorem 3.3. Hence, the proof is omitted. □

Corollary 3.7. Let the feasible point $\bar{\pi}$ be an LU-efficient solution to the problem (MIVP) and satisfies all the assumptions of Theorem 3.5. If we take enough large penalty parameter $\rho \geqq \max\{\xi_j,\ \forall\, j \in \mathfrak{J},\ |\zeta_l|,\ \forall\, l \in \mathfrak{L}\}$, then the feasible point $\bar{\pi}$ is an LU-efficient solution to the penalized problem (MIVP$_\rho$) with exact l_1 penalty function.

Now, we proceed to prove the converse of the above result.

Theorem 3.8. Let $\Gamma \subseteq \mathbb{R}^n$ be a compact set and the feasible point $\bar{\pi}$ be a weak LU-efficient solution to the penalized problem (MIVP$_\rho$) with exact l_1 penalty function. Moreover, suppose it satisfies the following additional assumptions:

(i) The components of the objective functions $\aleph_k^L$ and $\aleph_k^U$ are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η;

(ii) The constraints $\phi_j (j \in \mathfrak{J})$, $\psi_l (l \in \mathfrak{L}^+(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l > 0\})$ and $-\psi_l (l \in \mathfrak{L}^-(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l < 0\})$ are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η.

If we take enough large penalty parameter $\bar{\rho}$, then the feasible point $\bar{\pi}$ is weak LU-efficient solution to the problem (MIVP).

Proof. Let the feasible point $\bar{\pi}$ is a weak LU-efficient solution to the penalized problem (MIVP$_\rho$) with an exact l_1 penalty function. We discuss the two cases, one if $\bar{\pi} \in \Gamma$ and the other if $\bar{\pi} \notin \Gamma$. Let us assume that the feasible point $\bar{\pi} \in \Gamma$. By definition of weak LU-efficient solution, no $\pi \in \mathbb{X}$ exists that satisfy

$$P_k(\pi, \bar{\rho}) <_{\mathrm{LU}} P_k(\bar{\pi}, \bar{\rho}), \ \forall k \in \mathfrak{K},$$

that is,

$$\begin{matrix} P_k^L(\pi,\bar{\rho}) < P_k^L(\bar{\pi},\bar{\rho}) & \text{or,} \ P_k^L(\pi,\bar{\rho}) \leqq P_k^L(\bar{\pi},\bar{\rho}) & \text{or,} \ P_k^L(\pi,\bar{\rho}) < P_k^L(\bar{\pi},\bar{\rho}) \\ \\ P_k^U(\pi,\bar{\rho}) < P_k^U(\bar{\pi},\bar{\rho}) & P_k^U(\pi,\bar{\rho}) < P_k^U(\bar{\pi},\bar{\rho}) & P_k^U(\pi,\bar{\rho}) \leqq P_k^U(\bar{\pi},\bar{\rho}). \end{matrix}$$

The above inequalities can be rewritten as

$$\begin{cases} \aleph_k^L(\pi) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\pi) + \sum\limits_{l=1}^q |\psi_l(\pi)|\right] < \aleph_k^L(\bar{\pi}) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\pi) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\pi) + \sum\limits_{l=1}^q |\psi_l(\pi)|\right] < \aleph_k^U(\bar{\pi}) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \end{cases}$$

or,

$$\begin{cases} \aleph_k^L(\pi) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\pi) + \sum\limits_{l=1}^q |\psi_l(\pi)|\right] \leqq \aleph_k^L(\bar{\pi}) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\pi) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\pi) + \sum\limits_{l=1}^q |\psi_l(\pi)|\right] < \aleph_k^U(\bar{\pi}) + \bar{\rho}\left[\sum\limits_{j=1}^m \phi_j^+(\bar{\pi}) + \sum\limits_{l=1}^q |\psi_l(\bar{\pi})|\right], \end{cases} \tag{24}$$

or,

$$\begin{cases} \aleph_k^L(\pi)+\bar{\rho}\left[\sum\limits_{j=1}^{m}\phi_j^+(\pi)+\sum\limits_{l=1}^{q}|\psi_l(\pi)|\right] < \aleph_k^L(\bar{\pi})+\bar{\rho}\left[\sum\limits_{j=1}^{m}\phi_j^+(\bar{\pi})+\sum\limits_{l=1}^{q}|\psi_l(\bar{\pi})|\right], \\ \aleph_k^U(\pi)+\bar{\rho}\left[\sum\limits_{j=1}^{m}\phi_j^+(\pi)+\sum\limits_{l=1}^{q}|\psi_l(\pi)|\right] \leqq \aleph_k^U(\bar{\pi})+\bar{\rho}\left[\sum\limits_{j=1}^{m}\phi_j^+(\bar{\pi})+\sum\limits_{l=1}^{q}|\psi_l(\bar{\pi})|\right], \end{cases}$$

for each $k \in \mathfrak{K}$. Applying condition (10) in the above inequalities, we conclude that there does not exist $\pi \in \Gamma$ satisfying

$$\aleph_k^L(\pi) < \aleph_k^L(\bar{\pi}), \quad \text{or,} \;\; \aleph_k^L(\pi) \leqq \aleph_k^L(\bar{\pi}), \quad \text{or,} \;\; \aleph_k^L(\pi) < \aleph_k^L(\bar{\pi})$$

$$\aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}), \qquad \aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}), \qquad \aleph_k^U(\pi) \leqq \aleph_k^U(\bar{\pi}).$$

That is,

$$\aleph_k(\pi) <_{\mathrm{LU}} \aleph_k(\bar{\pi}), \text{ for each } k \in \mathfrak{K}. \tag{25}$$

Therefore, we can say that the feasible point $\bar{\pi}$ is a weak LU-efficient solution to the problem (MIVP). Using inequality (25), one can conclude that for any $\rho \geqq \bar{\rho}$, the feasible point $\bar{\pi}$ is a weak LU-efficient solution to the penalized problem (MIVP$_\rho$) with exact l_1 penalty function, which, in turn, is also a weak LU-efficient solution to the considered multiobjective interval-valued programming problem (MIVP).

Next, let us consider the other possibility that the feasible point $\bar{\pi} \notin \Gamma$. As the feasible point $\bar{\pi}$ is a weak LU-efficient solution to the penalized problem (MIVP$_\rho$) with exact l_1 penalty function, then by Theorem 2.20, there exist the Lagrange multipliers $\mu_k^L,\ \mu_k^U \geqq 0$ for all $k \in \mathfrak{K}$ (not all being zero simultanously) satisfying $\sum_{k=1}^{s} \mu_k^L + \mu_k^U = 1$, along with

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial P_k^L(\bar{\pi},\bar{\rho}) + \mu_k^U \partial P_k^U(\bar{\pi},\bar{\rho}). \tag{26}$$

Using the exact l_1 penalty function, we write the above inclusion as

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \left(\aleph_k^L(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m}\phi_j^+(\bar{\pi})+\sum_{l=1}^{q}|\psi_l(\bar{\pi})|\right]\right)$$
$$+\sum_{k=1}^{s} \mu_k^U \partial \left(\aleph_k^U(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m}\phi_j^+(\bar{\pi})+\sum_{l=1}^{q}|\psi_l(\bar{\pi})|\right]\right). \tag{27}$$

As the functions $\phi_j (j \in \mathfrak{J})$ are locally Lipschitz on $\mathbb{X}$, we conclude that the functions $\phi_j^+ (j \in \mathfrak{J})$ are also locally Lipschitz. Moreover, the Lagrange multipliers μ_k^L and μ_k^U are nonnegative. Using Corollary 2.6, we have

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^L \partial \left(\bar{\rho} \left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})| \right] \right)$$

$$+ \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \partial \left(\bar{\rho} \left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})| \right] \right).$$

Hence, the above condition yields

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \partial \left(\bar{\rho} \left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})| \right] \right) \sum_{k=1}^{s} \mu_k^L$$

$$+ \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi}) + \partial \left(\bar{\rho} \left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})| \right] \right) \sum_{k=1}^{s} \mu_k^U,$$

which imply that

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi})$$

$$+\partial \left(\bar{\rho} \left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})| \right] \right) \sum_{k=1}^{s} \left(\mu_k^L + \mu_k^U \right).$$

Applying the result $\sum_{k=1}^{s} \mu_k^L + \mu_k^U = 1$ in the above relation, we obtain

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi}) + \partial \left(\bar{\rho} \left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})| \right] \right).$$

Using Lemma 2.4, the above relation simplifies to

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi}) + \bar{\rho} \partial \left(\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})| \right] \right),$$

which, due to Proposition 2.5 gives

$$0 \in \sum_{k=1}^{s} \mu_k^L \partial \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \partial \aleph_k^U(\bar{\pi}) + \bar{\rho} \left(\left[\sum_{j=1}^{m} \partial \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\partial \psi_l(\bar{\pi})| \right] \right). \quad (28)$$

Since $\aleph_k^L$ and $\aleph_k^U$ are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η, we see that

$$\aleph_k^L(\pi) - \aleph_k^L(\bar{\pi}) \geqq \langle \alpha_k^L, \eta(\pi, \bar{\pi}) \rangle, \ \forall \alpha_k^L \in \partial \aleph_k^L(\bar{\pi}), \ \forall k \in \mathfrak{K}, \tag{29}$$

$$\aleph_k^U(\pi) - \aleph_k^U(\bar{\pi}) \geqq \langle \alpha_k^U, \eta(\pi, \bar{\pi}) \rangle, \ \forall \alpha_k^U \in \partial \aleph_k^U(\bar{\pi}), \ \forall k \in \mathfrak{K}. \tag{30}$$

hold for each $\pi \in \mathbb{X}$. Moreover, the constraints $\phi_j (j \in \mathfrak{J})$, $\psi_l (l \in \mathfrak{L}^+(\bar{\pi}))$ and $-\psi_l (l \in \mathfrak{L}^-(\bar{\pi}))$ are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η and locally Lipschitz on $\mathbb{X}$. Using Proposition 2.14, we can say that $\phi_j^+ (j \in \mathfrak{J})$ are also invex functions at a feasible point $\bar{\pi}$ in connection with η. Therefore, the following inequalities hold for all $\pi \in \mathbb{X}$:

$$\phi_j^+(\pi) - \phi_j^+(\bar{\pi}) \geqq \langle \beta_j^+, \eta(\pi, \bar{\pi}) \rangle, \ \forall \beta_j^+ \in \partial \phi_j(\bar{\pi}), \ \forall j \in \mathfrak{J}, \tag{31}$$

$$\psi_l(\pi) - \psi_l(\bar{\pi}) \geqq \langle \delta_l, \eta(\pi, \bar{\pi}) \rangle, \ \forall \delta_l \in \partial \psi_l(\bar{\pi}), \ \forall l \in \mathfrak{L}^+(\bar{\pi}), \tag{32}$$

$$-\psi_l(\pi) + \psi_l(\bar{\pi}) \geqq \langle -\delta_l, \eta(\pi, \bar{\pi}) \rangle, \ \forall \delta_l \in \partial \psi_l(\bar{\pi}), \ \forall l \in \mathfrak{L}^-(\bar{\pi}). \tag{33}$$

Multiplying the inequality (29) by $\mu_k^L \geq 0$ and the inequality (30) by $\mu_k^U \geq 0$, we get

$$\mu_k^L \aleph_k^L(\pi) - \mu_k^L \aleph_k^L(\bar{\pi}) \geqq \langle \mu_k^L \alpha_k^L, \eta(\pi, \bar{\pi}) \rangle, \ \forall \alpha_k^L \in \partial \aleph_k^L(\bar{\pi}), \ \forall k \in \mathfrak{K}, \tag{34}$$

$$\mu_k^U \aleph_k^U(\pi) - \mu_k^U \aleph_k^U(\bar{\pi}) \geqq \langle \mu_k^U \alpha_k^U, \eta(\pi, \bar{\pi}) \rangle, \ \forall \alpha_k^U \in \partial \aleph_k^U(\bar{\pi}), \ \forall k \in \mathfrak{K}. \tag{35}$$

Multiplying the inequalities (31) – (33) with the penalty parameter $\bar{\rho} > 0$, we have

$$\bar{\rho}\phi_j^+(\pi) - \bar{\rho}\phi_j^+(\bar{\pi}) \geqq \bar{\rho}\langle \beta_j^+, \eta(\pi, \bar{\pi}) \rangle, \ \forall \beta_j^+ \in \partial \phi_j(\bar{\pi}), \ \forall j \in \mathfrak{J}, \tag{36}$$

$$\bar{\rho}|\psi_l(\pi)| - \bar{\rho}|\psi_l(\bar{\pi})| \geqq \bar{\rho}\langle \delta_l, \eta(\pi, \bar{\pi}) \rangle, \ \forall \delta_l \in \partial |\psi_l(\bar{\pi})|, \ \forall l \in \mathfrak{L}, \tag{37}$$

On summing up both sides of the inequalities (34) – (37), we get

$$\begin{aligned}
&\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\pi) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\pi) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^+(\pi) + \sum_{l=1}^{q} |\psi_l(\pi)|\right] \\
&\qquad - \left(\sum_{k=1}^{s} \mu_k^L \aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s} \mu_k^U \aleph_k^U(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^+(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right]\right) \\
&\geqq \left\langle \sum_{k=1}^{s} \mu_k^L \alpha_k^L + \sum_{k=1}^{s} \mu_k^U \alpha_k^U + \bar{\rho}\sum_{j=1}^{m} \beta_j^+ + \bar{\rho}\sum_{l=1}^{q} \delta_l, \eta(\pi, \bar{\pi}) \right\rangle,
\end{aligned}$$

for all $\alpha_k^L \in \partial\aleph_k^L(\bar{\pi})$, $\alpha_k^U \in \partial\aleph_k^U(\bar{\pi})(k \in \mathfrak{K})$; $\beta_j^+ \in \partial\phi_j(\bar{\pi})(j \in \mathfrak{J})$ and $\delta_l \in \partial|\psi_l(\bar{\pi})|(l \in \mathfrak{L})$. Using condition (28), the above ineuuality yields

$$\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\pi) + \sum_{k=1}^{s}\mu_k^U\aleph_k^U(\pi) + \bar{\rho}\left[\sum_{j=1}^{m}\phi_j^+(\pi) + \sum_{l=1}^{q}|\psi_l(\pi)|\right]$$

$$-\left(\sum_{k=1}^{s}\mu_k^L\aleph_k^L(\bar{\pi}) + \sum_{k=1}^{s}\mu_k^U\aleph_k^U(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m}\phi_j^+(\bar{\pi}) + \sum_{l=1}^{q}|\psi_l(\bar{\pi})|\right]\right) \geqq 0, \quad (38)$$

for all $\pi \in \mathbb{X}$. Since $\pi \in \Gamma$, so with the help of (10), the inequality (38) reduces to

$$\sum_{k=1}^{s}\left(\mu_k^L\aleph_k^L(\pi) - \mu_k^L\aleph_k^L(\bar{\pi})\right) + \sum_{k=1}^{s}\left(\mu_k^U\aleph_k^U(\pi) - \mu_k^U\aleph_k^U(\bar{\pi})\right)$$

$$\geqq \bar{\rho}\left[\sum_{j=1}^{m}\phi_j^+(\bar{\pi}) + \sum_{l=1}^{q}|\psi_l(\bar{\pi})|\right], \quad (39)$$

for all $\pi \in \Gamma$. Since $\bar{\pi}$ is not feasible to the problem (MIVP), therefore, by (10), one can conclude that

$$\sum_{j=1}^{m}\phi_j^+(\bar{\pi}) + \sum_{l=1}^{q}|\psi_l(\bar{\pi})| > 0. \quad (40)$$

Let us take the penalty parameter $\bar{\rho}$ large enough for all $\pi \in \Gamma$ defined by

$$\bar{\rho} > \max\left\{\frac{\mu_k^L\aleph_k^L(\pi) + \mu_k^U\aleph_k^U(\pi) - \mu_k^L\aleph_k^L(\bar{\pi}) - \mu_k^U\aleph_k^U(\bar{\pi})}{\left[\sum_{j=1}^{m}\phi_j^+(\bar{\pi}) + \sum_{l=1}^{q}|\psi_l(\bar{\pi})|\right]} : k \in \mathfrak{K}\right\}. \quad (41)$$

Using (25) and (41), we can say $\bar{\rho} > 0$. Moreover, the feasible point $\bar{\pi}$ is a weak LU-efficient solution to the penalized problem (MIVP$_\rho$) with exact l_1 penalty functions. Therefore, by the definition of weak LU-efficient solution, there does not exist $\pi \in \mathbb{X}$ satisfying

$$P_k(\pi, \bar{\rho}) <_{\mathrm{LU}} P_k(\bar{\pi}, \bar{\rho}), \ \forall\ k \in \mathfrak{K},$$

which can be rephrased as

$$\aleph_k(\pi) + \bar{\rho}\left[\sum_{j=1}^{m}\phi_j^+(\pi) + \sum_{l=1}^{q}|\psi_l(\pi)|\right] <_{\mathrm{LU}} \aleph_k(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m}\phi_j^+(\bar{\pi}) + \sum_{l=1}^{q}|\psi_l(\bar{\pi})|\right], \ \forall\ k \in \mathfrak{K}.$$

As $\Gamma \subset \mathbb{X}$, there does not exist $\pi \in \Gamma$ satisfying

$$\aleph_k(\pi) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\pi) + \sum_{l=1}^{q} |\psi_l(\pi)|\right] <_{\mathrm{LU}} \aleph_k(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \ \forall\ k \in \mathfrak{K},$$

and if $\pi \in \Gamma$, then

$$\aleph_k(\pi) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\pi) + \sum_{l=1}^{q} |\psi_l(\pi)|\right] \geqq_{\mathrm{LU}} \aleph_k(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \ \forall\ k \in \mathfrak{K}.$$

If $\pi \in \Gamma$, we use (10) to get

$$\aleph_k(\pi) \geqq_{\mathrm{LU}} \aleph_k(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \ \forall\ k \in \mathfrak{K},$$

for all $\pi \in \Gamma$, that is,

$$\aleph_k^L(\pi) \geqq \aleph_k^L(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \ \forall\ \pi \in \Gamma, \ \forall\ k \in \mathfrak{K}, \tag{42}$$

and

$$\aleph_k^U(\pi) \geqq \aleph_k^U(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \ \forall\ \pi \in \Gamma, \ \forall\ k \in \mathfrak{K}. \tag{43}$$

Multiplying the inequality (42) by μ_k^L, (43) by μ_k^U, and summing up, we get

$$\mu_k^L \aleph_k^L(\pi) + \mu_k^U \aleph_k^U(\pi) \geqq \mu_k^L \aleph_k^L(\bar{\pi}) + \mu_k^U \aleph_k^U(\bar{\pi}) + \bar{\rho}\left[\sum_{j=1}^{m} \phi_j^{+}(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right], \ \forall\ \pi \in \Gamma, \ \forall\ k \in \mathfrak{K}. \tag{44}$$

With the help of inequality (44), we get

$$\bar{\rho} \leqq \frac{\mu_k^L \aleph_k^L(\pi) + \mu_k^U \aleph_k^U(\pi) - \mu_k^L \aleph_k^L(\bar{\pi}) - \mu_k^U \aleph_k^U(\bar{\pi})}{\left[\sum_{j=1}^{m} \phi_j^{+}(\bar{\pi}) + \sum_{l=1}^{q} |\psi_l(\bar{\pi})|\right]}, \ \forall\ \pi \in \Gamma, \ \forall\ k \in \mathfrak{K},$$

which contradicts (41). Hence $\bar{\pi} \notin \Gamma$ is not possible. Therefore, if $\pi \in \Gamma$, then the feasible point $\bar{\pi}$ is a weak LU-efficient solution to the problem (MIVP). Hence, the proof is complete. □

Theorem 3.9. Let $\Gamma \subseteq \mathbb{R}^n$ be a compact set and the feasible point $\bar{\pi}$ be an LU-efficient solution to the penalized problem (MIVP_ρ) with exact l_1 penalty function. Moreover, suppose it satisfies the following two additional assumptions:

(i) The components of the objective functions $\aleph_k^L$ and $\aleph_k^U (k \in \mathfrak{K})$ are strictly invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η;

(ii) The constraints $\phi_j (j \in \mathfrak{J})$, $\psi_l (l \in \mathfrak{L}^+(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l > 0\})$ and $-\psi_l (l \in \mathfrak{L}^-(\bar{\pi}) := \{l \in \mathfrak{L} : \bar{\zeta}_l < 0\})$ are invex at a feasible point $\bar{\pi} \in \mathbb{X}$ in connection with η.

If we take enough large penalty parameter $\bar{\rho}$, then the feasible point $\bar{\pi}$ is an LU-efficient solution to the problem (MIVP).

Proof. The proof of this theorem, being similar to that of Theorem 3.8, is omitted. □

Next, let us formulate an example of the nonsmooth multiobjective interval-valued problem based on the results established in the article and interpret it using the concept of the exact l_1 penalty function.

Example 3.10. Let us construct the nonsmooth multiobjective interval-valued programming problem:

$$
\begin{aligned}
(\text{MIVP}') \qquad \text{minimize} \qquad \aleph(\pi) &= \big(\aleph_1(\pi), \aleph_2(\pi)\big), \\
&= \big([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)]\big), \\
&= \Big(\Big[\pi_1^2 + \pi_2^2 + |\pi_2| - \pi_2,\ 4\pi_1^2 + 4\pi_2^2 + |\pi_2|\Big], \\
&\qquad \Big[\pi_1^4 + \pi_2^2 + e^{\pi_1} + |\pi_1| - \pi_1,\ 6\pi_1^4 + 4\pi_2^2 + e^{\pi_1}\Big]\Big), \\
\text{subject to} \quad \phi_1(\pi) &= \pi_1^2 - \pi_1 \leqq 0, \\
\phi_2(\pi) &= \pi_2^2 - \pi_2 \leqq 0, \\
\psi_1(\pi) &= \pi_2 - \pi_1 = 0; \ \pi \in \mathbb{R}^2.
\end{aligned}
$$

The set of all feasible solutions is given by $\Gamma = \{\pi = (\pi_1, \pi_2) \in \mathbb{R}^2 : 0 \leqq \pi_1 \leqq 1 \text{ and } 0 \leqq \pi_2 \leqq 1\}$ whereas $\bar{\pi} = (0, 0)$ is an LU-efficient solution of the problem (MIVP′). Moreover, we see that $\aleph_1$ and $\aleph_2$ are strictly invex functions at a point $\bar{\pi} \in \mathbb{R}^2$ in connection with $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$
\eta(\pi, \bar{\pi}) = \begin{bmatrix} \pi_1 + \bar{\pi}_2 \\ \bar{\pi}_1 + \pi_2 \end{bmatrix}.
$$

We also observe that the functions ϕ_1, ϕ_2, and ψ_1 are invex at a point $\bar{\pi} \in \mathbb{R}^2$ in connection with the same η.
To deal with the unconstrained interval-valued multiobjective problem, let us construct the penalized problem (MIVP_ρ) with exact l_1 penalty functions

$$
\begin{aligned}
(\text{MIVP}'_\rho) \text{ minimize } P(\pi, \rho) &= \Big(P_1(\pi, \rho), P_2(\pi, \rho)\Big) \\
&= \Big(\big[P_1^L(\pi, \rho), P_1^U(\pi, \rho)\big], \big[P_2^L(\pi, \rho), P_2^U(\pi, \rho)\big]\Big),
\end{aligned}
$$

$$
\begin{aligned}
=\Bigg(\Big[&\pi_1^2+\pi_2^2+|\pi_2|-\pi_2\\
&+\rho\Big(\max\{0,\pi_1^2-\pi_1\}+\max\{0,\pi_2^2-\pi_2\}+|\pi_2-\pi_1|\Big),\\
&4\pi_1^2+4\pi_2^2+|\pi_2|\\
&+\rho\Big(\max\{0,\pi_1^2-\pi_1\}+\max\{0,\pi_2^2-\pi_2\}+|\pi_2-\pi_1|\Big)\Big],\\
&\Big[\Big(\pi_1^4+\pi_2^2+e^{\pi_1}+|\pi_1|-\pi_1\\
&+\rho\Big(\max\{0,\pi_1^2-\pi_1\}+\max\{0,\pi_2^2-\pi_2\}+|\pi_2-\pi_1|\Big),\\
&6\pi_1^4+4\pi_2^2+e^{\pi_1}\\
&+\rho\Big(\max\{0,\pi_1^2-\pi_1\}+\max\{0,\pi_2^2-\pi_2\}+|\pi_2-\pi_1|\Big)\Big]\Bigg).
\end{aligned}
$$

Moreover, there exist Lagrange multipliers $\mu = (\mu^L,\ \mu^U) \in \mathbb{R}^2$, $\xi = (\xi^L,\ \xi^U) \in \mathbb{R}^2$, and $\zeta \in \mathbb{R}$ satisfying the KKT necessary conditions (6)−(8) at a point $\bar{\pi}$ satisfying $\alpha_1^L\mu_1^L + \alpha_1^U\mu_1^U - \xi_2 + \zeta_1 = 0$, $\alpha_2^L\mu_2^L + \mu_2^U - \xi_1 - \zeta_1 = 0$, and $\mu_1^L + \mu_1^U + \mu_2^L + \mu_2^U = 1$, where $\alpha_1^L = \{-2, 0\}$, $\alpha_1^U = \{-1, 1\}$ and $\alpha_2^L = \{-1, 1\}$. From the above equations, we see that $\max\{\xi_1,\ \xi_2,\ |\zeta_1|\} = 1$. Therefore, using Theorem 3.5, we conclude that if we take enough large penalty parameter $\rho \geqq 1$, then the feasible point $\bar{\pi} = (0, 0)$ is LU-efficient solution to the penalized problem (MIVP$'_\rho$) with exact l_1 penalty function.

Remark 3.11. Note that the exact l_1 penalty method transforms a constrained nonsmooth multiobjective interval-valued problem (MIVP$'$) to a simple unconstrained interval-valued problem (MIVP$'_\rho$). Also, the solutions to both the constrained and unconstrained problems are equal under certain assumptions. Hence, it can be seen that the complexity of the problem can be reduced for some classes of constraint problems using the exact l_1 penalty method, and the interrelation between two problems can give many valuable informations.

Let us look at an example of a nonsmooth multiobjective interval-valued programming problem when at least one function appearing in the objective function is not invex. Note that the set of LU-efficient solutions to the original constrained problem (MIVP) and its associated unconstrained optimization problem (MIVP$_\rho$) using the exact l_1 penalty functions are not equivalent.

Example 3.12. Consider the following nonsmooth multiobjective interval-valued problem:

$$
\begin{aligned}
(\text{MIVP}'') \qquad \text{minimize} \qquad \aleph(\pi) &= \big(\aleph_1(\pi), \aleph_2(\pi)\big),\\
&= \big([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)]\big),\\
&= \Big(\big[\pi^3,\ \pi^3+1\big],\ \big[\pi^3+1,\ \pi^3+2\big]\Big),\\
\text{subject to} \quad \phi_1(\pi) &= |\pi| - \pi - 2 \leqq 0;\ \pi \in \mathbb{R}.
\end{aligned}
$$

Here, $\Gamma = \{\pi \in \mathbb{R} : \pi \geqq -1\}$ is the set of all feasible solutions, and $\bar{\pi} = -1$ is an LU-efficient solution of the problem (MIVP$''$). Moreover, using Theorem 1 of Ben-Israel and Mond [9], it follows that the objective functions are not invex in connection with the real-valued function η defined on $\mathbb{R} \times \mathbb{R}$. However, using the exact l_1 penalty method, we obtain the following unconstrained interval-valued problem associated with the original problem (MIVP$''$):

$$\begin{aligned}
(\text{MIVP}''_\rho) \qquad \text{minimize} \quad P(\pi,\rho) =& \Big(P_1(\pi,\rho), P_2(\pi,\rho)\Big) \\
=& \Big(\big[P_1^L(\pi,\rho), P_1^U(\pi,\rho)\big], \big[P_2^L(\pi,\rho), P_2^U(\pi,\rho)\big]\Big), \\
=& \Bigg(\Big[\pi^3 + \rho\big(\max\{0, |\pi| - \pi - 2\}\big), \\
& \pi^3 + 1 + \rho\big(\max\{0, |\pi| - \pi - 2\}\big)\Big], \\
& \Big[\pi^3 + 1 + \rho\big(\max\{0, |\pi| - \pi - 2\}\big), \\
& \pi^3 + 2 + \rho\big(\max\{0, |\pi| - \pi - 2\}\big)\Big]\Bigg).
\end{aligned}$$

It is not difficult to show that (MIVP$''_\rho$) does not have a minimizer at $\bar{\pi} = -1$ for any $\rho > 0$. It is evident from the fact that the downward order of growth of the objective functions $\aleph_1^L$, $\aleph_1^U$, $\aleph_2^L$, and $\aleph_2^U$ exceed the upward growth of the constraint ϕ_1 at $\bar{\pi}$ when moving from $\bar{\pi}$ towards smaller values. Indeed, $\inf_{\pi\in\mathbb{R}} P_1^L(\pi,\rho) \to -\infty$, $\inf_{\pi\in\mathbb{R}} P_1^U(\pi,\rho) \to -\infty$, $\inf_{\pi\in\mathbb{R}} P_2^L(\pi,\rho) \to -\infty$, and $\inf_{\pi\in\mathbb{R}} P_2^U(\pi,\rho) \to -\infty$ when $\pi \to -\infty$ for any $\rho > 0$. Hence, for the penalty parameter $\rho > 0$, the equivalence between the set of LU-efficient solutions of the original problem (MIVP$''$) and the associated penalized problem (MIVP$''_\rho$) does not hold.

It follows from the above example that invexity is essential to prove the equivalence between the set of LU-efficient solutions in the original nonsmooth interval-valued multiobjective optimization problem and its associated penalized problem with the exact l_1 penalty function.

Next, we explore the application of the interval-valued vector exact l_1 penalty function method to solve the following optimization problem.

Example 3.13. An owner operates two car dealerships, both selling identical cars. The annual inventory expenses at each dealership are determined by the number of cars sold, yearly storage costs, and fixed expenses associated with ordering new vehicles from the manufacturer. It can be assumed that, on average, half of the π cars are stored in each of the two car dealerships. In this scenario, the yearly inventory expenses at each dealership are represented by the function considered at intervals as follows:

$$\begin{aligned}
\aleph(\pi) = \Bigg(&\Big[25\pi + \frac{10000}{\pi} + 8000, 25\pi + \frac{10000}{\pi} + 10000\Big], \\
&\Big[15\pi + \frac{12000}{\pi} + 12000, 15\pi + \frac{12000}{\pi} + 18000\Big]\Bigg).
\end{aligned}$$

Moreover, each order size must meet the following requirement:

$$\phi(\pi) = |\pi - 10| - 10.$$

The question is: In what quantity will the cars be ordered for both dealerships to minimize their costs? The mathematical representation of this challenging problem gives rise to a nonsmooth multiobjective interval-valued problem as follows:

$$\begin{aligned}
(\text{MIVP}''') \qquad \text{minimize} \qquad \aleph(\pi) &= \big(\aleph_1(\pi), \aleph_2(\pi)\big),\\
&= \big([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)]\big),\\
&= \bigg(\Big[25\pi + \frac{10000}{\pi} + 8000,\ 25\pi + \frac{10000}{\pi} + 10000\Big],\\
&\qquad \Big[15\pi + \frac{12000}{\pi} + 12000,\ 15\pi + \frac{12000}{\pi} + 18000\Big]\bigg),\\
\text{subject to} \quad \phi_1(\pi) &= |\pi - 10| - 10 \leqq 0,\\
\mathbb{X} &= \{\pi \in \mathbb{R} : \pi > 0\}.
\end{aligned}$$

Here, $\Gamma = \{\pi \in \mathbb{R} : |\pi - 10| - 10 \leqq 0\}$ represents the set of all feasible solutions and $\bar{\pi} = 20$ is an LU-efficient solution to the considered multiobjective interval-valued problem (MIVP‴). Further, it can be easily verified that the considered interval-valued objective functions are strictly invex and the constraint is invex at a point $\bar{\pi}$ in connection with $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\eta(\pi, \bar{\pi}) = \pi - \bar{\pi}.$$

We implement the exact l_1 penalty function method to address the considered multiobjective interval-valued problem (MIVP‴). First of all, we construct the interval-valued penalized problem (MIVP''_ρ) with exact l_1 penalty functions as follows:

$$\begin{aligned}
(\text{MIVP}'''_\rho) \ \text{minimize}\ P(\pi, \rho) &= \Big(P_1(\pi, \rho), P_2(\pi, \rho)\Big)\\
&= \Big(\big[P_1^L(\pi, \rho), P_1^U(\pi, \rho)\big], \big[P_2^L(\pi, \rho), P_2^U(\pi, \rho)\big]\Big),\\
&= \Big(\Big[25\pi + \frac{10000}{\pi} + 8000 + \rho\big(\max\{0, |\pi - 10| - 10\}\big),\\
&\qquad 25\pi + \frac{10000}{\pi} + 10000 + \rho\big(\max\{0, |\pi - 10| - 10\}\big)\Big],\\
&\qquad \Big[15\pi + \frac{12000}{\pi} + 12000 + \rho\big(\max\{0, |\pi - 10| - 10\}\big),\\
&\qquad 15\pi + \frac{12000}{\pi} + 18000 + \rho\big(\max\{0, |\pi - 10| - 10\}\big)\Big]\Big).
\end{aligned}$$

With the help of KKT conditions (6) – (8), we can show that there exist Lagrange multipliers $\mu = (\mu^L, \mu^U) \in \mathbb{R}^2$ and $\xi_1 \in \mathbb{R}$ with $\xi_1 = 15$. Therefore, by Theorem 3.5, if we take enough large penalty parameter, say $\rho \geqq 15$, then the feasible point $\bar{\pi} = 20$ being an LU-efficient solution to the considered multiobjective interval-valued problem (MIVP‴)

becomes an LU-efficient solution to the corresponding penalized problem (MIVP$''_\rho$) with exact l_1 penalty function. Since all the assumptions of Theorem 3.8 are satisfied, then the feasible point $\bar{\pi} = 20$ being an LU-efficient solution to the penalized problem (MIVP$'''_\rho$) for any $\rho \geqq 15$, is also a solution to the problem (MIVP$'''$). Hence, in order to minimize the cost, at least 15 cars should be ordered for both dealerships.

4. CONCLUSIONS

In this article, we studied a nonsmooth nonlinear multiobjective interval-valued programming problem (MIVP) having both equality and inequality constraints using an exact l_1 penalty function. Further, we derived the equivalence relation between weak LU-efficient solutions of the problem (MIVP) and the penalized problem (MIVP$_\rho$) with the exact l_1 penalty approach by assuming the functions to be invex in connection with η. Moreover, we constructed the Lagrange function for the interval-valued multiobjective problem and established the equivalence relation between the set of weak LU-efficient solutions of the problem (MIVP) and the penalized problem (MIVP$_\rho$) with the exact l_1 penalty function for the Lagrange function. Also, the converse of the above theorem is derived in the paper. Examples were given to demonstrate the application of the exact l_1 penalty method and the resulting outcome in this paper.

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