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Mathematica Bohemica, Vol. 150 (2025), No. 2, 161–185

Persistent URL: `http://dml.cz/dmlcz/152969`

150 (2025) MATHEMATICA BOHEMICA No. 2, 161–185

BOUNDEDNESS AND HÖLDER CONTINUITY OF WEAK SOLUTIONS OF THE NONLINEAR BOUNDARY-VALUE PROBLEM FOR ELLIPTIC EQUATIONS WITH GENERAL NONSTANDARD GROWTH CONDITIONS

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Received November 23, 2023. Published online September 5, 2024.
Communicated by Michela Eleuteri

Abstract. We study a nonlinear boundary-value problem for elliptic equations with critical growth conditions involving Lebesgue measurable functions. We prove global boundedness and Hölder continuity of weak solutions for this problem. Our results generalize the ones obtained by P. Winkert and his colleagues (2012) not only in the variable exponent case but also in the constant exponent case.

Keywords: nonstandard growth; nonlinear boundary condition; Hölder continuity; boundedness

MSC 2020: 35B65, 35D30, 35J66, 76A05

1. Introduction

In this paper we deal with the regularity properties of weak solutions for the elliptic boundary value problem

$$-\operatorname{div} A(x,u,\nabla u) + B(x,u,\nabla u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$A(x,u,\nabla u)\cdot\nu = C(x,u) \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz boundary $\partial\Omega$ and $\nu=\nu(x)$ denotes the outer unit normal to $\partial\Omega$ at $x\in\partial\Omega$. We assume that the coefficients $A\colon\ \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $B\colon\ \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, and $C\colon\ \partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory

DOI: 10.21136/MB.2024.0166-23

functions satisfying the following structural conditions:

$$A(x,z,\xi)\xi \geqslant a_0|\xi|^{p(x)} - a_1|z|^{p^*(x)} - a_2(x), \quad \text{for a.e. } x \in \Omega, \tag{1.3}$$

$$|A(x,z,\xi)| \leqslant a_3|\xi|^{p(x)-1} + a_4|z|^{p^*(x)/p'(x)} + a_5(x), \quad \text{for a.e. } x \in \Omega, \tag{1.4}$$

$$|B(x,z,\xi)| \leqslant b_0|\xi|^{p(x)/(p^*(x))'} + b_1|z|^{p^*(x)-1} + b_2(x), \quad \text{for a.e. } x \in \Omega, \tag{1.5}$$

$$|C(x,z)| \leqslant c_0|z|^{p_*(x)-1} + c_1(x), \quad \text{for a.e. } x \in \partial\Omega, \tag{1.6}$$

and for all $z \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$. Here p is a function such that

$$1 < p^- := \inf_{\Omega} p(x) \leqslant p^+ := \sup_{\Omega} p(x) < n \tag{1.7}$$

and

$$p^*(x) := \frac{np(x)}{n-p(x)} \quad \text{for } x \in \Omega, \quad p_*(x) := \frac{(n-1)p(x)}{n-p(x)} \quad \text{for } x \in \partial\Omega,$$

and $p'(x) = p(x)/(p(x)-1)$. Further, a_0, a_1, a_2, a_3, a_4, b_0, b_1, and c_0 are positive constants, and $a_2(\cdot)$, $a_5(\cdot)$, $b_2(\cdot)$, and $c_1(\cdot)$ are certain non-negative Lebesgue measurable functions; see (1.9), (1.10) and (1.11) for details.

Our setting includes as a special case the boundary-value problem with the $p(x)$-Laplacian:

$$-\Delta_{p(x)}u + B(x,u,\nabla u) = 0 \quad \text{in } \Omega, \quad |\nabla u|^{p(x)-2}\nabla u \cdot \nu = C(x,u) \quad \text{on } \partial\Omega.$$

Here the $p(x)$-Laplacian, which is defined by

$$\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

reduces to the well-known p-Laplacian in the case $p(x) \equiv p$.

In recent years increasing attention has been paid to the study of elliptic problems with variable exponents, which are also termed problems with nonstandard growth conditions, that appear in the study of non-Newtonian fluids with thermo-convective effects, electro-rheololgical fluids, nonlinear elasticity and image restoration; see, e.g., [3], [6], [27], [31].

There are some essential differences between variable exponent problems and constant exponent problems. The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous. Indeed, one can see that the inhomogeneity due to variable exponent is the main difficulty in generalizing the results for the constant exponent problem to the variable exponent one and it is a source of singular phenomena in the variable exponent problems (see, e.g., [13], [17], [21], [26], [30]).

Hölder continuity results for the weak solutions in the literature require that weak solutions belong to $L^\infty(\Omega)$. In the case of the Dirichlet boundary conditions, this technique was used by Ladyzhenskaya and Ural'tseva in [22] for quasilinear elliptic equations with standard growth conditions and generalized by many authors to nonstandard growth conditions; see, e.g., [2], [9], [12], [14], [15], [20], [23]. In particular, Ri and Yu (see [29]) considered general nonstandard growth conditions, while the peculiar fact is that the assumptions on the lower order terms are sharp and formulated themselves in terms of variable growth exponents.

Let us comment relevant known regularity results about the other boundary-value problems with $p(x)$-growth. In [10] Fan proved the global Hölder continuity of the bounded weak solutions to the problem (1.1), (1.2) under the assumptions that

$$\begin{aligned} A(x,z,\xi)\xi &\geqslant \lambda(|z|)|\xi|^{p(x)} - \wedge(|z|),\\ |A(x,z,\xi)| &\leqslant \wedge(|z|)(|\xi|^{p(x)-1}+1),\\ |B(x,z,\xi)| &\leqslant \wedge(|z|)(|\xi|^{p(x)}+1), \end{aligned}$$

and $C \in C(\partial\Omega \times \mathbb{R}, \mathbb{R})$, where $\lambda\colon [0,\infty) \to (0,\infty)$ is a nonincreasing continuous function and $\wedge\colon [0,\infty) \to (0,\infty)$ is a nondecreasing continuous function. Hence, one needs to show its boundedness in order to prove the Hölder continuity of a weak solution with help of the result in [10]. Gasiński and Papageorgiou (see [16]) prove global a priori bounds for weak solutions to the problem

$$-\Delta_{p(x)}u + B(x,u) = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $p \in C^1(\overline{\Omega})$ with $1 < \min\limits_{\overline{\Omega}} p(x)$ and the Carathéodory function $B\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the growth condition

$$|B(x,z)| \leqslant b_1|z|^{q(x)-1} + b_2$$

with positive constants b_1, b_2 and a subcritical exponent $q \in C(\overline{\Omega})$ such that $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$. In [28], Winkert and Zacher obtained the result that weak solutions to (1.1), (1.2) are bounded in Ω under the subcritical growth conditions similar to (1.3)–(1.6) in the case that a_i, b_j and c_l are all positive constants, but did not obtain Hölder continuity. Marino and Winkert (see [24]) generalized the boundedness result obtained by Winkert and Zacher to the critical growth conditions when p is a constant. We refer to [25] for similar results for elliptic systems with a nonlinear boundary condition. Consequently, in order to prove the Hölder continuity for weak solutions to the problem (1.1), (1.2) by combining Fan [10], statement (3) of Proposition 2.2 and boundedness results obtained by Winkert and his colleagues (see [25]), we need to assume that a_i and b_j in (1.3)–(1.5) are all positive constants and $C(x,z)$ belongs to $C(\partial\Omega \times \mathbb{R}, \mathbb{R})$, and unlike the Dirichet problem, it is impossible to assume that weak solutions of the problem (1.1), (1.2) are Hölder continuous on the boundary $\partial\Omega$. In our opinion, there is no paper which deals with the global Hölder continuity of weak solutions to the variable exponent elliptic Neumann boundary value problem (1.1), (1.2) with critical growth without the assumption that they are Hölder continuous on $\partial\Omega$. In this sense, we assert that there is no work concerned with the global boundedness or the global Hölder continuity for the problem (1.1), (1.2) under the critical growth conditions (1.3)–(1.6); see [19] and the references therein. For further results on the regularity of weak solutions we refer to Acerbi, Minginoe [1] for $C^{1,\alpha}$ regularity results, Borsuk [5] for L^∞-estimate for a singular $p(x)$-Laplacian problem in a conical domain; see also Baroni, Colombo, Mingione [45] and Harjulehto, Hästö, Lê, Naortio [18] for a survey on regularity of weak solutions and minima. DiBennedetto [7], Chapter 10 proved the boundedness and the Hölder continuity of weak solutions to the problem (1.1), (1.2) under the conditions that

$$\begin{aligned}
A(x,z,\xi)\xi &\geqslant a_0|\xi|^p - a_2(x),\\
|A(x,z,\xi)| &\leqslant a_3|\xi|^{p-1} + a_5(x),\\
|B(x,z,\xi)| &\leqslant b_0|\xi|^{p-1} + b_2(x),\\
C(x,z) &\equiv c_1(x)
\end{aligned}$$

for given positive constants a_0, a_3 and b_0, and given non-negative functions

$$\begin{cases} a_2 \in L^{(n+\varepsilon)/p}(\Omega), \quad a_5 \in L^{((n+\varepsilon)/p)p'}(\Omega), \quad b_2 \in L^{(n+\varepsilon)/p}(\Omega), \\ c_1 \in L^{((n-1)/(p-1))((n+\varepsilon)/n)}(\partial\Omega) \quad \text{for some } \varepsilon > 0, \end{cases} \tag{1.8}$$

where $1 < p \leqslant n$ is a constant.

The purpose of the present paper is to find sharp conditions on $a_2(\cdot)$, $a_5(\cdot)$, $b_2(\cdot)$ and $c_1(\cdot)$ in (1.3)–(1.6) for weak solutions of (1.1), (1.2) to be globally bounded and Hölder continuous in Ω without assuming that weak solutions are bounded or Hölder continuous on $\partial\Omega$. To our knowledge, the problem which is treated in this paper seems to be the most general nonlinear boundary-value problem with the variable critical exponent. We use the variable exponent spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, the definitions of which will be given in Section 2. The symbols of some common spaces used in this paper such as $L^\infty(\Omega)$, $L^\infty(\partial\Omega)$, $C^1(\overline{\Omega})$, $C(\overline{\Omega})$, $C(\partial\Omega)$, $C^{0,\alpha}(\overline{\Omega})$ and $C^{0,\alpha}(\partial\Omega)$ are standard. Now, we state the main results of this paper.

Our first main result is the following global boundedness:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let* (1.3)–(1.7) *be satisfied, where a_0, a_1, a_3, a_4, b_0, b_1 and c_0 are given positive constants, and a_2, a_5, b_2 and c_1 are non-negative measurable functions satisfying*

$$a_2, b_2 \in L^{s(\cdot)}(\Omega), \quad a_5 \in L^{s(\cdot)p'(\cdot)}, \quad c_1 \in L^{\mu(\cdot)}(\partial\Omega), \tag{1.9}$$

with functions s, μ such that

$$s(x) > \frac{n}{p(x)} \quad \text{for all } x \in \overline{\Omega} \text{ and } s \in C(\overline{\Omega}), \tag{1.10}$$

$$\mu(x) > \frac{n-1}{p(x)-1} \quad \text{for all } x \in \partial\Omega \text{ and } \mu \in C(\partial\Omega). \tag{1.11}$$

Let $p \in W^{1,\gamma}(\Omega)$ with a number $\gamma \in (n, \infty)$. Then, any weak solution of the problem (1.1), (1.2) *is of class $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$.*

Our second main result is the global Hölder continuity for bounded weak solutions:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Suppose that for all $z \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$*

$$A(x,z,\xi)\xi \geqslant a_0(|z|)|\xi|^{p(x)} - a_1(|z|)a_2(x) \quad \text{for a.e.} x \in \Omega, \tag{1.12}$$

$$|A(x,z,\xi)| \leqslant a_3(|z|)(|\xi|^{p(x)-1} + a_5(x)) \quad \text{for a.e. } x \in \Omega, \tag{1.13}$$

$$|B(x,z,\xi)| \leqslant b_0(|z|)(|\xi|^{p(x)} + b_2(x)) \quad \text{for a.e. } x \in \Omega, \tag{1.14}$$

$$|C(x,z)| \leqslant c_0(|z|)c_1(x) \quad \text{for a.e. } x \in \partial\Omega, \tag{1.15}$$

where $a_0\colon [0,\infty) \to (0,\infty)$ is a nonincreasing continuous function, a_1, a_3, b_0 and $c_0\colon [0,\infty) \to (0,\infty)$ are nondecreasing continuous functions and a_2, a_5, b_2 and c_1 are non-negative measurable functions satisfying the same conditions as (1.9)–(1.11) *and p is as in Theorem* 1.1. *Then, any bounded weak solution of the problem* (1.1), (1.2) *is of class $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*

As one can easily verify, the conditions (1.12)–(1.15) follow from the conditions (1.3)–(1.6). Hence, the following Hölder continuity result for general weak solutions follows from Theorems 1.1 and 1.2 immediately.

Corollary 1.3. *Suppose that all conditions of Theorem* 1.1 *are satisfied. Then any weak solution of the problem* (1.1), (1.2) *is of class $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*

R e m a r k 1.4. For brevity, we assumed that $p^+ < n$. In fact, if $p^+ \geqslant n$, then by replacing $p^*(x)$ and $p_*(x)$ in (1.3)–(1.6) with $q(x)$ and $r(x)$, respectively, we can obtain similar results as in Theorems 1.1 and 1.2, where $q \in C(\overline{\Omega})$ and $r \in C(\partial\Omega)$ are functions such that $p(x) < q(x) \leqslant \widetilde{p^*}(x)$ for all $x \in \overline{\Omega}$ and $p(x) < r(x) \leqslant \widetilde{p_*}(x)$ for all $x \in \partial\Omega$ with $\widetilde{p^*}(x)$ and $\widetilde{p_*}(x)$ defined by

$$\widetilde{p^*}(x) := \begin{cases} \dfrac{np(x)}{n-p(x)} & \text{if } p(x) < n \\ \infty & \text{if } p(x) \geqslant n \end{cases} \qquad \text{for all } x \in \overline{\Omega}$$

and

$$\widetilde{p_*}(x) := \begin{cases} \dfrac{(n-1)p(x)}{n-p(x)} & \text{if } p(x) < n \\ \infty & \text{if } p(x) \geqslant n \end{cases} \qquad \text{for all } x \in \partial\Omega.$$

R e m a r k 1.5. It is clear that the conditions (1.9)–(1.11) on $a_2(\cdot)$, $a_5(\cdot)$, $b_2(\cdot)$ and $c_1(\cdot)$ exactly coincide with the conditions (1.8) when $p(x) \equiv p$, and note that we do not assume that a weak solution is Hölder continuous on $\partial\Omega$ to see its global Hölder continuity. In this sense, (1.9)–(1.11) are optimal conditions for weak solutions of (1.1), (1.2) to be bounded and Hölder continuous in Ω, and our results generalize the ones obtained by Winkert and his colleagues (see [24], [25]) not only in the variable exponent case but also in the constant exponent case.

The proofs of Theorems 1.1 and 1.2 are based on ideas of the localization method and De Giorgi's iteration technique developed by Winkert and Zacher (see [28]) and also Yu and Ri (see [29]). Our goal here is to derive a new suitable Caccioppoli type inequality admitting only in some neighborhood of every $x \in \overline{\Omega}$ solutions of the boundary-value problem (1.1), (1.2) and to explain the decay of level sets of a solution, while they are more complicated than the case that structural conditions are subcritical growth or a_2, a_5, b_2 and c_1 are constants; see Remarks 3.2 and 3.4 for details. The present paper is organized in four sections. In Section 2, we introduce some notations and well-known results which will be used in next sections. We prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

2. Preliminaries

Let E be a bounded open set in $\mathbb{R}^n$, $n \geqslant 2$, and $C_+(\overline{E})$ be denoted by

$$C_+(\overline{E}) := \{f \in C(\overline{E})\colon\ f(x) > 1 \text{ for all } x \in \overline{E}\}.$$

For $p \in C_+(\overline{E})$, we define the variable exponent Lebesgue space $L^{p(\cdot)}(E)$ by

$$L^{p(\cdot)}(E) := \left\{u\colon\ E \to \mathbb{R} \text{ is a measurable function: } \int_E |u|^{p(x)}\,\mathrm{d}x < \infty\right\}$$

with the norm

$$\|u\|_{p(\cdot);E} := \|u\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0\colon \int_E \left|\frac{u}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant \infty\right\}$$

and the variable exponent Sobolev space $W^{1,p(\cdot)}(E)$ by

$$W^{1,p(\cdot)}(E) := \{u \in L^{p(\cdot)}(E)\colon\ |\nabla u| \in L^{p(\cdot)}(E)\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(E)} = \|\nabla u\|_{p(\cdot);E} + \|u\|_{p(\cdot);E},$$

where $\|\nabla u\|_{p(\cdot);E} := \||\nabla u|\|_{p(\cdot);E}$. $W_0^{1,p(\cdot)}(E)$ is the closure of $C_0^\infty(E)$ in the space $W^{1,p(\cdot)}(E)$. The spaces $L^{p(\cdot)}(E)$, $W^{1,p(\cdot)}(E)$ and $W_0^{1,p(\cdot)}(E)$ are Banach spaces. In the space $W_0^{1,p(\cdot)}(E)$, we can take

$$\|u\|'_{W^{1,p(\cdot)}(E)} = \|\nabla u\|_{p(\cdot);E}$$

as an equivalent norm; i.e., there is a positive constant C such that

$$\|u\|_{p(\cdot);E} \leqslant C\|\nabla u\|_{p(\cdot);E} \quad \text{for all } u \in W_0^{1,p(\cdot)}(E).$$

For any $u \in L^{p(\cdot)}(E)$ and $v \in L^{p'(\cdot)}(E)$, we have Hölder inequality

$$\left|\int_E uv\,\mathrm{d}x\right| \leqslant 2\|u\|_{p(\cdot);E}\|v\|_{p'(\cdot);E}.$$

There holds the inequality

$$\|u\|_{p(\cdot);E} \leqslant (1 + |E|)\|u\|_{q(\cdot);E} \quad \text{for any } u \in L^{q(\cdot)}(E)$$

if $p(x) \leqslant q(x)$ for a.e. $x \in E$, where $|E|$ is the Lebesgue measure of E. For a measurable function $f\colon\ E \to \mathbb{R}$, we put

$$\sup_E f := \operatorname{ess\,sup}_E f, \quad \inf_E f := \operatorname{ess\,inf}_E f, \quad \operatorname{osc}_E f := \sup_E f - \inf_E f,$$
$$f^{(k)}(x) := \max\{f(x) - k, 0\} \quad \text{for } k \in \mathbb{R}.$$

For brevity, we often use the notations

$$f_E^+ := \sup_E f, \quad f_E^- := \inf_E f,$$

in particular,

$$\|f\|_{p(\cdot)} := \|f\|_{p(\cdot);\Omega}, \quad f^+ := f_\Omega^+, \quad f^- := f_\Omega^-.$$

We have the inequalities

$$\min\{\|u\|_{p(\cdot);E}^{p_E^-}, \|u\|_{p(\cdot);E}^{p_E^+}\} \leqslant \int_E |u|^{p(x)}\,\mathrm{d}x$$
$$\leqslant \max\{\|u\|_{p(\cdot);E}^{p_E^-}, \|u\|_{p(\cdot);E}^{p_E^+}\} \quad \text{for all } u \in L^{p(\cdot)}(E)$$

and if $0 \leqslant a_E^- \leqslant a_E^+ < \infty$, $r \in L^\infty(E)$, $1 \leqslant a(x)r(x)$, $r(x) \geqslant 1$ for a.e. $x \in E$ and $u \in L^{a(\cdot)r(\cdot)}(E)$, then there holds the inequality

$$\||u|^{a(x)}\|_{r(\cdot);E} \leqslant \max\{\|u\|_{a(\cdot)r(\cdot);E}^{a_E^-}, \|u\|_{a(\cdot)r(\cdot);E}^{a_E^+}\}.$$

Let $B_\varrho(x_0)$ be an open ball in $\mathbb{R}^n$ of radius ϱ centered at $x_0 \in \mathbb{R}^n$ and put

$$\omega_n := |B_1(x_0)|, \quad \Omega_\varrho(x_0) := \Omega \cap B_\varrho(x_0), \quad (\partial\Omega)_\varrho(x_0) := \partial\Omega \cap B_\varrho(x_0).$$

Sometimes we may omit x_0 in the above notations. In order to have critical embedding in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$, we must assume more regularity on the function $p(\cdot)$. For this, let us denote by $C^{0,1/|\log t|}(\overline{\Omega})$ the set of all functions $f\colon \overline{\Omega} \to \mathbb{R}$ that are log-Hölder continuous on $\overline{\Omega}$, i.e.,

$$-|f(x) - f(y)| \log|x-y| \leqslant C_{\log} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } |x-y| \leqslant \frac{1}{2},$$

where $C_{\log}$ is a positive constant.

R e m a r k 2.1. It is well-known that if $p \in C^{0,1/|\log t|}(\overline{\Omega})$, then there is a constant $L > 0$ such that

$$R^{-\underset{\Omega_R}{\operatorname{osc}} p} \leqslant L \quad \text{for all } B_R \text{ with } \Omega_R \neq \emptyset;$$

see, e.g., [8], Lemma 4.1.6.

Lemma 2.2 ([8], Theorem 8.3.1). *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $p \in C^{0,1/|\log t|}(\overline{\Omega}) \cap C_+(\overline{\Omega})$ satisfy $1 \leqslant p^- \leqslant p^+ < n$. Then*

$$\|u\|_{p^*(\cdot)} \leqslant C\|u\|_{W^{1,p(\cdot)}(\Omega)} \quad \textit{for all } u \in W^{1,p(\cdot)}(\Omega),$$

where the constant C depends only on n, p^+, $C_{\log}$ and $|\Omega|$.

Lemma 2.3 ([11], Theorem 2.1). *Let Ω be as in Lemma 2.2 and let $p \in W^{1,\gamma}(\Omega)$ with $1 \leqslant p^- \leqslant p^+ < n < \gamma$. Then there is a continuous embedding*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p_*(\cdot)}(\partial\Omega).$$

R e m a r k 2.4. For Ω as in Lemma 2.2, it is obvious that

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\Omega) \subset C^{0,1-n/\gamma}(\overline{\Omega}) \subset C^{0,1/|\log t|}(\overline{\Omega}).$$

Lemma 2.5 ([22], Lemma 4.7 of Chapter 2). *Suppose that a sequence y_h ($h = 0,1,2,\ldots$) of non-negative numbers satisfies the recursion relation*

$$y_{h+1} \leqslant Cb^h y_h^{1+\varepsilon}, \quad h = 0,1,2,\ldots,$$

where C, ε and b are positive constants and $b > 1$. If $y_0 \leqslant C^{-1/\varepsilon} b^{-1/\varepsilon^2}$, then $y_h \to 0$ as $h \to \infty$.

Lemma 2.6 ([22], Remark after Lemma 3.5 of Chapter 2). *Let E be a bounded convex open set in $\mathbb{R}^n$, and let $u \in W^{1,1}(E)$. Then*

$$(l-k)|A_l|^{1-1/n} \leqslant \frac{\beta(\operatorname{diam} E)^n}{|E \setminus A_k|} \int_{A_k \setminus A_l} |\nabla u| \,\mathrm{d}x$$

for arbitrary k and l with $k < l$, where $A_k = \{x \in E\colon\ u(x) > k\}$ and $\beta = \beta(n) > 1$ is a constant depending only on n.

Definition 2.7. We say that a function $u \in W^{1,p(\cdot)}(\Omega)$ is a weak solution of the problem (1.1), (1.2) if

$$\int_\Omega (A(x,u,\nabla u)\nabla v + B(x,u,\nabla u)v) \,\mathrm{d}x = \int_{\partial\Omega} C(x,u)v \,\mathrm{d}\sigma \tag{2.1}$$

for every $v \in W^{1,p(\cdot)}(\Omega)$, where $\mathrm{d}\sigma$ is the surface measure on $\partial\Omega$.

Remark 2.8. Note that all terms on the left-hand side of (2.1) are well defined by virtue of the conditions (1.4), (1.5), (1.7), (1.9), (1.10) and Lemma 2.2, and the boundary integral on the right-hand side of (2.1) is well defined by virtue of the conditions (1.6), (1.7), (1.9), (1.11) and Lemma 2.3.

Definition 2.9. $u \in W^{1,p(\cdot)}(\Omega)$ is called a bounded weak solution of the problem (1.1), (1.2) if $u \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ and (2.1) holds for any $v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

We note that the integrals of all terms in the identity (2.1) are also finite for arbitrary $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ and $v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ if the conditions (1.9)–(1.11) and (1.13)–(1.15) are satisfied.

3. Boundedness of weak solutions

In this section, we prove Theorem 1.1. The proof is based on the localization method and De Giorgi's iteration technique.

3.1. Caccioppoli type inequality. For brevity, we put $f_0 := f^-_{\Omega_R}$, $f_1 := f^+_{\Omega_R}$ for a given function $f \in C(\overline{\Omega})$, and so do for $f \in C(\partial\Omega)$, i.e., $f_0 := f^-_{(\partial\Omega)_R}$, $f_1 := f^+_{(\partial\Omega)_R}$ below.

Lemma 3.1. *Let $\Omega\subset\mathbb{R}^n$, $n>2$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $p\in W^{1,\gamma}(\Omega)$ with $1<p^-\leqslant p^+<n<\gamma$. Suppose that (1.3)–(1.6) and (1.9)–(1.11) are satisfied. Let a function $u\in W^{1,p(\cdot)}(\Omega)$ be a weak solution of the problem (1.1), (1.2). Then there is a number $R>0$ such that*

$$\int_{A_{k,\tau}}|\nabla\omega|^{p(x)}\,\mathrm{d}x\leqslant C\biggl(\int_{A_{k,t}}\Bigl|\frac{\omega-k}{t-\tau}\Bigr|^{p^*(x)}\,\mathrm{d}x+\int_{A_{k,t}}|\omega|^{p^*(x)}\,\mathrm{d}x+\int_{\Gamma_{k,t}}|\omega|^{p_*(x)}\,\mathrm{d}\sigma+|A_{k,t}|^{1-1/\delta}\biggr)\tag{3.1}$$

for any $x_0\in\overline{\Omega}$, for all $k\geqslant 1$, and $0<\tau<t\leqslant R$, where $\omega=\pm u$, $A_{k,t}:=\{x\in\Omega_t(x_0)\colon\,\omega(x)>k\}$, $\Gamma_{k,t}:=\{x\in(\partial\Omega)_t\colon\,\omega(x)>k\}$, and δ is a number such that $\delta>n/p_0$, $p_0=p^-_{\overline{\Omega}_R}$ and C is a constant depending on n, a_0, a_1, $\|a_2\|_{s(\cdot)}$, a_3, a_4, $\|a_5\|_{s(\cdot)p'(\cdot)}$, b_0, b_1, $\|b_2\|_{s(\cdot)}$, c_0, $\|c_1\|_{\mu(\cdot)}$, $p(\cdot)$, $s(\cdot)$, $\mu(\cdot)$, x_0, R, Ω and $\partial\Omega$.

Proof. We fix $k\geqslant 1$ and $x_0\in\overline{\Omega}$. Since p, s and μ are continuous and satisfy (1.10) and (1.11), we may choose $R>0$ so small that

$$R\leqslant 1,\quad |B_R|\leqslant 1,\quad s_0>\frac{n}{p_0},\quad (p^*)_0>p_1\frac{s_0}{s_0-1},\quad \mu_0>\frac{n-1}{p_0-1}.\tag{3.2}$$

Let $\eta\in C^1(\mathbb{R}^n)$ be a function such that $0\leqslant\eta(x)\leqslant 1$, $|\nabla\eta(x)|\leqslant 2/(t-\tau)$ for $x\in\mathbb{R}^n$, $\eta(x)=1$ for $x\in B_\tau(x_0)$ and $\operatorname{supp}\eta\subset B_t(x_0)$ where $0<\tau<t\leqslant R$. Setting $\omega=\pm u$, we have $\omega^{(k)}\in W^{1,p(\cdot)}(\Omega)$ and so $v=\eta^{(p^*)^+}\omega^{(k)}\in W^{1,p(\cdot)}(\Omega)$. Taking this v as a test function in (2.1), we get

$$\begin{aligned}\int_{A_{k,t}}\eta^{(p^*)^+}A(x,u,\nabla u)\nabla\omega\,\mathrm{d}x=&-(p^*)^+\int_{A_{k,t}}\eta^{(p^*)^+-1}\nabla\eta A(x,u,\nabla u)(\omega-k)\,\mathrm{d}x\\&-\int_{A_{k,t}}B(x,u,\nabla u)(\omega-k)\eta^{(p^*)^+}\,\mathrm{d}x\\&+\int_{\partial\Omega}C(x,u)\omega^{(k)}\eta^{(p^*)^+}\,\mathrm{d}\sigma.\end{aligned}\tag{3.3}$$

We use (1.3)–(1.5) to get:

$$\begin{aligned}\pm\int_{A_{k,t}}\eta^{(p^*)^+}A(x,u,\nabla u)\nabla\omega\,\mathrm{d}x\geqslant{}& a_0\int_{A_{k,t}}\eta^{(p^*)^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x\\&-a_1\int_{A_{k,t}}|\omega|^{p^*(x)}\,\mathrm{d}x-2\|a_2\|_{s(\cdot)}|A_{k,t}|^{1-1/s_0},\end{aligned}\tag{3.4}$$

where the upper or lower sign is to be taken according to whether ω is $+u$ or $-u$, respectively,

$$
\begin{aligned}
(3.5)\quad & \Big|(p^*)^+\int_{A_{k,t}} \eta^{(p^*)^+-1}\nabla\eta A(x,u,\nabla u)(\omega-k)\,\mathrm{d}x\Big| \\
&\leqslant 2(p^*)^+\int_{A_{k,t}} \Big(a_3\eta^{(p^*)^+-1}|\nabla\omega|^{p(x)-1}\Big|\frac{\omega-k}{t-\tau}\Big| \\
&\quad + a_4(x)|\omega|^{p^*(x)/p'(x)}\Big|\frac{\omega-k}{t-\tau}\Big| + a_5(x)\Big|\frac{\omega-k}{t-\tau}\Big|\Big)\,\mathrm{d}x \\
&\leqslant \frac{a_0}{5}\int_{A_{k,t}} \eta^{(p^*)^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x \\
&\quad + C_1\int_{A_{k,t}}\Big(\Big|\frac{\omega-k}{t-\tau}\Big|^{p^*(x)} + |\omega|^{p^*(x)}\Big)\,\mathrm{d}x + |A_{k,t}|^{1-1/s_0},
\end{aligned}
$$

where we used the Young inequality and $|A_{k,t}|\leqslant 1$,

$$
\begin{aligned}
(3.6)\quad & \Big|\int_{A_{k,t}} B(x,u,\nabla u)(\omega-k)\eta^{(p^*)^+}\,\mathrm{d}x\Big| \\
&\leqslant \int_{A_{k,t}} (b_0\eta^{(p^*)^+}|\nabla\omega|^{p(x)/(p^*)'}|\omega| + b_1|\omega|^{p^*(x)} + \eta^{(p^*)^+}b_2(x)(\omega-k))\,\mathrm{d}x \\
&\leqslant \frac{a_0}{5}\int_{A_{k,t}} \eta^{(p^*)^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x + C_2\int_{A_{k,t}}|\omega|^{p^*(x)}\,\mathrm{d}x \\
&\quad + \int_{A_{k,t}} \eta^{(p^*)^+}b_2(x)(\omega-k)\,\mathrm{d}x.
\end{aligned}
$$

We estimate the last term on the right-hand side by using the Sobolev embedding. Setting

$$\widetilde{s_0} := \min\{s_0, n\},$$

we have $b_2\in L^{\widetilde{s_0}}(\Omega_t)$ and $\widetilde{s_0} > n/p_0$. Therefore, defining $\mathscr{P}$ by

$$\frac{n\mathscr{P}}{n-\mathscr{P}} = \frac{\widetilde{s_0}}{\widetilde{s_0}-1},$$

it follows that

$$1\leqslant \mathscr{P} < \frac{n}{n-p_0+1},$$

and so we have $1\leqslant\mathscr{P}<p_0<n$ and

$$\frac{n-p_0}{n}\frac{\mathscr{P}}{p_0-\mathscr{P}} < \frac{1}{p_0-1},$$

from which we get

$$1-\frac{1}{\widetilde{s_0}} < \Big(\frac{1}{\mathscr{P}}-\frac{1}{p_0}\Big)\frac{p_0}{p_0-1}. \tag{3.7}$$

Hence, by using the Sobolev embedding and the assumption on the support of η, we have

$$
\begin{aligned}
(3.8)\quad & \int_{A_{k,t}} \eta^{(p^*)^+} b_2(x)(\omega - k)\,\mathrm{d}x \\
&= \int_{\Omega_t} b_2(x)\eta^{(p^*)^+}\omega^{(k)}\,\mathrm{d}x \leqslant 2\|b_2\|_{s(\cdot)}\|\eta^{(p^*)^+}\omega^{(k)}\|_{n\mathscr{P}/(n-\mathscr{P});\Omega} \\
&\leqslant C_3\|b_2\|_{s(\cdot)}(\|\nabla(\eta^{(p^*)^+}\omega^{(k)})\|_{\mathscr{P};\Omega} + \|\eta^{(p^*)^+}\omega^{(k)}\|_{\mathscr{P};\Omega}) \\
&\leqslant C_3\|b_2\|_{s(\cdot)}(\|\nabla(\eta^{(p^*)^+}(\omega - k))\|_{\mathscr{P};A_{k,t}} + \|\omega\|_{\mathscr{P};A_{k,t}}) \\
&\leqslant C_3\|b_2\|_{s(\cdot)}(\|\nabla(\eta^{(p^*)^+}\omega^{(k)})\|_{p_0;A_{k,t}} + \|\omega\|_{p_0;A_{k,t}})|A_{k,t}|^{1/\mathscr{P}-1/p_0} \\
&\leqslant \frac{a_0}{5}\int_{A_{k,t}} \eta^{(p^*)^+}|\nabla\omega|^{p_0}\,\mathrm{d}x \\
&\quad + C_4\biggl(\int_{A_{k,t}} \Bigl|\frac{\omega - k}{t-\tau}\Bigr|^{p_0}\,\mathrm{d}x + \int_{A_{k,t}} |\omega|^{p_0}\,\mathrm{d}x + |A_{k,t}|^{(1/\mathscr{P}-1/p_0)p_0/(p_0-1)}\biggr) \\
&\leqslant \frac{a_0}{5}\int_{A_{k,t}} \eta^{(p^*)^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x \\
&\quad + C_5\biggl(\int_{A_{k,t}} \Bigl|\frac{\omega - k}{t-\tau}\Bigr|^{p^*(x)}\,\mathrm{d}x + \int_{A_{k,t}} |\omega|^{p^*(x)}\,\mathrm{d}x + |A_{k,t}|^{1-1/\tilde{s}_0}\biggr),
\end{aligned}
$$

where we used (3.7) and $|A_{k,t}| \leqslant 1$. Finally, we estimate the boundary integral on the right-hand side of (3.3). Using the condition (1.6), we have

$$
(3.9)\qquad \biggl|\int_{\partial\Omega} C(x,u)\omega^{(k)}\eta^{(p^*)^+}\,\mathrm{d}\sigma\biggr| \leqslant c_0\int_{\Gamma_{k,t}} |\omega|^{p_*(x)}\,\mathrm{d}\sigma + \int_{\partial\Omega} c_1(x)\eta^{(p^*)^+}\omega^{(k)}\,\mathrm{d}\sigma.
$$

The derivation of the estimate of the last term here is similar to the derivation of (3.8) but we give it for completeness. Defining $\mathscr{P}_\Gamma$ by

$$
(3.10)\qquad \frac{(n-1)\mathscr{P}_\Gamma}{n-\mathscr{P}_\Gamma} = \frac{\mu_0}{\mu_0 - 1}
$$

and recalling that $\mu_0 > (n-1)/(p_0-1)$, we obtain $1 < \mathscr{P}_\Gamma < n/(n-p_0+1)$, and so we have $1 < \mathscr{P}_\Gamma < p_0 < n$. Putting

$$
(3.11)\qquad s_\Gamma^0 := \frac{n(p_0-1)\mu_0}{(n-1)p_0},
$$

we find that $s_\Gamma^0 > n/p_0$ and that by (3.10)

$$
(3.12)\qquad \Bigl(\frac{1}{\mathscr{P}_\Gamma} - \frac{1}{p_0}\Bigr)\frac{p_0}{p_0-1} = 1 - \frac{1}{s_\Gamma^0}.
$$

Therefore, by the Sobolev embedding we have

$$
\begin{aligned}
(3.13)\quad \int_{\partial\Omega} c_1(x)\eta^{(p^*)^+}\omega^{(k)}\,\mathrm{d}\sigma
&\leqslant \|c_1\|_{\mu_0;\Gamma_{k,t}}\|\eta^{(p^*)^+}\omega^{(k)}\|_{\mu_0/(\mu_0-1);\partial\Omega}\\
&\leqslant C_6(\|\nabla(\eta^{(p^*)^+}\omega^{(k)})\|_{\mathscr{P}_\Gamma}+\|\eta^{(p^*)^+}\omega^{(k)}\|_{\mathscr{P}_\Gamma})\\
&\leqslant C_6(\|\nabla(\eta^{(p^*)^+}\omega^{(k)})\|_{p_0;A_{k,t}}+\|\omega\|_{p_0;A_{k,t}})|A_{k,t}|^{1/\mathscr{P}_\Gamma-1/p_0}\\
&\leqslant \frac{a_0}{5}\int_{A_{k,t}}\eta^{(p^*)^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x\\
&\quad + C_7\biggl(\int_{A_{k,t}}\Bigl|\frac{\omega-k}{t-\tau}\Bigr|^{p^*(x)}\,\mathrm{d}x+\int_{A_{k,t}}|\omega|^{p^*(x)}\,\mathrm{d}x+|A_{k,t}|^{1-1/s_\Gamma^0}\biggr),
\end{aligned}
$$

where we used (3.12). Substituting (3.4), (3.5), (3.6), (3.8), (3.9), (3.13) into (3.3) and putting $\delta=\min\{\widetilde{s}_0,s_\Gamma^0\}$, we find (3.1). The proof is completed. $\square$

R e m a r k 3.2. Note that if a_2, a_5, b_2 and c_1 in (1.3)–(1.6) are all constants, then $|A_{k,t}|^{1-1/\delta}$ in (3.1) is replaced by $|A_{k,t}|$ and such complicated treatments as done in (3.8) and (3.13) are not required; see, for example, Winkert and Zacher [28] or Marino and Winkert [24].

3.2. Boundedness of weak solutions.

Lemma 3.3. *Let all assumptions of Lemma* 3.1 *be satisfied. Then there exists a number $R>0$ such that for any $x_0\in\overline{\Omega}$*

$$
\sup_{\Omega_{R/2}(x_0)}|u|<M \quad \text{and} \quad \sup_{(\partial\Omega)_{R/2}(x_0)}|u|<M,
$$

where M is a positive constant depending on the same quantities as C in (3.1) *and on u.*

P r o o f. We put

$$
\widetilde{s}(x):=\min\{s(x),n\}\ \text{for}\ x\in\overline{\Omega}\quad\text{and}\quad s_\Gamma(x):=\frac{n(p(x)-1)}{(n-1)p(x)}\mu(x)\ \text{for}\ x\in\partial\Omega.
$$

It is clear that

$$
p(x)\frac{\widetilde{s}(x)}{\widetilde{s}(x)-1}<p^*(x)\quad\text{for all}\ x\in\overline{\Omega}
$$

and

$$
p(x)\frac{s_\Gamma(x)}{s_\Gamma(x)-1}<p^*(x)\quad\text{for all}\ x\in\partial\Omega
$$

by virtue of $\widetilde{s}(x) > n/p(x)$ for all $x \in \overline{\Omega}$ and $\mu(x) > (n-1)/(p(x)-1)$ for all $x \in \partial\Omega$. Therefore, there is a number $R > 0$ such that

$$\frac{(p^*)_0}{p_1}\Big(1 - \frac{1}{\widetilde{s}_0}\Big) > 1, \quad \frac{(p^*)_0}{p_1}\Big(1 - \frac{1}{s_\Gamma^0}\Big) > 1,$$

where s_Γ^0 is as in (3.11), so we have

$$\frac{(p^*)_0}{p_1}\Big(1 - \frac{1}{\delta}\Big) > 1, \tag{3.14}$$

where $\delta = \min\{\widetilde{s}_0, s_\Gamma^0\}$ is just as in (3.1). We may regard $(p_*)_0$ as

$$\frac{(p_*)_0}{p_1} > 1 \tag{3.15}$$

with R as above, since $p_*(x) > p(x)$ for all $x \in \partial\Omega$. Without loss of generality, we can suppose that this R is a number as in Lemma 3.1. Choose $k_0 \geqslant 1$ large enough so that

$$\int_{A_{k_0}} |\nabla u|^{p(x)}\,\mathrm{d}x + \int_{A_{k_0}} |u|^{p^*(x)}\,\mathrm{d}x + \int_{\Gamma_{k_0}} |u|^{p_*(x)}\,\mathrm{d}\sigma \leqslant 1, \tag{3.16}$$

where $A_{k_0} = \{x \in \Omega\colon\ |u(x)| > k_0\}$, $\Gamma_{k_0} := \{x \in \partial\Omega\colon\ |u| > k_0\}$, and put

$$\varrho_h = \Big(\frac{1}{2} + \frac{1}{2^{h+1}}\Big)R, \quad k_h = \Big(2 - \frac{1}{2^h}\Big)k_0, \quad h = 0, 1, 2, \ldots$$

Applying Lemma 3.1 to $t = \varrho_h$, $\tau = \varrho_{h+1}$ and $k = k_{h+1}$ for $h = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned}\int_{A_{k_{h+1},\varrho_{h+1}}} |\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant C\Big(&\frac{2^{(h+2)(p^*)^+}}{R^{(p^*)^+}}\int_{A_{k_{h+1},\varrho_h}} (\omega - k_{h+1})^{p^*(x)}\,\mathrm{d}x \\ &+ \int_{A_{k_{h+1},\varrho_h}} |\omega|^{p^*(x)}\,\mathrm{d}x \\ &+ \int_{\Gamma_{k_{h+1},\varrho_h}} |\omega|^{p_*(x)}\,\mathrm{d}\sigma + |A_{k_{h+1},\varrho_h}|^{1-1/\delta}\Big).\end{aligned} \tag{3.17}$$

Using that $0 < \omega(x) - k_{h+1} < \omega(x) - k_h$ and $\omega(x)/(2^{h+2}-1) = (1 - k_h/k_{h+1})\omega(x) \leqslant \omega(x) - k_h$, that is, $\omega(x) \leqslant (2^{h+2}-1)(\omega(x) - k_h)$ when $x \in A_{k_{h+1},\varrho_h}$ or $x \in \Gamma_{k_{h+1},\varrho_h}$, by (3.17) we get

$$\begin{aligned}\int_{A_{k_{h+1},\varrho_{h+1}}} |\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant C\bigg(&\frac{2^{(h+2)(p^*)^+}}{R^{(p^*)^+}}\int_{A_{k_{h+1},\varrho_h}} (\omega - k_h)^{p^*(x)}\,\mathrm{d}x \\ &+ 2^{(h+2)(p^*)^+}\int_{\Gamma_{k_{h+1},\varrho_h}} (\omega - k_h)^{p_*(x)}\,\mathrm{d}\sigma + |A_{k_{h+1},\varrho_h}|^{1-1/\delta}\bigg).\end{aligned} \tag{3.18}$$

It follows that
(3.19)

$$|A_{k_{h+1},\varrho_h}| \leqslant \int_{A_{k_{h+1},\varrho_h}} \Big(\frac{\omega-k_h}{k_{h+1}-k_h}\Big)^{p^*(x)}\,\mathrm{d}x \leqslant \frac{2^{(h+1)(p^*)^+}}{k_0{}^{(p^*)^-}} \int_{A_{k_h,\varrho_h}} (\omega-k_h)^{p^*(x)}\,\mathrm{d}x$$
$$\leqslant 2^{(h+1)(p^*)^+} \int_{A_{k_h,\varrho_h}} (\omega-k_h)^{p^*(x)}\,\mathrm{d}x.$$

Substituting (3.19) into (3.18) and using (3.16) we find that

$$\int_{A_{k_{h+1},\varrho_{h+1}}} |\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant C2^{(h+2)(p^*)^+}\bigg(\Big(\int_{A_{k_h,\varrho_h}} (\omega-k_h)^{p^*(x)}\,\mathrm{d}x\Big)^{1-1/\delta} + \int_{\Gamma_{k_h,\varrho_h}} (\omega-k_h)^{p_*(x)}\,\mathrm{d}\sigma\bigg). \tag{3.20}$$

On the other hand, taking $\eta\in C^1(\mathbb{R}^n)$ as in the proof of Lemma 3.1 with $\tau=\varrho_{h+1}$ and $t=\varrho_h$, we have by Lemma 2.2
(3.21)

$$\begin{aligned}
&\int_{A_{k_{h+1},\varrho_{h+1}}} (\omega-k_{h+1})^{p^*(x)}\,\mathrm{d}x\\
&\quad= \int_{\Omega_{\varrho_{h+1}}} (\eta\omega^{(k_{h+1})})^{p^*(x)}\,\mathrm{d}x \leqslant \|\eta\omega^{(k_{h+1})}\|_{p^*(\cdot);\Omega}^{(p^*)_0}\\
&\quad\leqslant C(\|\nabla(\eta\omega^{(k_{h+1})})\|_{p(\cdot);\Omega} + \|\eta\omega^{(k_{h+1})}\|_{p(\cdot);\Omega})^{(p^*)_0}\\
&\quad\leqslant C\Big(\|\nabla\omega\|_{p(\cdot);A_{k_{h+1},\varrho_h}} + \Big(\frac{2^{h+3}}{R}+1\Big)\|\omega-k\|_{p(\cdot);A_{k_{h+1},\varrho_h}}\Big)^{(p^*)_0}\\
&\quad\leqslant C\bigg(\Big(\int_{A_{k_{h+1},\varrho_h}} |\nabla\omega|^{p(x)}\,\mathrm{d}x\Big)^{1/p_1} + \Big(\frac{2^{h+3}}{R}+1\Big)\max\{\Upsilon^{1/p_0},\Upsilon^{1/p_1}\}\bigg)^{(p^*)_0}\\
&\quad\leqslant C2^{2(h+2)((p^*)^+)^2}\\
&\qquad\times\bigg(\Big(\int_{A_{k_h,\varrho_h}} |\nabla\omega|^{p(x)}\,\mathrm{d}x\Big)^{(p^*)_0/p_1} + \Big(\int_{A_{k_h,\varrho_h}} (\omega-k_h)^{p^*(x)}\,\mathrm{d}x\Big)^{(p^*)_0/p_1}\bigg),
\end{aligned}$$

where

$$\Upsilon = \int_{A_{k_{h+1},\varrho_h}} (\omega-k_h)^{p(x)}\,\mathrm{d}x$$

and we used that

$$\Big(\int_{A_{k_{h+1},\varrho_h}} (\omega-k_h)^{p(x)}\,\mathrm{d}x\Big)^{1/p_0} \leqslant \Big(\int_{A_{k_{h+1},\varrho_h}} (\omega-k_h)^{p^*(x)}\,\mathrm{d}x\Big)^{1/p_0} + |A_{k_{h+1},\varrho_h}|^{1/p_0},$$
$$|A_{k_{h+1},\varrho_h}| \leqslant 1$$

and (3.19). By using Lemma 2.3 we get similarly as above

(3.22)
$$\begin{aligned}
&\int_{\Gamma_{k_{h+1},\varrho_{h+1}}} (\omega - k_{h+1})^{p_*(x)}\,\mathrm{d}\sigma \\
&\quad\leqslant \int_{\partial\Omega} (\eta\omega^{(k_{h+1})})^{p_*(x)}\,\mathrm{d}\sigma \leqslant \|\eta\omega^{(k_{h+1})}\|_{p_*(\cdot);\partial\Omega}^{(p_*)_0} \\
&\quad\leqslant C(\|\nabla(\eta\omega^{(k_{h+1})})\|_{p(\cdot);\Omega} + \|\eta\omega^{(k_{h+1})}\|_{p(\cdot);\Omega})^{(p_*)_0} \\
&\quad\leqslant C2^{2(h+2)((p^*)^+)^2} \\
&\quad\quad\times \biggl(\Bigl(\int_{A_{k_h,\varrho_h}} |\nabla\omega|^{p(x)}\,\mathrm{d}x\Bigr)^{(p_*)_0/p_1} + \Bigl(\int_{A_{k_h,\varrho_h}} (\omega - k_h)^{p^*(x)}\,\mathrm{d}x\Bigr)^{(p_*)_0/p_1}\biggr).
\end{aligned}$$

Putting
$$\begin{aligned}
I_h := &\int_{A_{k_h,\varrho_h}} |\nabla\omega|^{p(x)}\,\mathrm{d}x + \int_{A_{k_h,\varrho_h}} (\omega - k_h)^{p^*(x)}\,\mathrm{d}x \\
&+ \int_{\Gamma_{k_h,\varrho_h}} (\omega - k_h)^{p_*(x)}\,\mathrm{d}\sigma, \quad h = 0,1,2,\dots
\end{aligned}$$
and using (3.20), (3.21) and (3.22), we arrive at

$$I_{h+1} \leqslant Cb^h({I_{h-1}}^{(p^*)_0/p_1(1-1/\delta)} + {I_{h-1}}^{(p_*)_0/p_1}), \tag{3.23}$$

where $b = 8^{((p^*)^+)^2} > 1$ and C is a positive constant independent of k_0 and h. Defining ε by
$$1 + \varepsilon = \min\Bigl\{\frac{(p^*)_0}{p_1}\Bigl(1 - \frac{1}{\delta}\Bigr), \frac{(p_*)_0}{p_1}\Bigr\}$$
it follows from (3.14) and (3.15) that $\varepsilon > 0$, and using (3.23), we have

$$I_{h+1} \leqslant Cb^h {I_{h-1}}^{1+\varepsilon} \tag{3.24}$$

since $I_h \leqslant 1$, $h = 0,1,2,\dots$, which follows from (3.16). Note by virtue of (3.24) that
$$\widetilde{I}_{h+1} \leqslant bC\widetilde{b}^h \widetilde{I}_h^{1+\varepsilon},$$
where $\widetilde{I}_h = I_{2h}$ and $\widetilde{b} = b^2$. We can choose $k_0 \geqslant 1$ large enough so that
$$\widetilde{I}_0 = I_0 \leqslant (bc)^{-1/\varepsilon}\widetilde{b}^{-1/\varepsilon^2},$$

and therefore, by Lemma 2.5 we get

$$\widetilde{I}_h = I_{2h} \to 0 \quad \text{as } h \to \infty. \tag{3.25}$$

From (3.24), we have also
$$\widetilde{\widetilde{I}}_{h+1} \leqslant bC\widetilde{b}^h {\widetilde{\widetilde{I}}_h}^{1+\varepsilon},$$

where $\widetilde{\widetilde{I}}_h = I_{2h+1}$ and $\widetilde{b}$ is as above. Choosing $k_0 \geqslant 1$ so that

$$\widetilde{\widetilde{I}}_0 = I_1 \leqslant \int_{A_{k_0}} |\nabla u|^{p(x)}\,\mathrm{d}x + \int_{A_{k_0}} |u|^{p^*(x)}\,\mathrm{d}x + \int_{\Gamma_{k_0}} |u|^{p_*(x)}\,\mathrm{d}\sigma \leqslant (bc)^{-1/\varepsilon}\widetilde{b}^{-1/\varepsilon^2},$$

by Lemma 2.5 we get

$$\widetilde{\widetilde{I}}_h = I_{2h+1} \to 0 \quad \text{as } h \to \infty. \tag{3.26}$$

Now (3.25) and (3.26) mean that $I_h \to 0$ as $h \to \infty$, and so we have

$$\sup_{\Omega_{R/2}} |u| \leqslant 2k_0, \qquad \sup_{(\partial\Omega)_{R/2}} |u| \leqslant 2k_0$$

with the above k_0. This completes the proof of the lemma. □

Remark 3.4. A key argument in the proof of Lemma 3.3 is the inequality (3.14), while this inequality is unneeded since the Caccioppoli type inequality (3.1) includes $|A_{k,t}|$ instead of $|A_{k,t}|^{1-1/\delta}$ when a_2, a_5, b_2 and c_1 in (1.3)–(1.6) are all constants.

Proof of Theorem 1.1. Since $\overline{\Omega}$ is compact, there exists a finite open cover $\{B_{R/2}(x_i)\}_{i=1}^m$ of $\overline{\Omega}$ with radius R as in Lemma 3.3, where $x_i \in \overline{\Omega}$, $i = 1, \ldots, m$. Then by applying the standard finite cover method and Lemma 3.3, we complete the proof of Theorem 1.1. □

Remark 3.5. The condition $p \in W^{1,\gamma}(\Omega)$, where $n < \gamma < \infty$, is due to the critical growth condition (1.6). We note that if we replace (1.6) by the subcritical growth condition $c(x,z) \leqslant c_0|z|^{q(x)-1} + c_1(x)$, where $q \in C(\partial\Omega)$ with $1 < q(x) < p_*(x)$ for all $x \in \partial\Omega$, or we study only local boundedness of weak solutions of (1.1), then it is sufficient to assume that $p \in C^{0,1/|\log t|}(\overline{\Omega}) \cap C_+(\overline{\Omega})$; see Lemma 2.2 and Fan [11], Corollary 2.4.

4. Hölder continuity of weak solutions

Let Ω be a Lipschitz bounded domain and let $z \in \partial\Omega$ be a given point. As our hypotheses are invariant under Lipschitz coordinate changes, without loss of generality we can assume that there exists a number $\widetilde{R} > 0$ such that

$$\Omega_R(z) = B_R^+(z), \quad (\partial\Omega)_R(z) = B_R^0(z) \quad \text{for all } R \in (0, \widetilde{R}], \tag{4.1}$$

where $B_R^+(z) = B_R(z) \cap \mathbb{R}^n_+$, $B_R^0(z) = B_R(z) \cap \mathbb{R}^n_0$, $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon x_n > 0\}$, $\mathbb{R}^n_0 = \{x = (x', 0) \colon x' \in \mathbb{R}^{n-1}\}$.

Definition 4.1. Let M, γ, γ_1, δ, r, R be positive constants with $\sigma \leqslant 2$, $r > 1$ and let $x_0 \in \partial\Omega$. We denote by $\widehat{\mathfrak{B}}_{p(\cdot)}(B_R^+(x_0), M, \gamma, \gamma_1, \delta, 1/r)$ the class of functions $u \in W^{1,p(\cdot)}(B_R^+)$ with $\|u\|_{\infty;B_R^+} \leqslant M$ such that, for $\omega = \pm u$, the following inequalities are valid for arbitrary $0 < \tau < t \leqslant R$:

$$\int_{A_{k,t}} |\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant \gamma \int_{A_{k,t}} \left|\frac{\omega - k}{t-\tau}\right|^{p(x)} \mathrm{d}x + \gamma_1 |A_{k,t}|^{1-1/r} \tag{4.2}$$

for $k \geqslant \sup\limits_{\Omega_t} \omega - \delta M$, where $A_{k,t} = \{x \in B_t^+(x_0)\colon\, \omega(x) > k\}$.

Without loss of generality we may assume that $L \geqslant 1$, $\gamma \geqslant 1$ in Remark 2.1 and (4.2). Let M be a positive constant. Since p is continuous on $\overline{\Omega}$, there exists a radius R_1 such that

$$M^{\operatorname{osc}\limits_{\Omega_{R_1}} p} \leqslant 2 \tag{4.3}$$

for any B_{R_1} with $\Omega_{R_1} \neq \emptyset$.

Lemma 4.2. *Let Ω be a bounded domain of $\mathbb{R}^n$, let $z \in \partial\Omega$, and let* (4.1) *be satisfied. Suppose that $p \in C^{0,1/|\log t|}(\overline{\Omega})$ satisfies* (1.7)*. Let $R_0 \in (0,1)$ be a number such that $R_0 \leqslant \min\{R_1, \widetilde{R}\}$ and $p_0\sigma_0 \geqslant n + \varepsilon_0$, where $\sigma_0 > 1$, $\varepsilon_0 > 0$ are given numbers and $p_0 = p^-_{B^+_{R_0(z)}}$. Suppose that $B_{R'}(x_0) \subset B_{R_0}(z)$, where $x_0 \in \partial\Omega$, $u \in W^{1,p(\cdot)}(B^+_{R'}(x_0)) \cap L^\infty(B^+_{R'}(x_0))$ and $u \in \widehat{\mathfrak{B}}_{p(\cdot)}(B_R^+(x_0), M, \gamma, \gamma_1, \delta, 1/r)$ for some $R \in (0, R']$ and $r \geqslant \sigma_0$. Then*

$$\operatorname*{osc}_{B_R^+(x_0)} u \leqslant C{R'}^{-\alpha} R^\alpha \tag{4.4}$$

where $\alpha = \min\{\varepsilon_0/(n+\varepsilon_0), -\log_4(1 - 2^{-s}l^{-1})\}$, $l = \max\{2, 2/\delta\}$, $s = s(n, p^+, p^-, \gamma, L, \varepsilon_0) > 2$ and $C = 4\max\Bigl\{l2^s(((\omega_n+1)(\gamma+1)+\gamma_1)/\gamma){R'}^{\varepsilon_0/(n+\varepsilon_0)}, \operatorname*{osc}\limits_{B^+_{R'}} u\Bigr\}$.

The proof of Lemma 4.2 is completely analogous to that of [29], Lemma 4.5, in which Yu and Ri obtained an interior Hölder estimate, the only difference being that we have to use the inequality (4.3) and Lemma 2.6 with $E = B_\varrho^+$ $(0 < \varrho < R)$ and $p_0\sigma_0$ has to be replaced by $n + \varepsilon_0$ in our proof.

Lemma 4.3. *Let Ω and $z \in \partial\Omega$ be as in Lemma 4.2 and let $p \in W^{1,\gamma}(\Omega)$, where $n < \gamma < \infty$, satisfy* (1.7)*. Suppose that A, B and C are satisfied the same conditions as in Theorem 1.2 and $u \in W^{1,p(\cdot)}(\Omega)$ is a bounded weak solution of* (1.1), (1.2) *with $\sup\limits_{\Omega}|u| \leqslant M$ and $\sup\limits_{\partial\Omega}|u| \leqslant M$, where M is a positive number. Then there exist a number $R_0 \in (0,1)$ with $|B_{R_0}| \leqslant 1$ and a number $\varepsilon_0 > 0$ such that for any*

$B_R(x_0) \subset B_{R_0}(z)$ *and* $\sigma_R := \min\Big\{s^-_{B^+_R}, \Big(n\Big(p^-_{B^+_R} - 1\Big)\Big/\Big((n-1)p^-_{B^+_R}\Big)\Big)\mu^-_{B^0_R}\Big\}$, *where* $x_0 \in \partial\Omega$, $u \in \widehat{\mathfrak{B}}_{p(\cdot)}(B^+_R(x_0), M, \gamma, \gamma_1, \delta, 1/\sigma_R)$ *and* $\sigma_R > (n+\varepsilon_0)/p^-_{B^+_R(x_0)}$, *where* $\delta = \min\{2, a_0(M)/(4Mb_0(M))\}$, $\gamma = \gamma(p^+, a_0(M), a_3(M))$ *and* $\gamma_1 = \gamma_1(n, p^+, p^-, \mu^+, \mu^-, a_0(M), a_1(M), \|a_2\|_{s(\cdot)}, a_3(M), \|a_5\|_{p'(\cdot)s(\cdot)}, \|b_2\|_{s(\cdot)}, c_0(M), \|c_1\|_{\mu(\cdot);\partial\Omega}, \Omega, \partial\Omega)$.

P r o o f. Since p and s are continuous on $\overline{\Omega}$, by (1.10) we can take numbers $\varepsilon' > 0$ and $R_2 > 0$ such that

$$s^-_{\Omega_{R_2}} > \frac{n+\varepsilon'}{p^-_{\Omega_{R_2}}} \quad \text{for all } B_{R_2} \text{ with } \Omega_{R_2} \neq \emptyset. \tag{4.5}$$

Similarly, putting

$$s^R_\Gamma(x) := \frac{n(p^-_{\Omega_R(x)} - 1)}{(n-1)p^-_{\Omega_R(x)}}\mu^-_{(\partial\Omega)_R(x)} \quad \text{for } x \in \partial\Omega$$

and using (1.11), we find that there exist numbers $\varepsilon'' > 0$ and $R_3 > 0$ such that

$$s^{R_3}_\Gamma > \frac{n+\varepsilon''}{p^-_{\Omega_{R_3}(x)}} \quad \text{for all } x \in \partial\Omega. \tag{4.6}$$

Setting

$$\mathscr{P}(x) := \frac{n\mu(x)}{n\mu(x) - n + 1} \quad \text{for } x \in \partial\Omega,$$

we have $\mathscr{P} \in C(\partial\Omega)$ and

$$1 < \mathscr{P}(x) < p(x) < n \quad \text{for all } x \in \partial\Omega.$$

Therefore we find that

$$\mathscr{P}(x) = \frac{n\mu'(x)}{n-1+\mu'(x)}, \text{ that is, } \mu'(x) = \frac{(n-1)\mathscr{P}(x)}{n - \mathscr{P}(x)} \tag{4.7}$$

and there is a number $R_4 > 0$ such that

$$\mathscr{P}^+_{(\partial\Omega)_R(x)} < p^-_{\Omega_R(x)} \quad \text{for all } 0 < R < R_4 \text{ and for all } x \in \partial\Omega. \tag{4.8}$$

Let $R_0 \in (0,1)$ be a number satisfying $R_0 \leqslant \min\{\widetilde{R}, R_1, R_2, R_3, R_4\}$ and $|B_{R_0}| \leqslant 1$, and let $B_R(x_0) \subset B_{R_0}(z)$, where $x_0 \in \partial\Omega$. Putting $\varepsilon_0 = \min\{\varepsilon', \varepsilon''\}$, from (4.5), (4.6) and (4.7) we have

$$\sigma_R > \frac{n+\varepsilon_0}{p^-_{B^+_R(x_0)}} \tag{4.9}$$

and

$$(\mu^-_{B^0_R(x_0)})' = \frac{(n-1)\mathscr{P}^+_{B^0_R(x_0)}}{n - \mathscr{P}^+_{B^0_R(x_0)}}. \tag{4.10}$$

Let $0 < \tau < t \leqslant R$ and let $\eta \in C^1(\mathbb{R}^n)$ be as in the proof of Lemma 3.1. Setting $v = \eta^{p^+}\omega^{(k)}$, where $k \geqslant \sup_{B_t^+} \omega - \delta M$, we have $v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, so we can take v as a test function in Definition 2.9. This yields

$$\int_{A_{k,t}} \eta^{p^+} A(x,u,\nabla u)\nabla\omega \,\mathrm{d}x + p^+ \int_{A_{k,t}} \eta^{p^+-1}\nabla\eta A(x,u,\nabla u)(\omega - k)\,\mathrm{d}x + \int_{A_{k,t}} B(x,u,\nabla u)(\omega-k)\eta^{p^+}\,\mathrm{d}x = \int_{B_t^0} C(x',u)\omega^{(k)}\eta^{p^+}\,\mathrm{d}x'. \tag{4.11}$$

We use (1.9)–(1.14) to get:

$$\begin{aligned} \pm \int_{A_{k,t}} \eta^{p^+} A(x,u,\nabla u)\nabla\omega\,\mathrm{d}x \\ \geqslant a_0(M)\int_{A_{k,t}} \eta^{p^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x - 2a_1(M)\|a_2\|_{s(\cdot)}|A_{k,t}|^{1-1/s^-_{B_R^+}}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\left|p^+\int_{A_{k,t}} \eta^{p^+}\nabla\eta A(x,u,\nabla u)(\omega-k)\,\mathrm{d}x\right| \\ &\leqslant 2p^+a_3(M) \\ &\quad\times \int_{A_{k,t}} \Big(\varepsilon\eta^{p^+}|\nabla\omega|^{p(x)} + (\varepsilon^{1-p(x)}+1)\Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)} + {a_4}^{p'(x)}\Big)\,\mathrm{d}x \\ &\leqslant \frac{a_0(M)}{4}\int_{A_{k,t}} \eta^{p^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x \\ &\quad + C(p^+,a_0(M),a_3(M))\int_{A_{k,t}} \Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x \\ &\quad + C(p^+,p^-,a_3(M),\|a_5\|_{s(\cdot)p'(\cdot)})|A_{k,t}|^{1-1/s^-_{B_R^+}}, \end{aligned} \tag{4.13}$$

where we used the Hölder inequality and Young inequality with $\varepsilon = a_0(M)/(8p^+a_3(M))$,

$$\begin{aligned} &\left|\int_{A_{k,t}} B(x,u,\nabla u)(\omega-k)\eta^{p^+}\,\mathrm{d}x\right| \\ &\leqslant \frac{a_0(M)}{4}\int_{A_{k,t}} \eta^{p^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x + C(a_0(M),\|b_2\|_{s(\cdot)})|A_{k,t}|^{1-1/s^-_{B_R^+}} \end{aligned} \tag{4.14}$$

and where we took into account that $0 \leqslant \omega(x) - k \leqslant \delta M \leqslant a_0(M)/(4b_0(M))$ for $x \in A_{k,t}$. Finally, we estimate the boundary integral on the right-hand side of (4.11). For brevity, we write $\mathscr{P}_1$, s_0, p_0, μ_0 instead of $\mathscr{P}^+_{B^0_R(x_0)}$, $s^-_{B^+_R(x_0)}$, $p^-_{B^+_R(x_0)}$, $\mu^-_{B^0_R(x_0)}$, respectively. Thus, taking into account (4.8), (4.10) and using (1.9), (1.15), the Hölder inequality, the boundary trace imbedding and the Young inequality, we find that

$$
\begin{aligned}
(4.15)\quad & \left|\int_{B_t^0} C(x',u)\omega^{(k)}\eta^{p^+}\,\mathrm{d}x'\right| \\
&\leqslant c_0(M)\int_{B_t^0} c_1(x')\omega^{(k)}\eta^{p^+}\,\mathrm{d}x' \leqslant C\|\omega^{(k)}\eta^{p^+}\|_{(n-1)\mathscr{P}_1/(n-\mathscr{P}_1);\partial\Omega} \\
&\leqslant C(\|\nabla(\omega^{(k)}\eta^{p^+})\|_{\mathscr{P}_1;\Omega}+\|\omega^{(k)}\eta^{p^+}\|_{\mathscr{P}_1;\Omega}) \\
&\leqslant C\Big(\|\nabla\omega\eta^{p^+}\|_{\mathscr{P}_1;A_{k,t}}+\Big\|\frac{\omega-k}{t-\tau}\Big\|_{\mathscr{P}_1;A_{k,t}}\Big) \\
&\leqslant C\Big(\|\nabla\omega\eta^{p^+}\|_{p_0;A_{k,t}}+\Big\|\frac{\omega-k}{t-\tau}\Big\|_{p_0;A_{k,t}}\Big)|A_{k,t}|^{1/\mathscr{P}_1-1/p_0} \\
&\leqslant \frac{a_0(M)}{4}\Big(\int_{A_{k,t}}\eta^{p^+}|\nabla\omega|^{p(x)}\,\mathrm{d}x+\int_{A_{k,t}}\Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x\Big)+C|A_{k,t}|^{1-1/\sigma},
\end{aligned}
$$

where $\sigma = n(p_0-1)\mu_0/((n-1)p_0)$ and we used that $|A_{k,t}| \leqslant 1$. Combining (4.12)–(4.15) with (4.11) and using the definition of σ_R, we conclude that

$$\int_{A_{k,\tau}}|\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant \int_{A_{k,t}}\eta^{p^+}|\nabla\omega|^{p(x)} \leqslant \gamma\int_{A_{k,t}}\Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x+\gamma_1|A_{k,t}|^{1-1/\sigma_R}.$$

The lemma is proved. □

Next we consider the local Hölder continuity of weak solutions of (1.1). We shall refer to a function $u \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ such that

$$\int_\Omega (A(x,u,\nabla u)\nabla v + B(x,u,\nabla u)v)\,\mathrm{d}x = 0$$

for an arbitrary function $v \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ as to a bounded weak solution of the equation (1.1). The following class which is useful for proving the interior Hölder continuity was defined by Ri and Yu (see [29]).

Definition 4.4. Let M, γ, γ_1, δ, r be positive constants with $\delta \leqslant 2$, $r > 1$ and $B_R(y) \subset \Omega$. We say that a function u belongs to the class $\mathfrak{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r)$ if $u \in W^{1,p(\cdot)}(B_R)$, $\|u\|_{\infty;B_R} \leqslant M$ and the functions $\omega(x) = \pm u(x)$ satisfy the inequalities

$$\int_{A_{k,\tau}}|\nabla\omega|^{p(x)}\,\mathrm{d}x \leqslant \gamma\int_{A_{k,t}}\Big|\frac{\omega-k}{t-\tau}\Big|^{p(x)}\,\mathrm{d}x+\gamma_1|A_{k,t}|^{1-1/r}$$

for arbitrary $0 < \tau < t \leqslant R$ and k such that $k \geqslant \sup\limits_{B_t(y)} \omega - \delta M$.

Lemma 4.5. *Let $p \in C^{0,1/|\log t|}(\overline{\Omega})$ satisfy* (1.7) *let $R_0 \in (0,1)$ be a number satisfying $R_0 \leqslant R_1$. Let $\varepsilon_0 > 0$ and $r > 1$ be numbers such that $p_0 r \geqslant n + \varepsilon_0$, where $p_0 = p^-_{\Omega_{R_0}(x_0)}$ and $x_0 \in \overline{\Omega}$. Let $B_{R'}(y) \subset \Omega_{R_0}(x_0)$, and $u \in W^{1,p(\cdot)}(B_{R'}) \cap L^\infty(B_{R'})$ and $\sup\limits_{B_{R'}} |u| \leqslant M$. Suppose that $u \in \mathfrak{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r)$ for any $R \in (0, R']$. Then there exists a constant $s = s(n, \gamma, p^+, p^-, L, \varepsilon_0) > 2$ such that, for arbitrary $0 < R \leqslant R'$,*

$$\operatorname*{osc}_{B_R(y)} u \leqslant C_0 R'^{-\alpha} R^\alpha,$$

where L is as in Remark 2.1 *and*

$$C_0 = 4 \max\Big\{ \frac{(\omega_n + 1)(\gamma + 1) + \gamma_1}{\gamma} l 2^s R'^{\varepsilon_0/(n+\varepsilon_0)}, \operatorname*{osc}_{B_{R'}} u \Big\}, \quad l = \max\Big\{2, \frac{2}{\delta}\Big\},$$
$$\alpha = \min\Big\{\frac{\varepsilon_0}{n + \varepsilon_0}, -\log_4(1 - l^{-1}2^{-s})\Big\}.$$

Lemma 4.6. *Let $p \in C(\overline{\Omega})$ satisfy* (1.7). *Let A and B satisfy* (1.9), (1.10), (1.12)–(1.14) *and let u be a bounded weak solution of* (1.1) *such that $\sup\limits_{\Omega} |u| \leqslant M$. Then for any ball $B_R(y) \subset\subset \Omega$ with $|B_R| \leqslant 1$*

$$u \in \mathfrak{B}_{p(\cdot)}\Big(B_R(y), M, \gamma, \gamma_1, \delta, \frac{1}{s^-_{B_R}}\Big),$$

where $\delta = \min\{2, a_0(M)/(3Mb_0(M))\}$, $\gamma = \gamma(p^+, a_0(M), a_3(M))$, $\gamma_1 = \gamma_1(p^+, p^-, a_0(M), a_1(M), \|a_2\|_{s(\cdot)}, a_3(M), \|a_5\|_{p'(\cdot)s(\cdot)}, \|b_2\|_{s(\cdot)})$.

The proofs of Lemmas 4.5 and 4.6 are similar to those of Lemmas 4.2 and 4.3 (see also the proofs of Lemmas 4.5 and 4.6 in [29]), respectively, and are therefore omitted.

P r o o f of Theorem 1.2. Consider now the collection of the balls $B_{R(z)}(z)$ for all $z \in \partial\Omega$, where $R(z) = \frac{1}{3}R_0$ with R_0 corresponding to z as in Lemma 4.3. A finite subset $B_{R(z_i)}(z_i)$, $i = 1, \ldots, N$, of this collection covers $\partial\Omega$. It follows from Lemma 4.3 that u belongs to $\widehat{\mathfrak{B}}_{p(\cdot)}(B^+_R(x_0), M, \gamma, \gamma_1, \delta, 1/\sigma_R)$, where $x_0, R > 0$ and σ_R satisfy $x_0 \in \partial\Omega, B_R(x_0) \subset B_{R(z_i)}(z_i)$ and $\sigma_R > (n + \varepsilon_0)/p^-_{B^+_R(x_0)}$. We denote C corresponding to $z = z_i$ in (4.4) by C_i. Put

$$\overline{R} := \min\{R(z_1), \ldots, R(z_N)\}.$$

Let $x, y \in \Omega$, $\beta = \operatorname{dist}(y, \partial\Omega)$. First, suppose that $\beta < \overline{R}$. When $|x - y| < \beta$, putting $R = |x - y|$ we have $B_R(y) \subset\subset \Omega$. Thus, by Lemma 4.6 we get $u \in \mathfrak{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/s^-_{B_R})$, where $s^-_{B_R} > (n + \varepsilon_0)/p^-_{B_R}$ by using (4.5). By Lemma 4.5 with $x_0 = y, R_0 = R' = \beta$, we have

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leqslant C_0 \beta^{-\alpha} \leqslant 4 \max\Big\{C, \beta^{-\alpha} \operatorname*{osc}_{B_\beta(y)} u\Big\}, \tag{4.16}$$

where C depends on s as in Lemma 4.5 and γ, γ_1, l as in Lemma 4.6, and we used that $\varepsilon_0/(n+\varepsilon_0)-\alpha \geqslant 0$ and $\beta<1$. We estimate $\beta^{-\alpha} \operatorname*{osc}\limits_{B_\beta(y)} u$. Choosing $x_0 \in \partial\Omega$ such that $|x_0-y|=\beta$, then $B_\beta(y) \subset B_{2\beta}(x_0)$ and so we have

$$\beta^{-\alpha} \operatorname*{osc}_{B_\beta} u \leqslant 2^\alpha \operatorname*{osc}_{\Omega_{2\beta}(x_0)} u \cdot (2\beta)^{-\alpha}. \tag{4.17}$$

Since $x_0 \in B_{R(z_i)}(z_i)$ for some $1 \leqslant i \leqslant N$, we get $B_{2\beta}(x_0) \subset B_{2R(z_i)}(x_0) \subset B_{3R(z_i)}(z_i)$. Using Lemma 4.2 with $R=2\beta$, $R'=2R(z_i)$, $R_0=3R(z_i)$ and $z=z_i$, we therefore have

$$\operatorname*{osc}_{\Omega_{2\beta}(x_0)} u \cdot (2\beta)^{-\alpha} \leqslant C_i(2R(z_i))^{-\alpha} \leqslant C \tag{4.18}$$

where $C=C(n,p(\cdot),s(\cdot),\mu(\cdot),M,a_0(M),a_1(M),a_3(M),b_0(M),c_0(M),\|a_4\|_{p'(\cdot)s(\cdot)}$, $\|b_1\|_{s(\cdot)},\|c_1\|_{\mu(\cdot);\partial\Omega},\Omega)$. Substituting (4.17) and (4.18) into (4.16), we obtain

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leqslant C. \tag{4.19}$$

Let $\beta \leqslant |x-y| < \overline{R}$. By using the notations introduced above, we have $B_R(y) \subset B_{2R}(x_0)$ and

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leqslant \operatorname*{osc}_{\Omega_R(y)} u \cdot R^{-\alpha} \leqslant 2^\alpha \operatorname*{osc}_{\Omega_{2R}(x_0)} u \cdot (2R)^{-\alpha}.$$

Replacing β by R and repeating the arguments which give (4.18), we thus have (4.19).

Next, suppose that $\overline{R} \leqslant \beta$. When $|x-y|<\overline{R}$, putting $R=|x-y|$, $R'=\overline{R}$ and using Lemmas 4.5 and 4.6, we get (4.19). If $|x-y| \geqslant \overline{R}$, (4.19) is obvious. This completes the proof of Theorem 1.2. □

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