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WEIGHTED CALDERÓN-HARDY SPACES

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Abstract. We present the weighted Calderón-Hardy spaces on Euclidean spaces and investigate their properties. As an application we show, for certain power weights, that the iterated Laplace operator is a bijection from these spaces onto classical weighted Hardy spaces. The main tools to achieve our result are an atomic decomposition of weighted Hardy spaces furnished by the author, fundamental solutions of iterated Laplacian and pointwise inequalities for certain maximal functions.

Keywords: weighted Calderón-Hardy space; weighted Hardy space; atomic decomposition; Laplace operator

MSC 2020: 42B25, 42B30

1. Introduction

It is well known that classical Hardy spaces $H^p(\mathbb{R}^n)$ with $0 < p \leqslant 1$ play an important role in the harmonic analysis and PDEs. Many important operators are better behaved on Hardy spaces $H^p(\mathbb{R}^n)$ than on Lebesgue spaces $L^p(\mathbb{R}^n)$ in the range $0 < p \leqslant 1$.

The Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p < \infty$, were first defined by Stein and Weiss in [30] in terms of the theory of harmonic functions on $\mathbb{R}^n$. Afterward, Fefferman and Stein in [9] introduced real variable methods into this subject and characterized the Hardy spaces by means of maximal functions. This second approach brought greater flexibility to the theory.

The spaces $H^p(\mathbb{R}^n)$ can also be characterized by atomic decompositions. Roughly speaking, every distribution $f \in H^p$ can be expressed in the form

$$f = \sum_j \lambda_j a_j, \tag{1}$$

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where the a_j's are p-atoms, $\{\lambda_j\} \in l^p$ and $\|f\|_{H^p}^p \approx \sum_j |\lambda_j|^p$. For $0 < p \leqslant 1$, an p-atom is a function $a(\cdot)$ supported on a cube Q such that

$$\|a\|_\infty \leqslant |Q|^{-1/p} \quad \text{and} \quad \int x^\alpha a(x)\,\mathrm{d}x = 0 \quad \forall\,|\alpha| \leqslant n\Big(\frac{1}{p} - 1\Big).$$

Such decompositions were obtained by Coifman in [4] for the case $n = 1$ and by Latter in [17] for the case $n \geqslant 1$. These decompositions allow to study the behavior of certain operators on $H^p(\mathbb{R}^n)$ by focusing one's attention on individual atoms. In principle, the continuity of an operator T on $H^p(\mathbb{R}^n)$ can often be proved by estimating Ta when $a(\cdot)$ is an atom. For more results about Hardy spaces see [18], [29], [32], [33], and [15].

Gatto, Jiménez and Segovia in [12], by using the atomic decomposition (1) for members of $H^p(\mathbb{R}^n)$, solved the equation

$$\Delta^m F = f \quad \text{for } f \in H^p(\mathbb{R}^n). \tag{2}$$

Moreover, they characterized the solution set of (2). These sets result are the Calderón-Hardy spaces $\mathcal{H}^p_{q,2m}(\mathbb{R}^n)$, which were defined by them for this purpose. More precisely, they proved that the iterated Laplace operator Δ^m is a bijective mapping from Calderón-Hardy spaces $\mathcal{H}^p_{q,2m}(\mathbb{R}^n)$ onto Hardy spaces $H^p(\mathbb{R}^n)$. They also investigated the properties of these spaces and obtained an atomic decomposition for their elements.

The one-dimensional case with weights was studied by Ombrosi in [21], there he introduced the one-sided Calderón-Hardy spaces $\mathcal{H}^{p,+}_\alpha((x_{-\infty}, \infty), w)$ for weights w in a Sawyer class and investigated their properties (see also [23]). In [25], Perini studied the boundedness of one-sided fractional integrals on these spaces. Ombrosi, Perini and Testoni obtained in [22] a complex interpolation theorem between one-sided Calderón-Hardy spaces. With the appearing of the theory of variable exponents the Hardy type spaces received a new impetus (see [24], [16], [8], [5], [20], and [7]). In this setting, the author in [26] defined the variable Calderón-Hardy spaces $\mathcal{H}^{p(\cdot)}_{q,\gamma}(\mathbb{R}^n)$, and studied the behavior of the iterated Laplace operator Δ^m on these spaces obtaining analogous results to those of Gatto, Jiménez and Segovia.

Recently, Auscher and Egert in [1] presented results on elliptic boundary value problems where the theory of Hardy spaces associated with operators plays a key role (see also [2] and references therein).

The purpose of this paper is to define the weighted Calderón-Hardy spaces $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n, w)$ for $0 < p \leqslant 1 < q < \infty$, $\gamma > 0$, and investigate their properties. In Section 5 of this work we prove our main results. These are contained in the Theorems 21 and 22. The threshold $n(2m + n/q)^{-1}$ in the results of Theorems 21 and 22 is optimal in the sense of Theorem 17.

This paper is organized as follows. In Section 3 we give the basics of weighted Lebesgue theory, establish a Fefferman-Stein vector-valued inequality for the Hardy-Littlewood maximal operator and also recall the atomic decomposition of weighted Hardy spaces given in [27]. In Section 4 we define the weighted Calderón-Hardy spaces and investigate their properties. The iterated Laplacian is also presented. Theorems 21 and 22 are proved in Section 5.

2. Notation

The symbol $A \lesssim B$ stands for the inequality $A \leqslant cB$ for a constant c. We denote by $Q(x_0, r)$ the cube centered at $x_0 \in \mathbb{R}^n$ with side lenght r. Given a cube $Q = Q(x_0, r)$ and $\delta > 0$, we set $\delta Q = Q(x_0, \delta r)$. For a measurable subset $E \subseteq \mathbb{R}^n$ we denote by $|E|$ and χ_E the Lebesgue measure of E and the characteristic function of E, respectively. As usual we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space (i.e., the space of tempered distributions). A distribution u acting on an element $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is denoted by (u, φ). Δ and δ stand for the Laplacian and the Dirac's delta on $\mathbb{R}^n$, respectively. If α is the multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $\alpha! = \alpha_1! \ldots \alpha_n!$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\partial^\alpha \varphi(x) = \partial^{|\alpha|}\varphi/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})(x)$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Given a real number $s \geqslant 0$, we write $\lfloor s \rfloor$ for the integer part of s.

Throughout this paper, C denotes a positive constant, not necessarily the same at each occurrence.

3. Preliminaries

In this section we present weighted Lebesgue spaces and weighted Hardy spaces. For more information about these spaces the reader can consult [6], [11], [14] and [10], [31].

3.1. Weighted Lebesgue spaces. A weight w is a non-negative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere, i.e., the weights are allowed to be zero or infinity only on a set of Lebesgue measure zero.

Given a weight w and $0 < p < \infty$, we denote by $L^p(\mathbb{R}^n, w)$ the spaces of all functions f defined on $\mathbb{R}^n$ satisfying $\|f\|^p_{L^p(\mathbb{R}^n,w)} := \int_{\mathbb{R}^n} |f(x)|^p w(x)\, \mathrm{d}x < \infty$. When $p = \infty$, we have that $L^\infty(\mathbb{R}^n, w) = L^\infty(\mathbb{R}^n)$ with $\|f\|_{L^\infty(\mathbb{R}^n,w)} = \|f\|_{L^\infty(\mathbb{R}^n)}$. If E is a measurable set, we use the notation $w(E) = \int_E w(x)\, \mathrm{d}x$. It is easy to check that $w(E) = 0$ if and only if $|E| = 0$.

It is well known that the harmonic analysis on weighted spaces is relevant if the weights w belong to the class $\mathcal{A}_p$. Before defining the class $\mathcal{A}_p$, we first introduce the Hardy-Littlewood maximal operator.

Let f be a locally integrable function on $\mathbb{R}^n$. The function

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y,$$

where the supremum is taken over all cubes Q containing x, is called the uncentered Hardy-Littlewood maximal function of f. We say that a weight $w \in \mathcal{A}_1$ if there exists $C > 0$ such that

$$M(w)(x) \leqslant Cw(x), \quad \text{a.e. } x \in \mathbb{R}^n;$$

the best possible constant is denoted by $[w]_{\mathcal{A}_1}$. Equivalently, a weight $w \in \mathcal{A}_1$ if there exists $C > 0$ such that for every cube Q

$$\frac{1}{|Q|} \int_Q w(x) \, \mathrm{d}x \leqslant C \operatorname*{ess\,inf}_{x \in Q} w(x).$$

For $1 < p < \infty$, we say that a weight $w \in \mathcal{A}_p$ if there exists $C > 0$ such that for every cube Q

$$\left(\frac{1}{|Q|} \int_Q w(x) \, \mathrm{d}x \right) \left(\frac{1}{|Q|} \int_Q [w(x)]^{-1/(p-1)} \, \mathrm{d}x \right)^{p-1} \leqslant C.$$

It is well known that $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$ for all $1 \leqslant p_1 < p_2 < \infty$. Also, if $w \in \mathcal{A}_p$ with $1 < p < \infty$, then there exists $1 < q < p$ such that $w \in \mathcal{A}_q$. This leads us to the following definition.

Definition 1. Given $w \in \mathcal{A}_p$ with $1 < p < \infty$, we define the critical index of w by

$$q_w = \inf\{q > 1 \colon \ w \in \mathcal{A}_q\}.$$

Remark 2. The index q_w is related to the vanishing moment condition satisfied by the atoms for the atomic decompositions of the weighted Hardy spaces (see Definition 7 below).

The $\mathcal{A}_p$-weights, $1 < p < \infty$, give the following characterization for the Hardy-Littlewood maximal operator M:

$$\int_{\mathbb{R}^n} [Mf(x)]^p w(x) \, \mathrm{d}x \leqslant C \int_{\mathbb{R}^n} |f(x)|^p w(x) \, \mathrm{d}x$$

for all $f \in L^p(\mathbb{R}^n, w)$ if and only if $w \in \mathcal{A}_p$ (see Theorem 9 in [19]).

A weight w satisfies reverse Hölder's inequality with exponent $s > 1$, denoted by $w \in RH_s$, if there exists $C > 0$ such that for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q [w(x)]^s \,\mathrm{d}x\right)^{1/s} \leqslant C\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x;$$

the best possible constant is denoted by $[w]_{RH_s}$. We observe that if $w \in RH_s$, then by Hölder's inequality, $w \in RH_t$ for all $1 < t < s$, and $[w]_{RH_t} \leqslant [w]_{RH_s}$. Moreover, if $w \in RH_s$, $s > 1$, then $w \in RH_{s+\varepsilon}$ for some $\varepsilon > 0$. This gives the following definition.

Definition 3. Given $w \in RH_s$ with $s > 1$, we define the critical index of w for the reverse Hölder condition by

$$r_w = \sup\{r > 1\colon\ w \in RH_r\}.$$

Remark 4. The index r_w is used to determine the size condition satisfied by the atoms for the atomic decompositions of the weighted Hardy spaces (see Definition 7).

In view of Corollary 7.3.4 in [14], we define the class $\mathcal{A}_\infty$ by $\mathcal{A}_\infty = \bigcup\limits_{1\leqslant p<\infty} \mathcal{A}_p$. Since $w \in \mathcal{A}_\infty$ if and only if $w \in RH_s$ for some $s > 1$, then it follows that $1 < r_w \leqslant \infty$ for all $w \in \mathcal{A}_\infty$.

The following lemma states a Fefferman-Stein vector-valued inequality for the Hardy-Littlewood maximal operator on $L^p(\mathbb{R}^n, w)$. This lemma is crucial to get Theorem 21.

Lemma 5. *Let $1 < p < \infty$. Then for $u \in (1,\infty)$ and $w \in \mathcal{A}_p$ we have that*

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^u\right)^{1/u}\right\|_{L^p(\mathbb{R}^n,w)} \lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^u\right)^{1/u}\right\|_{L^p(\mathbb{R}^n,w)}$$

holds for all sequences of bounded measurable functions with compact support $\{f_j\}_{j=1}^{\infty}$.

Proof. This lemma follows from Theorem 9 in [19], and Corollary 3.12 in [6]. □

3.2. Weighted Hardy spaces. We give the definition of weighted Hardy spaces and the atomic decomposition of these spaces developed by the author in [27]. We topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of semi-norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) = \sum_{|\beta|\leqslant N}\sup_{x\in\mathbb{R}^n}(1+|x|)^N|\partial^\beta\varphi(x)|,$$

for each $N \in \mathbb{N}$. We set $\mathcal{F}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n)\colon\ p_N(\varphi) \leqslant 1\}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{M}_{\mathcal{F}_N}$ the grand maximal operator given by

$$\mathcal{M}_{\mathcal{F}_N} f(x) = \sup_{t>0} \sup_{\varphi\in\mathcal{F}_N} |(t^{-n}\varphi(t^{-1}\cdot) * f)(x)|,$$

where N is a large and fixed integer.

Definition 6. Let $0 < p < \infty$. The weighted Hardy space $H^p(\mathbb{R}^n, w)$ is the set of all $f \in S'(\mathbb{R}^n)$ for which $\mathcal{M}_{\mathcal{F}_N} f \in L^p(\mathbb{R}^n, w)$. In this case the "norm" of f in $H^p(\mathbb{R}^n, w)$ is defined as

$$\|f\|_{H^p(\mathbb{R}^n,w)} := \|\mathcal{M}_{\mathcal{F}_N} f\|_{L^p(\mathbb{R}^n,w)}.$$

It is known that if $1 < p < \infty$ and $w \in \mathcal{A}_p$, then $H^p(\mathbb{R}^n, w) \simeq L^p(\mathbb{R}^n, w)$. In the range $0 < p \leqslant 1$ these spaces are not comparable.

Now, we introduce our atoms. We recall that our definition of atom differs from that given in [10] and [31].

Definition 7 (w-(p, p_0, d)-atom)**.** Let $w \in \mathcal{A}_\infty$ with critical index q_w and critical index r_w for reverse Hölder condition. Let $0 < p \leqslant 1$,

$$\max\{1, p(r_w/(r_w - 1))\} < p_0 \leqslant \infty,$$

and $d \in \mathbb{Z}$ such that $d \geqslant \lfloor n(q_w/p - 1)\rfloor$. We say that a function $a(\cdot)$ is a w-(p, p_0, d)-atom centered in $x_0 \in \mathbb{R}^n$ if

(A1) $a \in L^{p_0}(\mathbb{R}^n)$ with support in the cube $Q = Q(x_0, r)$.
(A2) $\|a\|_{L^{p_0}(\mathbb{R}^n)} \leqslant |Q|^{1/p_0}[w(Q)]^{-1/p}$.
(A3) $\int x^\alpha a(x)\,\mathrm{d}x = 0$ for all multi-index α such that $|\alpha| \leqslant d$.

Indeed, a w-(p, p_0, d)-atom $a(\cdot)$ belongs to $H^p(\mathbb{R}^n, w)$ (see Lemma 2.8 in [27]).

Remark 8. We observe that the condition $\max\{1, p(r_w/(r_w - 1))\} < p_0 < \infty$ implies that $w \in RH_{(p_0/p)'}$. If $r_w = \infty$, then $w \in RH_t$ for each $1 < t < \infty$. So, if $r_w = \infty$ and since $\lim\limits_{t\to\infty} t/(t - 1) = 1$, we put $r_w/(r_w - 1) = 1$. For example, if $w \equiv 1$, then $q_w = 1$ and $r_w = \infty$ and the definition of atom in this case coincide with the definition of atom in the classical Hardy spaces.

The set

$$\widehat{\mathcal{D}}_0 = \{\varphi \in \mathcal{S}(\mathbb{R}^n)\colon\ \widehat{\varphi} \in C_c^\infty(\mathbb{R}^n) \text{ and } 0 \notin \operatorname{supp}(\widehat{\varphi})\}$$

is dense in $H^p(\mathbb{R}^n, w)$, $0 < p < \infty$, for every $w \in \mathcal{A}_\infty$. Indeed, given a weight $w \in \mathcal{A}_\infty$ let ν be the measure defined by $d\nu = w(x)\,\mathrm{d}x$, by (9) in Proposition 7.1.5 in [14] we have that such measure ν is doubling, then the density of $\widehat{\mathcal{D}}_0$ in $H^p(\mathbb{R}^n, w)$ follows from Theorem 1 in [31].

Finally, the atomic decomposition for $H^p(\mathbb{R}^n,w)$, $0<p\leqslant 1$, established in [27], it is as follows.

Theorem 9 (Theorem 2.9 in [27]). *Let $f\in\widehat{\mathcal{D}}_0$, and $0<p\leqslant 1$. If $w\in\mathcal{A}_\infty$, then there exist a sequence of w-(p,p_0,d)-atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum\limits_j|\lambda_j|^p\leqslant c\|f\|^p_{H^p(\mathbb{R}^n,w)}$ such that $f=\sum\limits_j\lambda_ja_j$, where the convergence is in $L^{p_0}(\mathbb{R}^n)$.*

The novelty in this theorem is the convergence in $L^{p_0}(\mathbb{R}^n)$-norm of the weighted atomic series.

Remark 10. Since $\widehat{\mathcal{D}}_0$ is dense in $H^p(\mathbb{R}^n,w)$, $0<p\leqslant 1$, a routine argument allows us to ensure that every member of $H^p(\mathbb{R}^n,w)$ has a weighted atomic decomposition as in Theorem 9, where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$.

The atoms in Theorem 2.9, page 235 in [27], are supported on balls; this theorem still holds if we consider atoms supported on cubes instead of balls.

4. Weighted Calderón-Hardy spaces

4.1. Basics of weighted Calderón-Hardy spaces $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n,w)$. Let $L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $1<q<\infty$, be the space of all measurable functions g on $\mathbb{R}^n$ that belong locally to L^q for compact sets of $\mathbb{R}^n$. We endowed $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ with the topology generated by the seminorms

$$|g|_{q,Q}=\left(|Q|^{-1}\int_Q|g(y)|^q\,\mathrm{d}y\right)^{1/q},$$

where Q is a cube in $\mathbb{R}^n$ and $|Q|$ denotes its Lebesgue measure.

For $g\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, we define a maximal function $\eta_{q,\gamma}(g;x)$ as

$$\eta_{q,\gamma}(g;x)=\sup_{r>0}r^{-\gamma}|g|_{q,Q(x,r)},$$

where γ is a positive real number and $Q(x,r)$ is the cube centered at x with side length r. This type of maximal function was introduced by Calderón in [3].

Let k be a non-negative integer and $\mathcal{P}_k$ the subspace of $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ formed by all the polynomials of degree at most k. We denote by E^q_k the quotient space of $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ by $\mathcal{P}_k$. For $G\in E^q_k$ we define the seminorm $\|G\|_{q,Q}=\inf\{|g|_{q,Q}\colon\ g\in G\}$. The family of all these seminorms induced on E^q_k is the quotient topology.

Given a positive real number γ, we can write $\gamma=k+t$, where k is a non-negative integer and $0<t\leqslant 1$. This decomposition is unique.

For $G\in E^q_k$, we define the maximal function $N_{q,\gamma}(G;x)$ by

$$N_{q,\gamma}(G;x)=\inf\{\eta_{q,\gamma}(g;x)\colon\ g\in G\}.$$

The maximal function $N_{q;\gamma}(G;\cdot)$ associated with a class G in E_k^q is lower semicontinuous (see Lemma 6 in [3]).

Next, we define the weighted Calderón-Hardy spaces on $\mathbb{R}^n$.

Definition 11. Let $w \in \mathcal{A}_\infty$, $0 < p \leqslant 1 < q < \infty$ and $\gamma > 0$. We say that an element $G \in E_k^q$ belongs to the weighted Calderón-Hardy space $\mathcal{H}_{q,\gamma}^p(\mathbb{R}^n, w)$ if the maximal function $N_{q,\gamma}(G;\cdot) \in L^p(\mathbb{R}^n, w)$. The "norm" of G in $\mathcal{H}_{q,\gamma}^p(\mathbb{R}^n, w)$ is defined as

$$\|G\|_{\mathcal{H}_{q,\gamma}^p(\mathbb{R}^n,w)} := \|N_{q,\gamma}(G;\cdot)\|_{L^p(\mathbb{R}^n,w)}.$$

Lemma 12. *Let $G \in E_k^q$ with $N_{q,\gamma}(G;x_0) < \infty$, for some $x_0 \in \mathbb{R}^n$. Then*

(i) *there exists a unique $g \in G$ such that $\eta_{q,\gamma}(g;x_0) < \infty$ and, therefore, $\eta_{q,\gamma}(g;x_0) = N_{q,\gamma}(G;x_0)$;*

(ii) *for any cube Q, there is a constant c depending on x_0 and Q such that if g is the unique representative of G given in* (i), *then*

$$\|G\|_{q,Q} \leqslant |g|_{q,Q} \leqslant C\eta_{q,\gamma}(g;x_0) = CN_{q,\gamma}(G;x_0).$$

The constant c can be chosen independently of x_0 provided that x_0 varies in a compact set.

P r o o f. The proof is similar to the one given in Lemma 3 in [12]. □

Corollary 13. *If $\{G_j\}$ is a sequence of elements of E_k^q converging to G in $\mathcal{H}_{q,\gamma}^p(\mathbb{R}^n, w)$, then $\{G_j\}$ converges to G in E_k^q.*

P r o o f. For any cube Q, by (ii) of Lemma 12, we have

$$\|G - G_j\|_{q,Q} \leqslant C[w(Q)]^{-1/p}\|\chi_Q(\cdot)N_{q,\gamma}(G - G_j;\cdot)\|_{L^p(\mathbb{R}^n,w)} \leqslant C\|G - G_j\|_{\mathcal{H}_{q,\gamma}^p(\mathbb{R}^n,w)},$$

which proves the corollary. □

Lemma 14. *Let $\{G_j\}$ be a sequence in E_k^q such that for a given point x_0, the series $\sum\limits_j N_{q,\gamma}(G_j;x_0)$ is finite. Then*

(i) *the series $\sum\limits_j G_j$ converges in E_k^q to an element G and*

$$N_{q,\gamma}(G;x_0) \leqslant \sum_j N_{q,\gamma}(G_j;x_0);$$

(ii) *if g_j is the unique representative of G_j satisfying $\eta_{q,\gamma}(g_j;x_0) = N_{q,\gamma}(G_j;x_0)$, then $\sum\limits_j g_j$ converges in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ to a function g that is the unique representative of G satisfying $\eta_{q,\gamma}(g;x_0) = N_{q,\gamma}(G;x_0)$.*

P r o o f. The proof is similar to the one given in Lemma 4 in [12]. □

Proposition 15. *If $g \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and there is a point x_0 such that $\eta_{q,\gamma}(g; x_0) < \infty$, then $g \in \mathcal{S}'(\mathbb{R}^n)$.*

Proof. We first assume that $x_0 = 0$. Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k > \gamma + n$ we have that $|\varphi(y)| \leqslant p_k(\varphi)(1 + |y|)^{-k}$ for all $y \in \mathbb{R}^n$. So

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} g(y)\varphi(y)\,\mathrm{d}y \right| &\leqslant p_k(\varphi) \int_{Q(0,1)} |g(y)|(1 + |y|)^{-k}\,\mathrm{d}y \\
&\quad + p_k(\varphi) \sum_{j=0}^{\infty} \int_{Q(0,2^{j+1})\setminus Q(0,2^j)} |g(y)|(1 + |y|)^{-k}\,\mathrm{d}y \\
&\lesssim p_k(\varphi)\eta_{q,\gamma}(g; 0) + p_k(\varphi)\eta_{q,\gamma}(g; 0) \sum_{j=0}^{\infty} 2^{j(\gamma+n-k)}.
\end{aligned}$$

Being $k > \gamma + n$ it follows that $g \in \mathcal{S}'(\mathbb{R}^n)$. For the case $x_0 \neq 0$ we apply the translation operator τ_{x_0} defined by $(\tau_{x_0} g)(x) = g(x + x_0)$ and use the fact that $\eta_{q,\gamma}(\tau_{x_0} g; 0) = \eta_{q,\gamma}(g; x_0)$. $\square$

Proposition 16. *Let $w \in \mathcal{A}_\infty$. Then the space $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n, w)$ is complete.*

Proof. It is enough to show that $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n, w)$ has the Riesz-Fisher property: given any sequence $\{G_j\}$ in $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n, w)$ such that

$$\sum_j \|G_j\|^p_{\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n,w)} < \infty,$$

the series $\sum\limits_j G_j$ converges in $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n, w)$. For $1 \leqslant l$ fixed we have

$$\begin{aligned}
\left\| \sum_{j=l}^{k} N_{q,\gamma}(G_j; \cdot) \right\|^p_{L^p(\mathbb{R}^n,w)} &\leqslant \sum_{j=l}^{k} \|N_{q,\gamma}(G_j; \cdot)\|^p_{L^p(\mathbb{R}^n,w)} \leqslant \sum_{j=l}^{\infty} \|G_j\|^p_{\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n,w)} \\
&=: \alpha_l < \infty
\end{aligned}$$

for all $k \geqslant l$. Thus, for all $k \geqslant l$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\alpha_l^{-1/p} \sum_{j=l}^{k} N_{q,\gamma}(G_j; x) \right)^p w(x)\,\mathrm{d}x \\
&\qquad \leqslant \int_{\mathbb{R}^n} \left(\left\| \sum_{j=l}^{k} N_{q,\gamma}(G_j; \cdot) \right\|^{-1}_{L^p(\mathbb{R}^n,w)} \sum_{j=l}^{k} N_{q,\gamma}(G_j; x) \right)^p w(x)\,\mathrm{d}x = 1.
\end{aligned}$$

It follows from Fatou's lemma as $k \to \infty$ that

$$\int_{\mathbb{R}^n} \left(\alpha_l^{-1/p} \sum_{j=l}^{\infty} N_{q,\gamma}(G_j; x) \right)^p w(x)\,\mathrm{d}x \leqslant 1,$$

then for all $l \geqslant 1$

$$\left\| \sum_{j=l}^{\infty} N_{q,\gamma}(G_j;\cdot) \right\|^p_{L^p(\mathbb{R}^n,w)} \leqslant \alpha_l = \sum_{j=l}^{\infty} \|G_j\|^p_{\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n,w)} < \infty. \tag{3}$$

Taking $l = 1$ in (3) and since $w \in \mathcal{A}_\infty$, we obtain that $\sum\limits_j N_{q,\gamma}(G_j;x)$ is finite for a.e. $x \in \mathbb{R}^n$. Then by (i) of Lemma 14, the series $\sum\limits_j G_j$ converges in E^q_k to an element G. Now

$$N_{q,\gamma}\left(G - \sum_{j=1}^{k} G_j; x\right) \leqslant \sum_{j=k+1}^{\infty} N_{q,\gamma}(G_j;x),$$

from this and (3), we get

$$\left\| G - \sum_{j=1}^{k} G_j \right\|^p_{\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n,w)} \leqslant \sum_{j=k+1}^{\infty} \|G_j\|^p_{\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n,w)},$$

since the right-hand side tends to 0 as $k \to \infty$, the series $\sum\limits_j G_j$ converges to G in $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n,w)$. $\square$

The proof of the next result is an adaptation of the proof of Theorem 2 given in [12].

Theorem 17. *Let $m \geqslant 1$ and $w_a(x) = |x|^a$ with $-n < a < n(s-1)$ and $1 < s < \infty$. If $p \leqslant (n + \min\{a,0\})/(2m + n/q)$ and $p \leqslant 1$, then $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a) = \{0\}$.*

Proof. We observe that for $1 < s < \infty$ and $-n < a < n(s-1)$ the non-negative function $w_a(x) = |x|^a$, $x \in \mathbb{R}^n \setminus \{0\}$, is a weight in the Muckenhoupt class $\mathcal{A}_s$ (see page 506 in [14]). Let $G \in \mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a)$ and assume $G \neq 0$. Then there exists $g \in G$ that is not a polynomial of degree less or equal to $2m-1$. It is easy to check that there exist a positive constant C and a cube $Q = Q(0,r)$ with $r > 1$ such that

$$\int_Q |g(y) - P(y)|^q \,\mathrm{d}y \geqslant C > 0$$

for every $P \in \mathcal{P}_{2m-1}$.

Let x be a point such that $|x| > \sqrt{n}r$ and let $\delta = 4|x|$. Then $Q(0,r) \subset Q(x,\delta)$. If $h \in G$, then $h = g - P$ for some $P \in \mathcal{P}_{2m-1}$ and

$$\delta^{-2m} |h|_{q,Q(x,\delta)} \geqslant C|x|^{-2m-n/q}.$$

So $N_{q,2m}(G;x) \geqslant C|x|^{-2m-n/q}$, for $|x| > \sqrt{n}r$. Since

$$p \leqslant (n + \min\{a,0\})/(2m + n/q)$$

and $-n < a < n(s-1)$, we have

$$\int_{\mathbb{R}^n} [N_{q,2m}(G;x)]^p w_a(x)\,\mathrm{d}x \geqslant C \int_{|x|>\sqrt{n}r} |x|^{-(2m+n/q)p}|x|^a\,\mathrm{d}x = \infty.$$

Then, we get a contradiction. Thus $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a) = \{0\}$, for $p \leqslant (n+\min\{a,0\})/(2m+n/q)$. $\square$

4.2. The iterated Laplacian. The Laplace operator or Laplacian Δ on $\mathbb{R}^n$ is defined by

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

For $m \in \mathbb{N}$ we define the iterated Laplacian by $\Delta^m := \Delta \circ \ldots \circ \Delta$ (m times). Given $g \in \mathcal{S}'(\mathbb{R}^n)$, the iterated Laplacian Δ^m acts on the distribution g by means of the formula

$$(\Delta^m g, \varphi) = (g, \Delta^m \varphi) \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Thus $\Delta^m g \in \mathcal{S}'(\mathbb{R}^n)$ when $g \in \mathcal{S}'(\mathbb{R}^n)$.

Let Φ be the function defined on $\mathbb{R}^n \setminus \{0\}$ by

$$\Phi(x) = \begin{cases} C_1 |x|^{2m-n} \ln|x| & \text{if } n \text{ is even and } 2m-n \geqslant 0, \\ C_2 |x|^{2m-n} & \text{otherwise.} \end{cases} \tag{4}$$

Such a function is a fundamental solution of Δ^m (see pages 201–202 in [13]); i.e., $\Delta^m \Phi = \delta$ in $\mathcal{S}'(\mathbb{R}^n)$.

Lemma 18 (Lemma 8 in [12])**.** *If Φ is the kernel defined in* (4) *and $|\alpha| = 2m$, then $(\partial^\alpha \Phi)(x)$ is a C^∞ homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$ such that*

$$\int_{|x|=1} (\partial^\alpha \Phi)(x)\,\mathrm{d}x = 0.$$

Given a bounded function $a(\cdot)$ with compact support, its potential b, defined as

$$b(x) = \int_{\mathbb{R}^n} \Phi(x-y) a(y)\,\mathrm{d}y,$$

is a locally bounded function and $\Delta^m b = a$ in the sense of distributions. For these potentials, we have the following two results.

Lemma 19. *Let $a(\cdot)$ be an (p, p_0, d)-atom with $d = \max\{\lfloor n(q_w/p - 1)\rfloor, 2m-1\}$ and assume that $Q(x_0, r)$ is the cube containing the support of $a(\cdot)$ in the definition of (p, p_0, d)-atom. If*

$$b(x) = \int_{\mathbb{R}^n} \Phi(x-y)a(y)\,\mathrm{d}y,$$

then for $|x - x_0| \geqslant \sqrt{n}r$ and every multiindex α, there exists C_α such that

$$|(\partial^\alpha b)(x)| \leqslant C_\alpha r^{2m+n}[w(Q)]^{-1/p}|x-x_0|^{-n-\alpha}$$

holds.

P r o o f. Since $a(\cdot)$ has vanishing moments up to the order $d \geqslant 2m-1$, we have

$$\begin{aligned}(5)\quad (\partial^\alpha b)(x) &= \int_{Q(x_0,r)} (\partial^\alpha \Phi)(x-y)a(y)\,\mathrm{d}y \\ &= \int_{Q(x_0,r)} \Biggl[(\partial^\alpha\Phi)(x-y) - \sum_{|\beta|\leqslant 2m-1} (\partial^{\alpha+\beta}\Phi)(x-x_0)\frac{(x_0-y)^\beta}{\beta!}\Biggr]a(y)\,\mathrm{d}y,\end{aligned}$$

where $\sum\limits_{|\beta|\leqslant 2m-1} (\partial^{\alpha+\beta}\Phi)(x-x_0)(x_0-y)^\beta/\beta!$ is the degree $2m-1$ Taylor polynomial of the function $y \to (\partial^\alpha\Phi)(x-y)$ about x_0. By the standard estimate of the remainder term of the Taylor expansion there exists ξ between y and x_0 such that

$$(\partial^\alpha\Phi)(x-y) - \sum_{|\beta|\leqslant 2m-1} (\partial^{\alpha+\beta}\Phi)(x-x_0)\frac{(x_0-y)^\beta}{\beta!} = \sum_{|\beta|=2m} (\partial^{\alpha+\beta}\Phi)(x-\xi)\frac{(x_0-y)^\beta}{\beta!}.$$

If $|x-x_0| \geqslant \sqrt{n}r$, it follows that $|x-\xi| \geqslant \frac{1}{2}|x-x_0|$ since $|x_0-\xi| \leqslant \frac{1}{2}\sqrt{n}r$. Then, taking into account that for $|\beta| = 2m$, $\partial^{\alpha+\beta}\Phi$ is a homogeneous function of degree $-n-\alpha$, we obtain, for $|x-x_0| \geqslant \sqrt{n}r$ and $y \in Q(x_0, r)$, that

$$(6)\qquad \Biggl|(\partial^\alpha\Phi)(x-y) - \sum_{|\beta|\leqslant 2m-1} (\partial^{\alpha+\beta}\Phi)(x-x_0)\frac{(x_0-y)^\beta}{\beta!}\Biggr| \leqslant Cr^{|\beta|}|x-x_0|^{-n-\alpha}.$$

Finally, from (5) and (6) we obtain, for $|x-x_0| \geqslant \sqrt{n}r$, that

$$\begin{aligned}|(\partial^\alpha b)(x)| &\leqslant Cr^{|\beta|}|x-x_0|^{-n-\alpha}\int_{Q(x_0,r)} |a(y)|\,\mathrm{d}y \\ &\leqslant Cr^{|\beta|}|x-x_0|^{-n-\alpha}\|a\|_{p_0}|Q|^{1-1/p_0} \\ &\leqslant Cr^{2m+n}[w(Q)]^{-1/p}|x-x_0|^{-n-\alpha}.\end{aligned}$$

□

The following pointwise inequality is crucial to obtain Theorem 21 below.

Proposition 20. *Let $a(\cdot)$ be a w-(p,p_0,d)-atom with $d=\max\{\lfloor n(q_w/p-1)\rfloor, 2m-1\}$ and assume that $Q=Q(x_0,r)$ is the cube containing the support of $a(\cdot)$ in the definition of w-(p,p_0,d)-atom. If $b(x)=\int_{\mathbb{R}^n}\Phi(x-y)a(y)\,\mathrm{d}y$, then for all $x\in\mathbb{R}^n$, all $0<\mu<2m$ and all $q>1$*

$$\begin{aligned}(7)\qquad N_{q,2m}(B;x)\lesssim[w(Q)]^{-1/p}[M(\chi_Q)(x)]^{\Upsilon/n}+\chi_{4\sqrt{n}Q}(x)M(a)(x)\\ +\chi_{4\sqrt{n}Q}(x)[M(M^q(a))(x)]^{1/q}+\chi_{4\sqrt{n}Q}(x)\sum_{|\alpha|=2m}T^*_\alpha(a)(x),\end{aligned}$$

where $\Upsilon=2m+n/q-\mu$, B is the class of b in E^q_{2m-1}, M is the Hardy-Littlewood maximal operator and $T^*_\alpha(a)(x)=\sup\limits_{\varepsilon>0}\bigl|\int_{|x-y|>\varepsilon}(\partial^\alpha\Phi)(x-y)a(y)\,\mathrm{d}y\bigr|$.

Proof. We point out that the argument used in the proof of Proposition 15, page 1021 in [26], to obtain the pointwise inequality (4) in [26], works in this setting as well, but considering now conditions (A1), (A2) and (A3) given in Definition 7 of w-(p,p_0,d)-atom. These conditions are similar to those of the atoms in variable context (see Definition 1.4, page 3669 in [20]). Then this observation and Lemma 19 allow us to get (7). $\square$

5. Main result

We observe that if $G\in\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)$, then $N_{q,2m}(G;x_0)<\infty$ for some $x_0\in\mathbb{R}^n$. By (i) in Lemma 12 there exists $g\in G$ such that $N_{q,2m}(G;x_0)=\eta_{q,2m}(g;x_0)$; from Proposition 15 it follows that $g\in\mathcal{S}'(\mathbb{R}^n)$. So $\Delta^m g$ is well defined in the sense of distributions. On the other hand, since any two representatives of G differ in a polynomial of degree at most $2m-1$, we get that $\Delta^m g$ is independent of the representative $g\in G$ chosen. Therefore, for $G\in\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)$, we define $\Delta^m G$ as the distribution $\Delta^m g$, where g is any representative of G.

5.1. Solution of the equation $\Delta^m F=f$ for $f\in H^p(\mathbb{R}^n,w)$. In this section, we shall prove that the space $\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)$ is the solution set of the equation

$$\Delta^m F=f,\quad\text{for } f\in H^p(\mathbb{R}^n,w).$$

This is contained in the following two results. In the sequel, given $0<\mu<2m$ we consider $\Upsilon=2m+n/q-\mu$.

Theorem 21. *Let $1<q<\infty$, $n(2m+n/q)^{-1}<p\leqslant1$ and let $0<\mu<2m$ be such that $n<\Upsilon p$. If $w\in\mathcal{A}_{(\Upsilon/n)p}$, then the iterated Laplace operator Δ^m is a* *surjective mapping from $\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)$ onto $H^p(\mathbb{R}^n,w)$. Moreover, there exist two positive constants C_1 and C_2 such that*

$$C_1\|G\|_{\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)} \leqslant \|\Delta^m G\|_{H^p(\mathbb{R}^n,w)} \leqslant C_2\|G\|_{\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)} \tag{8}$$

holds for all $G\in\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)$.

Proof. Let $G\in\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)$. Since $N_{q,2m}(G;x)$ is finite for a.e. $x\in\mathbb{R}^n$, by (i) in Lemma 12 and Proposition 15, the unique representative g of G (which depends on x) satisfying $\eta_{q,2m}(g;x)=N_{q,2m}(G;x)$ is a function in $L^q_{\rm loc}(\mathbb{R}^n)\cap\mathcal{S}'(\mathbb{R}^n)$. Thus, if $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $\int\varphi(x)\,\mathrm{d}x\neq 0$, from Lemma 6 in [12] we get

$$M_\varphi(\Delta^m G)(x)\leqslant Cp_{2m+n}(\varphi)N_{q,2m}(G;x).$$

Thus $\Delta^m G\in H^p(\mathbb{R}^n,w)$ and

$$\|\Delta^m G\|_{H^p(\mathbb{R}^n,w)}\leqslant C\|G\|_{\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)}. \tag{9}$$

Now we shall see that the operator Δ^m is onto. Given $w\in\mathcal{A}_{(\Upsilon/n)p}$ and $f\in H^p(\mathbb{R}^n,w)$, by Remark 10, there exist a sequence of non-negative numbers $\{\lambda_j\}_{j=1}^\infty$ and a sequence of cubes $\{Q_j\}_{j=1}^\infty$ and w-(p,p_0,d)-atoms a_j supported on Q_j, such that $f=\sum\limits_{j=1}^\infty\lambda_j a_j$ and

$$\sum_{j=1}^\infty\lambda_j^p\lesssim\|f\|^p_{H^p(\mathbb{R}^n,w)}. \tag{10}$$

For each $j\in\mathbb{N}$ we put $b_j(x)=\int_{\mathbb{R}^n}\Phi(x-y)a_j(y)\,\mathrm{d}y$. From Proposition 20 we have

$$\begin{aligned}N_{q,2m}(B_j;x)&\lesssim[w(Q_j)]^{-1/p}[M(\chi_{Q_j})(x)]^{\Upsilon/n}+\chi_{4\sqrt{n}Q_j}(x)M(a_j)(x)\\&\quad+\chi_{4\sqrt{n}Q_j}(x)[M(M^q(a_j))(x)]^{1/q}+\chi_{4\sqrt{n}Q_j}(x)\sum_{|\alpha|=2m}T^*_\alpha(a_j)(x).\end{aligned}$$

So

$$\begin{aligned}\sum_{j=1}^\infty\lambda_j N_{q,2m}(B_j;x)\lesssim{}&\sum_{j=1}^\infty\lambda_j\frac{[M(\chi_{Q_j})(x)]^{\Upsilon/n}}{[w(Q_j)]^{1/p}}\\&+\sum_{j=1}^\infty\lambda_j\chi_{4\sqrt{n}Q_j}(x)M(a_j)(x)\\&+\sum_{j=1}^\infty\lambda_j\chi_{4\sqrt{n}Q_j}(x)[M(M^q(a_j))(x)]^{1/q}\\&+\sum_{j=1}^\infty\lambda_j\chi_{4\sqrt{n}Q_j}(x)\sum_{|\alpha|=2m}T^*_\alpha(a_j)(x)\\={}&\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.\end{aligned}$$

To study I, by hypothesis, we have that $0 < p \leqslant 1$ and $\Upsilon p > n$. Then

$$\begin{aligned}\|\mathrm{I}\|_{L^p(\mathbb{R}^n,w)} &= \Big\| \sum_{j=1}^{\infty} \lambda_j [w(Q_j)]^{-1/p} [M(\chi_{Q_j})(\cdot)]^{\Upsilon/n} \Big\|_{L^p(\mathbb{R}^n,w)} \\ &= \Big\| \Big\{ \sum_{j=1}^{\infty} \lambda_j [w(Q_j)]^{-1/p} [M(\chi_{Q_j})(\cdot)]^{\Upsilon/n} \Big\}^{n/\Upsilon} \Big\|^{\Upsilon/n}_{L^{(\Upsilon/n)p}(\mathbb{R}^n,w)} \\ &\lesssim \Big\| \Big\{ \sum_{j=1}^{\infty} \lambda_j [w(Q_j)]^{-1/p} \chi_{Q_j}(\cdot) \Big\}^{n/\Upsilon} \Big\|^{\Upsilon/n}_{L^{(\Upsilon/n)p}(\mathbb{R}^n,w)} \\ &= \Big\| \sum_{j=1}^{\infty} \lambda_j [w(Q_j)]^{-1/p} \chi_{Q_j}(\cdot) \Big\|_{L^p(\mathbb{R}^n,w)} \lesssim \Big(\sum_{j=1}^{\infty} \lambda_j^p \Big)^{1/p} \lesssim \|f\|_{H^p(\mathbb{R}^n,w)},\end{aligned}$$

the first inequality follows from $w \in \mathcal{A}_{(\Upsilon/n)p}$ and Lemma 5, the condition $0 < p \leqslant 1$ gives the second inequality, and (10) gives the last one.

Next, we estimate $\|\mathrm{II}\|^p_{L(\mathbb{R}^n,w)}$. Since $0 < p \leqslant 1$, we have

$$\begin{aligned}\|\mathrm{II}\|^p_{L^p(\mathbb{R}^n,w)} &= \Big\| \sum_{j=1}^{\infty} \lambda_j \chi_{4\sqrt{n}Q_j}(\cdot) M(a_j)(\cdot) \Big\|^p_{L^p(\mathbb{R}^n,w)} \\ &\lesssim \sum_{j=1}^{\infty} \lambda_j^p \int_{\mathbb{R}^n} \chi_{4\sqrt{n}Q_j}(x) [M(a_j)(x)]^p w(x)\,\mathrm{d}x.\end{aligned}$$

Applying Holder's inequality with p_0/p, where $\max\{1, p(r_w/(r_w-1))\} < p_0 < \infty$, we get

$$\lesssim \sum_{j=1}^{\infty} \lambda_j^p \Big(\int_{4\sqrt{n}Q_j} [w(x)]^{(p_0/p)'}\,\mathrm{d}x \Big)^{1-p/p_0} \Big(\int_{\mathbb{R}^n} [M(a_j)(x)]^{p_0}\,\mathrm{d}x \Big)^{p/p_0}. \tag{11}$$

By Remark 8 we have that $w \in RH_{(p_0/p)'}$. Then (9) in Proposition 7.1.5 in [14], the boundedness of the maximal operator M on $L^{p_0}(\mathbb{R}^n)$, the condition (A2) of the atom $a_j(\cdot)$, and (10) allow us to obtain

$$\|\mathrm{II}\|_{L^p(\mathbb{R}^n,w)} \lesssim \Big(\sum_{j=1}^{\infty} \lambda_j^p \Big)^{1/p} \lesssim \|f\|_{H^p(\mathbb{R}^n,w)}.$$

To study III, we apply once again Hölder's inequality with p_0/p and obtain

$$\begin{aligned}&\|\mathrm{III}\|^p_{L^p(\mathbb{R}^n,w)} \\ &\quad\lesssim \sum_{j=1}^{\infty} \lambda_j^p \int_{\mathbb{R}^n} \chi_{4\sqrt{n}Q_j}(x) [M(M^q(a_j))(x)]^{p/q} w(x)\,\mathrm{d}x \\ &\quad\lesssim \sum_{j=1}^{\infty} \lambda_j^p \Big(\int_{4\sqrt{n}Q_j} [w(x)]^{(p_0/p)'}\,\mathrm{d}x \Big)^{1-p/p_0} \Big(\int_{\mathbb{R}^n} [M(M^q(a_j))(x)]^{p_0/q}\,\mathrm{d}x \Big)^{p/p_0}.\end{aligned}$$

Since we also can take $p_0 > q$, we have that the maximal operator M is bounded on $L^{p_0/q}(\mathbb{R}^n)$, and

$$\|\mathrm{III}\|^p_{L^p(\mathbb{R}^n,w)} \lesssim \sum_{j=1}^{\infty} \lambda_j^p \left(\int_{4\sqrt{n}Q_j} [w(x)]^{(p_0/p)'} \, \mathrm{d}x \right)^{1-p/p_0} \left(\int_{\mathbb{R}^n} [M(a_j)(x)]^{p_0} \, \mathrm{d}x \right)^{p/p_0}.$$

Applying the same reasoning as the one carried out after of (11) on the right-hand side hand of this inequality, one can conclude that

$$\|\mathrm{III}\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f\|_{H^p(\mathbb{R}^n,w)}.$$

Now, we study IV. By Lemma 18 above and Theorem 4 (c) in [28], we have that the operator T^*_α is bounded on $L^{p_0}(\mathbb{R}^n)$. Proceeding as in the estimate of II, we get

$$\|\mathrm{IV}\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f\|_{H^p(\mathbb{R}^n,w)}.$$

Thus, the weighted estimates of I, II, III and IV give

$$\left\| \sum_{j=1}^{\infty} \lambda_j N_{q,2m}(B_j; \cdot) \right\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f\|_{H^p(\mathbb{R}^n,w)}.$$

Hence

$$\sum_{j=1}^{\infty} \lambda_j N_{q,2m}(B_j; x) < \infty \quad \text{a.e. } x \in \mathbb{R}^n, \tag{12}$$

and

$$\left\| \sum_{j=M+1}^{\infty} \lambda_j N_{q,2m}(B_j; \cdot) \right\|_{L^p(\mathbb{R}^n,w)} \to 0 \quad \text{as } M \to \infty. \tag{13}$$

From (12) and Lemma 14 it follows that there exists a function G such that $\sum\limits_{j=1}^{\infty} k_j B_j = G$ in E^q_{2m-1} and

$$N_{q,2m}\left(\left(G - \sum_{j=1}^{M} k_j B_j \right); x \right) \leqslant C \sum_{j=M+1}^{\infty} k_j N_{q,2m}(B_j; x).$$

This pointwise estimate together with (13) imply

$$\left\| G - \sum_{j=1}^{M} k_j B_j \right\|_{\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)} \to 0 \quad \text{as } M \to \infty.$$

By Proposition 16, we have that $G \in \mathcal{H}^p_{q,2m}(\mathbb{R}^n, w)$ and $G = \sum\limits_{j=1}^{\infty} \lambda_j B_j$ in $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w)$. By (9) we have that Δ^m is a continuous operator from $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w)$

into $H^p(\mathbb{R}^n, w)$, then we get

$$\Delta^m G = \sum_j \lambda_j \Delta^m B_j = \sum_j \lambda_j a_j = f,$$

in $H^p(\mathbb{R}^n, w)$. This shows that Δ^m is onto $H^p(\mathbb{R}^n, w)$. Moreover,
(14)

$$\|G\|_{\mathcal{H}^p_{q,2m}(\mathbb{R}^n,w)} \lesssim \Big\|\sum_{j=1}^{\infty} k_j N_{q,2m}(B_j;\cdot)\Big\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f\|_{H^p(\mathbb{R}^n,w)} = \|\Delta^m G\|_{H^p(\mathbb{R}^n,w)}.$$

Finally, (9) and (14) give (8). This completes the proof. $\square$

The following theorem generalizes Theorem 1 given in [12].

Theorem 22. *Let* $w_a(x) = |x|^a$, $1 < q \leqslant r < \infty$, $n(2m + n/q)^{-1} < p \leqslant 1$, *and let* $0 < \mu < 2m$ *be such that* $n < \Upsilon p$. *If* $0 \leqslant a < \min\{np/r, n((\Upsilon/n)p - 1)\}$, *then the iterated Laplace operator* Δ^m *is a bijective mapping from* $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a)$ *onto* $H^p(\mathbb{R}^n, w_a)$, *and* (8) *holds with* $w = w_a$.

Proof. To establish the injectivity of Δ^m in $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a)$, we need to introduce the space $\mathcal{N}^{r,q}_{2m}$ (see page 564 in [3]). Let $1 < q \leqslant r < \infty$ and $m \geqslant 1$. We say that $g \in \mathcal{N}^{r,q}_{2m}$ if $g \in L^q_{\rm loc}(\mathbb{R}^n)$ and for a.e. $x \in \mathbb{R}^n$ there exists a polynomial $p_x(\cdot) \in \mathcal{P}_{2m-1}$ such that for a measurable set $\mathcal{O}$ with $|\mathcal{O}| < \infty$, $x \to \eta_{q,2m}((g + p_x)(\cdot); x) \in L^r(\mathbb{R}^n \setminus \mathcal{O})$. We observe that if $G \in E^q_{2m-1}$ and $N_{q,2m}(G;\cdot) \in L^r(\mathbb{R}^n \setminus \mathcal{O})$, where $1 < q \leqslant r < \infty$ and $|\mathcal{O}| < \infty$, then by (i) of Lemma 12 we have that $G \subset \mathcal{N}^{r,q}_{2m}$. Since $\Delta^m G = 0$ means that $\Delta^m g = 0$ for all $g \in G$, by Lemma 9 in [3], it suffices to show that for such G there exists a measurable set $\mathcal{O}$ such that $|\mathcal{O}| < \infty$ and $N_{q,2m}(G;\cdot) \in L^r(\mathbb{R}^n \setminus \mathcal{O})$. Given $G \in \mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a)$, let $\mathcal{O}$ be defined by

$$\mathcal{O} = \{x \in \mathbb{R}^n \colon\ [w_a(x)]^{1/p} N_{q,2m}(G;x) > 1\}.$$

Since $N_{q,2m}(G;\cdot) \in L^p(\mathbb{R}^n, w_a)$, it follows that $|\mathcal{O}| < \infty$. Now, for $r \geqslant q$, $0 \leqslant a < np/r$, and since $p \leqslant 1 < q$, we obtain

$$\begin{aligned}\int_{\mathcal{O}^c} [N_{q,2m}(G;x)]^r \,\mathrm{d}x &= \int_{\mathcal{O}^c} [N_{q,2m}(G;x)]^r [w_a(x)]^{r/p} [w_a(x)]^{-r/p} \,\mathrm{d}x \\ &\leqslant \int_{\mathcal{O}^c} [N_{q,2m}(G;x)]^p w_a(x) [w_a(x)]^{-r/p} \,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^n} [N_{q,2m}(G;x)]^p w_a(x) \,\mathrm{d}x + \int_{|x|\leqslant 1} [w_a(x)]^{-r/p} \,\mathrm{d}x < \infty.\end{aligned}$$

Thus, $G \subset \mathcal{N}^{r,q}_{2m}$ for all $G \in \mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a)$. In particular, this gives the injectivity of Δ^m in $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a)$.

To finish, we observe that $0 \leqslant a < \min\{np/r, n((\Upsilon/n)p - 1)\}$ implies that $w_a \in \mathcal{A}_{(\Upsilon/n)p}$ (see Example 7.1.7 in [14]). So, by applying Theorem 21 with $w = w_a$, we obtain the surjectivity of Δ^m onto $H^p(\mathbb{R}^n, w_a)$ and also (8). $\square$

Thus, Theorem 22 allows us to conclude that given $f \in H^p(\mathbb{R}^n, w_a)$, where $0 \leqslant a < \min\{np/r, n((\Upsilon/n)p - 1)\}$, the equation $\Delta^m F = f$ has a unique solution in $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w_a)$, namely: $F := (\Delta^m)^{-1} f$.

Remark 23. Theorem 22 does not hold in general for $\mathcal{H}^p_{q,\gamma}(\mathbb{R}, w)$ when γ is not a natural number. In [21], Ombrosi gave an example where Theorem 22 is not true for $\mathcal{H}^p_{q,\gamma}(\mathbb{R}, w)$ with $w \equiv 1$ and the operator $(\mathrm{d}/\mathrm{d}x)^\gamma$, when $0 < \gamma < 1$ and $(\gamma + 1/q)p > 1$.

Remark 24. An open question if there exists a weight w that not be a power weight such that Δ^m be injective in $\mathcal{H}^p_{q,2m}(\mathbb{R}^n, w)$. A more interesting question is to replace the iterated Laplace operator Δ^m for another and to obtain analogous results to those established in Theorems 21 and 22.

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