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A COMMON GENERALIZATION OF KELLEY'S THEOREM (CONCERNING MEASURES ON A BOOLEAN ALGEBRA) AND OF VON NEUMANN'S MINIMAX THEOREM

by

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Let $(X,+)$ be a commutative semigroup with 0, let ω : $: X \longrightarrow [0,\infty]$ be subadditive ($\omega(x+y) \leq \omega(x) + \omega(y)$) and monotone ($\omega(x) \leq \omega(x+y)$), and let A denote the family of all additive functionals $\xi : X \longrightarrow [0,\infty]$ ($\xi(x+y) = \xi(x) + \xi(y)$). For every subset $Y \subset X$ we define the number

$K_\omega(Y) = \inf\{\frac{1}{k}\,\omega(\sum_{i=1}^{k} y_i) : y_1,\ldots,y_k \in Y\}$, where y_i are not necessarily distinct; with the convention that $\inf \emptyset = \infty$.

Theorem. We have

$K_\omega(Y) = \sup\{\inf_{y\in Y} \xi(y) : A \ni \xi \leq \omega\}$, the supremum being attained.

Corollary 1. (i)$\Longleftrightarrow$(ii),

(i) there exists $\xi \in A$ with $\xi \leq \omega$ and $\xi(y) > 0$ for all $y \in Y$,

(ii) there are $Y_n \subset X$ with $Y \subset \bigcup_{n=1}^{\infty} Y_n$ and $K_\omega(Y_n) > 0$ for all $n = 1,2,\ldots$

Corollary 2. Let $(X,+,\cdot)$ be a semilinear space, and let ω be, moreover, positively homogeneous (i.e. $\omega(\alpha x) = \alpha\omega(x)$ for $x \in X$, $\alpha \in R^+$). Then

$$K_\omega(Y) = \inf \{\omega(y) : y \in \operatorname{conv} Y\} =$$
$$= \sup \{\inf_{y\in Y} \xi(y) : L \ni \xi \leq \omega\},$$ the supremum being attained.

Here L denotes the family of all $\xi \in A$ that are positively homogeneous (semilinear).

Corollary 3 (J.L. Kelley). Let $\mathcal{A}$ be a Boolean algebra, and for every non-empty subset $\mathcal{B} \subset \mathcal{A}$ define the number

$$I(\mathcal{B}) = \inf \left\{\frac{i(\xi)}{n(\xi)}\right\},$$ where ξ runs over all

finite collections of elements of $\mathcal{B}$ (not necessarily distinct), $n(\xi)$ is the number of elements of ξ, and $i(\xi)$ is the maximal number of elements of ξ with non-empty intersection. Then

$$I(\mathcal{B}) = \sup \{\inf_{B\in\mathcal{B}} m(B)\},$$ the supremum, taken over all finitely additive measures $m: \mathcal{A} \longrightarrow [0,1]$, being attained.

There exists a strictly positive ($m(A) > 0$ for $A \neq 0$) finitely additive measure m on $\mathcal{A}$ iff there are $\mathcal{B}_n \subset \mathcal{A}$ with $\mathcal{A} \setminus \{0\} \subset \bigcup_{n=1}^{\infty} \mathcal{B}$ and $I(\mathcal{B}_n) > 0$ for all $n = 1,2,\ldots$

Corollary 4 (J.von Neumann). Let S and T be arbitrary non-empty finite sets. Then for every function $h: S \times T \longrightarrow (-\infty, \infty]$ we have

$$\inf\Big\{\max_{s\in S}\sum_{t\in T} f(t)h(s,t) : f: T \longrightarrow [0,1],\ \sum_{t\in T} f(t) = 1\Big\} =$$

$$= \sup\Big\{\min_{t\in T}\sum_{s\in S} g(s)h(s,t) : g: S \longrightarrow [0,1],\ \sum_{s\in S} g(s) = 1\Big\},$$

the supremum and infimum being attained.