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A COMMON GENERALIZATION OF KELLEY'S THEOREM (CONCERNING
MEASURES ON A BOOLEAN ALGEBRA) AND OF VON NEUMANN'S

MINIMAX THEOREM

by

Marek WILHEIM

Let $(X, +)$ be a commutative semigroup with 0, let $\omega : X \rightarrow [0, \infty]$ be subadditive ($\omega(x + y) \leq \omega(x) + \omega(y)$) and monotone ($\omega(x) \leq \omega(x + y)$), and let Λ denote the family of all additive functionals $\xi : X \rightarrow [0, \infty]$ ($\xi(x + y) = \xi(x) + \xi(y)$). For every subset $Y \subset X$ we define the number

$$K_\omega(Y) = \inf \left\{ \frac{1}{k} \omega\left(\sum_{i=1}^k y_i\right) : y_1, \dots, y_k \in Y \right\}, \text{ where}$$

y_i are not necessarily distinct; with the convention that $\inf \emptyset = \infty$.

Theorem. We have

$$K_\omega(Y) = \sup \left\{ \inf_{y \in Y} \xi(y) : \Lambda \ni \xi \leq \omega \right\}, \text{ the supremum being attained.}$$

Corollary 1. (i) $\iff$ (ii),

(i) there exists $\xi \in \Lambda$ with $\xi \leq \omega$ and $\xi(y) > 0$ for all $y \in Y$,

(ii) there are $Y_n \subset X$ with $Y \subset \bigcup_{n=1}^{\infty} Y_n$ and $K_\omega(Y_n) > 0$ for all $n = 1, 2, \dots$

Corollary 2. Let $(X, +, \cdot)$ be a semilinear space, and let ω be, moreover, positively homogeneous (i.e. $\omega(\alpha x) = \alpha \omega(x)$ for $x \in X$, $\alpha \in \mathbb{R}^+$). Then

$$\begin{aligned} K_{\omega}(Y) &= \inf \{ \omega(y) : y \in \text{conv } Y \} = \\ &= \sup \{ \inf_{y \in Y} \xi(y) : L \ni \xi \leq \omega \} , \text{ the supremum being} \\ &\text{attained.} \end{aligned}$$

Here L denotes the family of all $\xi \in A$ that are positively homogeneous (semilinear).

Corollary 3 (J.L. Kelley). Let $\mathcal{A}$ be a Boolean algebra, and for every non-empty subset $\mathcal{B} \subset \mathcal{A}$ define the number

$$I(\mathcal{B}) = \inf \left\{ \frac{i(\xi)}{n(\xi)} \right\}, \text{ where } \xi \text{ runs over all}$$

finite collections of elements of $\mathcal{B}$ (not necessarily distinct), $n(\xi)$ is the number of elements of ξ , and $i(\xi)$ is the maximal number of elements of ξ with non-empty intersection. Then

$I(\mathcal{B}) = \sup \{ \inf_{B \in \mathcal{B}} m(B) \}$, the supremum, taken over all finitely additive measures $m: \mathcal{A} \rightarrow [0, 1]$, being attained.

There exists a strictly positive ($m(A) > 0$ for $A \neq \emptyset$) finitely additive measure m on $\mathcal{A}$ iff there are $\mathcal{B}_n \subset \mathcal{A}$ with $\mathcal{A} \setminus \{0\} \subset \bigcup_{n=1}^{\infty} \mathcal{B}_n$ and $I(\mathcal{B}_n) > 0$ for all $n = 1, 2, \dots$

Corollary 4 (J.von Neumann). Let S and T be arbitrary non-empty finite sets. Then for every function $h: S \times T \rightarrow (-\infty, \infty]$ we have

$$\inf \left\{ \max_{s \in S} \sum_{t \in T} f(t)h(s,t) : f: T \rightarrow [0,1], \sum_{t \in T} f(t) = 1 \right\} =$$

$$= \sup \left\{ \min_{t \in T} \sum_{s \in S} g(s)h(s,t) : g: S \rightarrow [0,1], \sum_{s \in S} g(s) = 1 \right\},$$

the supremum and infimum being attained.