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ON CHOQUET'S THEORY

Michael Neumann

We start with a refined version of a convergence principle due to Stephen Simons which admits interesting applications in Choquet's theory concerning arbitrary cones of upper semicontinuous functions on a compact Hausdorff space. Furthermore, in a certain sense our convergence theorem may be incorporated into this context. In view of various consequences it seems remarkable to point out that the theory presented here is not restricted to the Choquet boundary case, it works whenever certain boundary conditions are satisfied. The following survey contains no proofs. Further information as well as references may be taken from my paper "Varianten zum Konvergenzsatz von Simons und Anwendungen in der Choquettheorie", Arch.Math. 1977.

Let Q denote the set of all sequences $\sigma=(\sigma_n)_n$ with components $\sigma_n \stackrel{>}{=} o$ such that $\sigma_1 + \sigma_2 + \sigma_3 + \ldots = 1$, and let P consist of all those $\sigma \in Q$ which fulfil $\sigma_n = o$ for almost all $n=1,2,\ldots$. In the following X denotes a compact Hausdorff topological space. First we consider a sequence $f_n \in USC(X)$ of upper semicontinuous functions $f_n : X \to [-\infty, \infty[$ which is uniformly bounded above: $f_n \stackrel{<}{=} c < \infty$ for all $n=1,2,\ldots$. In this situation countable convex combinations are well defined and upper semicontinuous, for each $\sigma \in Q$ define $f^\sigma \in USC(X)$ by $f^\sigma(x) := \stackrel{\sim}{\Sigma} \sigma_n f_n(x) \in [-\infty,c]$ n=1

Next consider a convex cone TCUSC(X) containing the real constants. For $\lambda, \mu \in \text{Prob}(X)$ we write $\lambda \prec \mu$ iff $\lambda_*(f) \stackrel{\leq}{=} \mu_*(f)$ for all fcT. In this very general context there are two notions of maximality with respect to \prec : $\lambda \in \text{Prob}(X)$ is called maximal iff $\lambda \prec \mu$ implies $\mu \prec \lambda$, and

strictly maximal iff $\lambda \downarrow \mu$ even implies $\lambda = \mu$. Of course these notions coincide for example in the case of maximum-stable $T \subset C(X)$ which separates the points of X, in particular in the classical situation of Choquet's theory. Anyway, by Zorn's lemma every $\mu \in Prob(X)$ is dominated by a maximal $\lambda \in Prob(X)$. On the other hand, the Hahn-Banach theorem leads to a useful description which works only for strictly maximal measures: For $\lambda \in Prob(X)$ define $\hat{\lambda}(g) := \inf\{-\lambda_*(f) : f \in T, g \leq -f\}$ for all $g \in USC(X)$, then λ turns out to be strictly maximal iff $\lambda_* = \hat{\lambda}$ on USC(X). Further let us introduce the Choquet boundary $Ch(X,T) := \{a \in X : \delta_{a} \downarrow \mu \Rightarrow \sup \mu \subseteq [a]\}$ where $[a] := \{x \in X : f(x) = f(a) \text{ for all } f \in T\}$. Finally let us carefully enlarge the given cone:

$$\begin{split} & \text{Q(T)} := \{ \text{f}^{\sigma} \colon \text{ } \text{f}_{\text{n}} \in \text{T, } \text{ } \text{f}_{\text{n}} \stackrel{\leq}{=} \text{c} < \infty \text{ for all } \text{n} = 1, 2, \dots, \sigma \in \text{Q} \} \\ & \text{I(T)} := \{ \text{inf } \text{f}_{\text{n}} \colon \text{f}_{\text{n}} \in \text{T for n} = 1, 2, \dots, \text{f}_{\text{n}} \text{ pointwise } + \} \end{split}$$

Obviously we obtain convex cones such that $\mathbb{R}\subset \mathbb{T}\subset \mathbb{C}(\mathbb{T})\subset \mathbb{T}(\mathbb{T})$. Now trivial combination of our convergence principle and the definition of $\hat{\lambda}$ yield

2. Lemma. Let Y⊂X be a maximum-boundary for Q(T). And consider $g_n \in USC(X)$ such that $g_n \ge c>-\infty$ for all n=1,2,.... Then for each $\lambda \in Prob(X)$ we have inf liminf $g_n(Y) \le \liminf_{n\to\infty} \Lambda(g_n)$.

Applying this lemma to suitable characteristic functions one easily obtains the following generalization of the well known Choquet-Bishop-de Leeuw theorem.

- 3. Theorem. For a strictly maximal $\lambda \in \text{Prob}(X)$ and an F_{σ} -subset YCX we have λ (Y)=1, if Y is a maximum-boundary for Q(T), in particular if Ch(X,T)CY.
- 4. Special Case. Let T separate the points of X and consider an F_0 -subset Y=X. Then Ch(X,T)=Y iff Y is a maximum-boundary for Q(T).

This corollary extends for instance the theorem concerning the existence of the Silov boundary. As counterexamples even in the classical situation of Choquet's theory show our result cannot be improved very much. However, with some more effort we obtain the following extremely useful characterization of those F_{σ} -sets which are maximum-boundaries for Q(T).

- 5. Theorem. For an F_{σ} -subset YCX the following properties are equivalent:
 - i) [a] $\Pi Y \neq \emptyset$ for all a $\in Ch(X,T)$.
 - ii) Y is a maximum-boundary for Q(T).
- iii) Y is a supremum-boundary for I(T).
 - iv) For arbitrary $f_n \in T$, $f_n = c < \infty$ we have $\inf \max_{\sigma \in P} f^{\sigma} \le \sup_{\gamma \in Y} \limsup_{n \to \infty} f_n(\gamma)$.
 - v) For arbitrary $f_n \in T$, $f_n \leq c < \infty$ and all $\lambda \in Prob(X)$ we have limsup $\lambda_*(f_n) \leq \sup_{y \in Y} \lim_{n \to \infty} f_n(y)$.

Since in general situations the existence of suitable strictly maximal measures is rather dubious the following extension of theorem 3 is of interest.

<u>6. Theorem.</u> Consider an F_{σ} -subset Y=X which is a maximum-boundary for Q(T). Then for every $\mu \in \text{Prob}(X)$ there is a $\lambda \in \text{Prob}(X)$ such that $\mu \prec \lambda$ and $\lambda (Y) = 1$.

We conclude with an extended version of the Riesz representation theorem recently found by Benno Fuchssteiner. Our method of proving this theorem as well as the preceding one is based on our results 3 and 5 via a suitable state space embedding. The main idea is not very difficult and may find applications elsewhere.

- 7. Example. Consider an arbitrary nonvoid set S and a convex cone T of upper bounded functions $f:S\rightarrow [-\infty,\infty[$ such that T contains the real constants. The subsequent properties are equivalent:
 - i) T is a Dini cone: For all pointwise decreasing sequences $(f_n)_n$ in T we have inf $\sup_{n \in \mathbb{N}} f_n(s) = \sup_{s \in S} \inf_n f_n(s)$.
- ii) Each state on T admits an integral representation: For every additive and positive-homogeneous functional λ:T→[-∞,∞[such that λ(f) ≤ sup f for all f∈T there exists a probability measure τ on the σ-algebra generated on S by T such that λ(f)≤∫fdτ for all f∈T.