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PROBLEMS CONCERNING WEAK ASPLUND SPACES

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A Banach space E is a weak Asplund (WA) space provided every continuous convex function on E is Gateaux differentiable at each point of some dense G_δ subset of E. The central problem is to give a complete description of the class of spaces with this property. It is convenient to single out two related classes of spaces: Say that E is a Gateaux differentiability space (GDS) or has the Gateaux differentiability property (GDP) if each continuous convex function on E is Gateaux differentiable on a dense set (not necessarily a G_δ). We also say that E has the support function differentiability property (sfDP) if every support function on E is Gateaux differentiable on a dense set. [Call p a support function if there exists a bounded nonempty subset A of E^* such that $p(x) = \sup\{\langle x^*, x\rangle : x^* \in A\}$, $x \in E$; this is clearly convex, continuous and positive homogeneous.] There are obvious inclusions between these three classes of spaces. One reason for introducing the GDP is that we have been unable to resolve the following question.

Problem 1. Is the set of points of Gateaux differentiability of a continuous convex function necessarily a G_δ set? A Borel set? Universally measurable? What if it is assumed to be dense?

The reason for introducing the sfDP is that it can be characterized in a manner completely analogous to the known characterization of Asplund spaces. Recall that $x^* \in K \subseteq E^*$ is a weak* exposed point of K if there exists $x \neq 0$ in E such that

$$\langle x^*, x\rangle > \langle y^*, x\rangle \quad \text{whenever} \quad y^* \in K, \quad y^* \neq x^*.$$

Proposition 1. The Banach space E has the sfDP if and only if every weak* compact convex subset of E^* is the weak* closed convex hull of its weak* exposed points.

Problem 2. Does the sfDP imply the GDP?

One way of resolving this question would use the following result.

Proposition 2. Let R denote the real line. If E x R has the sfDP, then E has the GDP. (Analogously, if every support function on E x R is Gateaux differentiable on a dense G_δ set, then E is a weak Asplund space.)

Problem 3. If E has the sfDP, must E x R have the same property?

Problem 4. Are any of the above three differentiability properties preserved under finite products?

Proposition 3. Each of the following assertions is equivalent to the sfDP: (1) Every nonempty weak* compact convex subset of E^* has at least one weak* exposed point.

(2) Every support functional on E has at least one point of Gateaux differentiability.

The following result yields an interesting necessary condition for the sfDP.

Proposition 4. (Stegall, Larman) If every nonempty weak* compact convex subset K of E^* has at least one extreme point which is a G_δ point of K in the weak* topology (a weak* exposed point has this property), then every bounded sequence in E^* has a weak* convergent subsequence.

Problem 5. Is the conclusion to Proposition 4 a sufficent condition for E to have the sfDP?

The best sufficient condition to date is that due to Edgar Asplund [Acta Math. 1968]:

Proposition 5. If E is a subspace of a weakly compactly generated (WCG) space, then E is a WA space.
(This uses Asplund's theorem and the fact that a subspace of a space whose dual norm is strictly convex has the same property.)

Problem 6. If E is a Lindelöf space in its weak topology, is it a WA space?
(Recall that a subspace of a WCG space is weakly Lindelöf.)

Stegal [The RNP in conjugate Banach spaces, II, Trans. Amer. Math. Soc. (to appear)] has a simple proof that WCG spaces are WA spaces, using the Davis-Figiel-Johnson-Pełczyński factorization theorem for weakly compact operators, and the following result.

Proposition 6. If E is an Asplund space and if there exists a continuous linear map $T:E \to F$ having dense range, then F is a weak Asplund space.

It is easy to prove the following analogue.

Proposition 7. If $T:E \to F$ is bounded, linear and has dense range and if E has the GDP [sfDP], then F has the GDP [sfDP].

Recall Asplund's result:

Proposition 8. If T maps E linearly and continuously onto F, and if E is a WA space, then so is F.

A concrete question we have been unable to resolve is the following:

Problem 7. For which compact Hausdorff spaces X is the continuous function space $C(X)$ a WA space? When will $C(X)$ have the GDP or the sfDP?

A necessary condition is that every nonempty closed subset of X have a dense set of relative G_δ points. (This implies that X is sequentially compact.)

The following is a long-standing problem.

Problem 8. Is the existence of an equivalent norm on E which is Gateaux differentiable (at nonzero points) either necessary or sufficient for E to be a WA space?

The study of these spaces suffers from a lack of examples. The two which we list here are not very surprising.

Examples (1) If Γ is an infinite set, then there is a continuous seminorm on $\ell_\infty(\Gamma)$ which is nowhere Gateaux differentiable. (On ℓ_∞, use $p(x) = \limsup |x_n|$.)

(2) If Γ is uncountable, then the norm in $\ell_1(\Gamma)$ is nowhere Gateaux differentiable.

In regards to the first example, note that every weak* lower semicontinuous convex continuous function on $\ell_\infty(\Gamma)$ is Fréchet differentiable on a dense G_δ, since $\ell_1(\Gamma)$ has the RNP. [Collier, Pacific J. Math. 64 (1976)].

The following "simple" question is still open.

Problem 9. Does a subspace of a WA space or a GDS have the same property?

An affirmative answer to the next question would be surprising.

Problem 10. If E^* is a WA space or a GDS, must E have the same property?