Edward Grzegorek

On sets always of the first category

In: Zdeněk Frolík (ed.): Abstracta. 7th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1979. pp. 20–24.

Persistent URL: http://dml.cz/dmlcz/701141

ON SETS ALWAYS OF THE FIRST CATEGORY

by

E. GRZEGOREK

We obtain topological analogies of some results from [2] and [3] concerning measures. We work in ZFC set theory.

$|S|$ denotes the cardinality of a set S. If f is a function from a set S into a set T and $\mathcal{F}$ is a family of subsets of S then by $f(\mathcal{F})$ we denote the family $\{f(F): F\in\mathcal{F}\}$ of subsets of T. Let $\mathcal{C}$ be a separable σ-field on S (i.e. $\mathcal{C}$ is a countably additive algebra of subsets of S such that $\mathcal{C}$ is countably generated and $\{s\}\in\mathcal{C}$ for every $s\in S$). We will write $\mathcal{C}\in\mathcal{N}$ iff there is no continuous probability on $\mathcal{C}$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are σ-fields on S then $\sigma(\mathcal{C}_1, \mathcal{C}_2)$ denotes the σ-field on S generated by $\mathcal{C}_1\cup\mathcal{C}_2$. We will write $\mathcal{C}\in\mathcal{Q}$ iff there is no metrizable separable without isolated points topology $\mathcal{T}$ on S such that $\mathcal{B}(\mathcal{T})\supset\mathcal{C}$ and $S\notin I(\mathcal{T})$, where $\mathcal{B}(\mathcal{T})$ is the usual Borel σ-field on S w.r.t. (with respect to) $\mathcal{T}$ and $I(\mathcal{T})$ denotes the σ-ideal of the first category subsets of S w.r.t. $\mathcal{T}$. By $\mathcal{B}$ we will denote the usual Borel σ-field on R (= the real line) w.r.t. the usual topology on R.

REMARK. If in the above definition of the class $\mathcal{Q}$ we will replace $I(\mathcal{T})$ by $I^*(\mathcal{T})$, where $I^*(\mathcal{T})$ denotes the σ-ideal of subsets of S which are always of the first category w.r.t. $\mathcal{T}$, then we obtain the same class $\mathcal{Q}$. It can be also observed that if $X \subset S$, $\mathcal{C}$ is a separable σ-field on S and $\mathcal{C} \in \mathcal{Q}$ then $\mathcal{C} \cap X$ is a separable σ-field on X such that $\mathcal{C} \in \mathcal{Q}$.

Recall that a subset X of S is always of the first category w.r.t. metrizable separable topology $\mathcal{T}$ on S iff each dense in itself subset Y of X is of the first category on itself w.r.t. the topology on Y induced by $\mathcal{T}$.

It is known the following

THEOREM 1 ([3]). There exist a separable σ-field $\mathcal{C}$ on a set S such that $\mathcal{C} \notin \mathcal{N}$ and a permutation φ of S such that $\sigma(\mathcal{C}, \varphi(\mathcal{C})) \in \mathcal{N}$.
If all subsets of R of cardinality $< 2^{\aleph_0}$ are Lebesgue measurable then we can additionally have $S = R$ and $\mathcal{C} = \mathcal{B}$.

We give sketch of the proof of the following topological analogue of Theorem 1.

THEOREM 2. There exist a separable σ-field $\mathcal{C}$ on a set S such that $\mathcal{C} \notin \mathcal{Q}$ and a permutation φ of S such that $\sigma(\mathcal{C}, \varphi(\mathcal{C})) \in \mathcal{Q}$.
If all metrizable separable spaces without isolated points of cardinality $< 2^{\aleph_0}$ are of the first category then we can additionally have $S = R$ and $\mathcal{C} = \mathcal{B}$.

Proof. Our proof is a modification of Prikry's method from [4]. Let $m = \min\{n$: there exists a separable σ-field

$\mathcal{C}$ on a set T such that $|T| = n$ and $\mathcal{C} \notin Q\}$. Hence there exists a set T with $|T| = m$ and a metrizable separable without isolated points topology $\mathcal{T}_0$ on T such that $T \notin I(\mathcal{T}_0)$.

It can be easily observed the following

LEMMA. Let $\mathcal{T}$ be a metrizable separable without isolated points topology on a set T such that $|T| = m$. Then $T' \subset T$ and $|T'| < m$ imply $T' \in I(\mathcal{T})$.

Let $\{t_\xi : \xi \in m\}$ be an one-to-one enumeration of T. Let for every $\xi \in m$, F_ξ be an F_σ (w.r.t. $\mathcal{T}$) subset of T such that $F_\xi \in I(\mathcal{T})$ and $F_\xi \supset \{t_{\xi'} : \xi' \leq \xi\}$. Put $Z = \bigcup_{\xi \in m} \{t_\xi\} \times F_\xi$. We have $Z \subset m \times T$. It can be proved (compare [4], [1] and [2]) that there exists a separable σ-field $\mathcal{C}$ on m such that Z belongs to the product σ-field $\mathcal{C} \otimes \mathcal{B}(\mathcal{T})$. We claim $\mathcal{C} \in Q$. If not, let $\mathcal{T}_1$ be a metrizable separable without isolated points topology on m such that $\mathcal{B}(\mathcal{T}_1) \supset \mathcal{C}$ and $m \notin I(\mathcal{T}_1)$. Applying Kuratowski-Ulam category version of Fubini's theorem and our Lemma to $\mathcal{T}_0$, $\mathcal{T}_1$ and Z one can easily obtain a contradiction (compare [4] and [3]). So $\mathcal{C} \in Q$.

Let S be a set such that $|S| = m$ and let $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$ and $|S_1| = |S_2|$. Since $|S_1| = m$ and $|S_2| = m$ it follows that there exist a separable σ-field $\mathcal{C}_1$ on S_1 such that $\mathcal{C}_1 \in Q$ and a separable σ-field $\mathcal{C}_2$ on S_2 such that $\mathcal{C}_2 \notin Q$. Let $\mathcal{C}$ be the σ-field on S generated by $\mathcal{C}_1 \cup \mathcal{C}_2$. Since $\mathcal{C} \cap S_2 = \mathcal{C}_2$ and $\mathcal{C}_2 \notin Q$ we have, by Remark, $\mathcal{C} \notin Q$. Let φ be a permutation of S

such that $\varphi(S_1) = S_2$. It can be chacked that our $\mathcal{C}$ and φ have the required properties.

We omit the easy proof of the additional claim of Theorem 2.

If $\mathcal{J}$ is a σ-ideal on R then we put $\mathcal{J}^+ = \{A \subset R:$ for every $B \subset R$ such that there exists 1 – 1 $\mathcal{B}$-measurable function $f: B \longrightarrow A$ we have $B \in \mathcal{J}\}$.

Observe that $\mathcal{J}^+$ is a σ-ideal on R and $\mathcal{J}^+ \subset \mathcal{J}$.

Denote by $\mathcal{L}_0$ the σ-ideal of subsets of R of the Lebesgue measure zero, by I the σ-ideal of the first category subsets of R w.r.t. usual topology on R and by I^* the σ-ideal of subsets of R which are always of the first category w.r.t. usual topology on R.

Let $m_0 = \min\{n: \exists(Y \subset R)(|Y| = n \text{ and } Y \notin \mathcal{L}_0\}$,

$m_1 = \min\{n: \exists(Y \subset R)(|Y| = n \text{ and } Y \notin I\}$, and

$m_2 = \min\{n: \exists(Y \subset R)(|Y| = n \text{ and } Y \notin I^*)\}$.

We omit the proof of the following

THEOREM 3. There exist $A_i \subset R$ $(i \leqslant 3)$ such that $|A_i| = m_i$ $(i \leqslant 2)$, $|A_3| = \min\{m_0, m_1\}$, $A_0 \in \mathcal{L}_0^+$, $A_1 \in I^+$, $A_2 \in (I^*)^+$ and $A_3 \in \mathcal{L}_0^+ \cap I^+$.

If is known that the σ-ideal $\mathcal{L}_0^+$ is equal to the σ-ideal of so called universal null subsets of R [5].

The part of Theorem 3 concerning A_0 was proved in fact in [2].

COROLLARY. There exists $A, B \subset R$ such that $|A| = |B|$, $A \in I^*$ and $B \notin I^*$.

A similar result for universal null subsets of R can be found in [2]. In connection with I^* it is worth to mention that Morgan II has proved [3a] that there exists a linear set every homeomorphic image of which is in I, but which is not in I^*.

REFERENCES

[1] R.H. Bing, W.W. Bledsoe, and R.D. Mauldin, Sets generated by rectangles, Pacific J. Math. 51(1974), 27-36.

[2] E. Grzegorek, Solution of a problem of Banach on σ-fields without continuous measures, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., to appear.

[3] E. Grzegorek and C. Ryll-Nardzewski, On universal null sets, preprint 1978, submitted for publication in Proc. Amer. Math. Soc..

[4] K. Prikry, On images of the Lebesgue measure, I, preprint 1977.

[5] E. Szpilrajn-Marczewski, Sur les ensembles et les fonctions absolument measurables, Compt. rend. Soc. Sci. Lett. Varsovie, Cl. III, 30(1937) (Polish). An English translation, by John C. Morgan II, of this article is available.

[3a] John C. Morgan II, On sets every homeomorphic image of which has the Baire property, preprint.