K.-D. Kürsten

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K.D. Kürsten, Leipzig

If f is an almost-periodic (a.p.) function, then for $s \in \mathbb{R}$ there exists the mean value

$$a(s) = \lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f(t)\, e^{-ist}\, dt .$$

We denote by $S(f) = \{s;\ a(s) \neq 0\}$ the spectrum of f. A Banach space valued function $F: \mathbb{R} \to X$ is called weakly a.p. if for every $\varphi \in X^*$ $\varphi \circ F$ is a.p. The spectrum of F is the union $\bigcup S(\varphi \circ F)$ where φ runs over X^*. The following theorem was proved by M.I. Kadec and K.D. Kürsten /2/.

Theorem: The spectrum of every Banach space valued weakly a.p. function is countable.

Let us consider the space AP of a.p. functions as a subspace of $L_\infty(\mathbb{R}, dt)$.

Lemma 1: A subset $M \subset AP$ is norm separable iff the union $\bigcup S(f)$ where f runs over M is countable.

This follows immediately from well known properties of a.p. function

Lemma 2: Every $\sigma(L_\infty, L_1)$-compact convex subset of AP is norm separable.

Sketch of proof: If $M \subset AP$ is convex, w^*-compact and nonseparable, Then using methods of /5/ one obtains a subset $\Delta \subset M$ such that every norm separable subset of Δ is countable and such that (Δ, w^*) is homeomorphic to $\{0,1\}^N$. Transforming the Haar measure of $\{0,1\}^N$ we obtain a measure m on Δ. The set of a.p. functions $\{w^*\text{-}\int g(f) f\, dm(f);\ g \in L_1(m)\}$ is norm separable and it follows from Bochner's approximation theorem (see /3/) that this set is contained in the image of a separable norm one projection in AP. This Projection P can be given as a limit of a double sequence of w^*-continuous operators and this allows us to show, that for m-almost all $f \in \Delta$ $Pf=f$, what is impossible.

Proof of theorem: Given a weakly a.p. function $F: \mathbb{R} \to X$. We consider the operator B defined by

$$L_1(\mathbb{R}) \ni h \to B(h) = \text{Pettis-}\int F(t)\, h(t)\, dt \in X.$$

Then $B^* \varphi = \varphi \circ F \in AP$. By Lemma 2 B^* has separable range and the theorem follows from lemma 1.

Let us give some examples to the following question, connected with lemma 2: For which Banach spaces X and subspaces $Y \subset X^*$ every w^*-compact (convex) subset of Y is norm separable?

1.) Let $l_1[0,1] \subset C[0,1]^*$ be the closed linear hull of point - measures. Then every w^*-compact convex subset of $l_1[0,1]$ is separable.
2.) V.I.Rybakov /4/ proved (using some special set - theoretic constructions) that there exist a Banach space C(K) and an uncountable set Λ such that $l_1(\Lambda) \subset C(K)^*$ and such that every w^*-compact subset of $l_1(\Lambda)$ is separable. He also proved, that in such situation the identity map $(l_1(\Lambda), w^*) \to (l_1(\Lambda), \|\cdot\|)$ is universally measurable.
3.) Using some modifications of methods of /5/ it can be proved, that there is a subset $\Gamma \subset [0,1]$ of cardinality continuum such that every $\sigma(l_1(\Gamma), C[0,1])$-compact subset of $l_1(\Gamma)$ is norm - separable.

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