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MODEL THEORETIC APPROACH TO TOPOLOGICAL FUNCTORS, II.

by

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This paper is a sequel of [6]. Most of results here presented will appear in the forthcoming author's paper [7].

Under a concrete category $(\mathcal{A},U)$ we will mean a category $\mathcal{A}$ equipped with a faithful functor $U: \mathcal{A} \longrightarrow$ Set satisfying the following two conditions:

(1) If $A \in \mathcal{A}$, X is a set and $f: UA \longrightarrow X$ a bijection, then there is $B \in \mathcal{A}$ and an isomorphism $g: A \longrightarrow B$ such that $Ug = f$

(2) If $A \in \mathcal{A}$ and $f: A \longrightarrow A$ is an isomorphism such that Uf is the identity, then f is the identity.

Under a functor $F: (\mathcal{A},U) \longrightarrow (\mathcal{B},V)$ between concrete categories we will mean a functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ such that $V.F = U$.

A type is given by a class of function symbols and a class of relation symbols. Their arities are arbitrary cardinals. The infinitary first-order language $L_{\infty,\infty}(\tau)$ of type τ includes a proper class V of variables and besides the usual logical symbols it admits infinitary conjunctions, disjunctions and quantifiers. A class of sentences of $L_{\infty\infty}(\tau)$ is called a theory of type τ. We denote by $(\mathcal{A}_\tau,U_\tau)$ or $(\mathcal{A}_T,U_T)$ the concrete category of all τ-structures or T-models resp. These categories need not be legitimate, i.e. they need not form a class. A theory having a representative set of n-ary atomic formulas for each cardinal n will be called normal. If T is normal, then $(\mathcal{A}_T,U_T)$ is a legitimate category and even it is strongly fibre-small in the sense of [1].

If $(\mathcal{A},U)$ is a concrete category and n a cardinal, then U^n will denote the functor Set(n,U-). Subfunctors of U^n will be called n-ary relation symbols interpretable in $(\mathcal{A},U)$ and natural transformations

$U^n \longrightarrow U$ n-ary <u>function symbols interpretable in $(\mathcal{A}, U)$</u>. It is motivated by the fact that any relation or function symbol of type τ determines a subfunctor of U^n or a natural transformation $U^n \longrightarrow U$ resp. Let ϱ_U be the collection of all relation and function symbols interpretable in $(\mathcal{A}, U)$. We emphasize that ϱ_U need not be a type because it need not be a class.

Let $\sigma \subseteq \varrho_U$ be a type. There is a functor $G_\sigma: (\mathcal{A}, U) \longrightarrow (\mathcal{A}_\sigma, U_\sigma)$ such that if $A \in \mathcal{A}$, then the σ-structure $G_\sigma(A)$ has the underlying set UA, the n-ary relation on UA corresponding to $R \in Rel_n(\sigma)$ equals to $R(A)$ and the n-ary function $f: (UA)^n \longrightarrow UA$ corresponding to $f \in Fnt_n(\sigma)$ is the component f_A of the natural transformation f. Let T_σ be the theory of type σ consisting of all sentences which hold in all σ-structures $G_\sigma(A)$ for $A \in \mathcal{A}$. Clearly we get the functor $G_\sigma: (\mathcal{A}, U) \longrightarrow (\mathcal{A}_{T_\sigma}, U_{T_\sigma})$.

We may restrict ourselves in the formation of T_σ to some specified kind of sentences. This yields a general method of getting suitable completions or hulls of $(\mathcal{A}, U)$. E.g. (with size conditions aside), if σ consists of all function symbols from ϱ_U and $T \subseteq T_\sigma$ of all atomic sentences, then T is the Linton's equational theory of U and $G: (\mathcal{A}, U) \longrightarrow (\mathcal{A}_T, U_T)$ is the equational completion of $(\mathcal{A}, U)$ (see [5]).

If $A \in \mathcal{A}$, then $R_A(X) = \{Uf \,/\, f: A \longrightarrow X\}$ defines a subfunctor R_A of U^{UA}. Let $\tau_U \subseteq \varrho_U$ be the type consisting of R_A where A carries over mutually non-isomorphic objects $A \in \mathcal{A}$ such that UA is a cardinal. Then G_{τ_U} is a full embedding and it is important that whenever $(\mathcal{A}, U)$ is strongly fibre-small, then T_{τ_U} is normal and $(\mathcal{A}, U)$ isomorphic to $(\mathcal{A}_{T_{\tau_U}}, U_{T_{\tau_U}})$.

Further, if T consist of all universal Horn sentences without equality (their specification follows) from T_{τ_U}, then $(\mathcal{A}_T, U_T)$ is the Mac Neille completion of $(\mathcal{A}, U)$ (in the sense of Herrlich [3]). It proves the conjecture from [6].

Theorem: A concrete category ($\mathcal{A}$,U) is (absolutely) topological iff it is isomorphic to the category of models of a relational normal universal Horn theory T without equality of some type τ .

Relational means that τ contains relation symbols only and universal Horn theory without equality consists of sentences bearing this name, i.e. arising from formulas $\bigwedge_{i \in I} R_i(x_i) \longrightarrow R(x)$, where $R_i \in Rel_{n_i}(\tau)$, $R \in Rel_n(\tau)$, $x_i \in V^{n_i}$ and $x \in V^n$, by universal quantification of all their variables.

Similarly, using τ_U and a suitable kind of sentences one can treat (epi-monosource)-topological categories (in the sense of [4]) or semi-topological categories (in the sense of [8]). In either case we get a completion playing the role of Mac Neilles one in the event of topological categories.

A relational theory T of type τ will be called <u>reflexive</u> if for any relation symbol R of τ $T \models (\forall x)R(x,x,\dots x,\dots)$ holds where $x \in V$. It is <u>transitive</u> if for any cardinal n and any $R \in Rel_n(\tau)$ $T \models (\forall x)[(\bigwedge_{i \in n} R(x_{i,1},x_{i,2},\dots x_{i,j},\dots) \wedge \bigwedge_{j \in n} R(x_{1,j},x_{2,j},\dots x_{i,j},\dots)) \longrightarrow R(x_{1,1},x_{2,2},\dots x_{i,i},\dots)]$ holds where $x = (x_{i,j}) \in V^{n \times n}$. Motivating is the case of a binary relation symbol R.

Proposition: Let T be a relational, normal, reflexive and transitive universal Horn theory without equality. Then ($\mathcal{A}_T$,U_T) is a cartesian closed topological category.

The author conjectures that this proposition can be converted. Namely, one is tempted to seek for a type $\sigma \leq \tau_U$ such that ($\mathcal{A}_\sigma$,U_σ) is (in general non-legitimate) cartesian closed topological hull of ($\mathcal{A}$,U) and its legitimacy corresponds to strict fibre-smalness of ($\mathcal{A}$,U) in the sense of Adámek and Koubek [2] (i.e. model theoretically recover their theorem).

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