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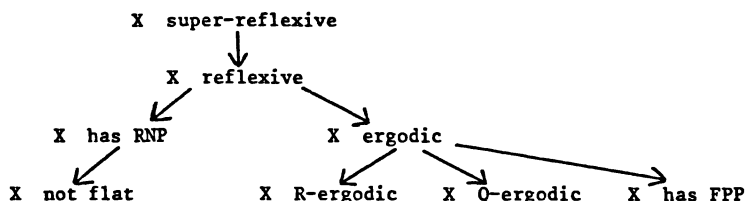
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On flatness and some ergodic super-properties of Banach spaces

A.J. Pach

Let X be a Banach space. The following implications are well-known:



Recall that X is called super-P iff, for each Banach space Y , $Y \triangleleft X$ implies that Y is P, where $Y \triangleleft X$ means that for each finite-dimensional subspace F of Y and for each $\varepsilon > 0$ there is a subspace F' of X with $d(F, F') < 1 + \varepsilon$.

The following results were known already:

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- (1) X is super-reflexive iff X is super-non-flat. [3]
 - (2) X is super-reflexive iff X is super-R-ergodic. [1]
 - (3) X is super-reflexive iff X is super-Q-ergodic. [2]
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Now we unify these results:

Theorem 1 [5]. A Banach space X is not super-reflexive iff there is a Banach space Y with $Y \triangleleft X$, Y completely flat, Y not R-ergodic, Y not Q-ergodic, and Y fails to have the FPP.

Let us first define some of the used concepts.

Definition 2. A Banach space X is ergodic (for isometries) iff for every isometry $T : X \rightarrow X$ and for every $x \in X$, the sequence $\left(\frac{1}{n} \sum_{i=1}^n T^{i-1} x \right)_{n=1}^{\infty}$ converges.

Definition 3. A real infinite matrix $(\rho_{i,j})_{i,j=1}^{\infty}$ is an R-matrix iff

$$(4) \quad \sum_{j=1}^{\infty} \rho_{i,j} \not\rightarrow 0 \text{ if } i \rightarrow \infty \text{ and}$$

$$(5) \quad \lim_{i \rightarrow \infty} \rho_{i,j} = 0 \text{ for each } j \in \mathbb{N}$$

Condition (4) means that $\sum_{j=1}^{\infty} \rho_{i,j}$ exists for each $i \in \mathbb{N}$, and the sequence $\left(\sum_{j=1}^{\infty} \rho_{i,j}\right)_{i=1}^{\infty}$ diverges or converges to a limit $\neq 0$.

Definition 4. A Banach space X is R-ergodic (for isometries) iff for each isometry $T : X \rightarrow X$ and for each $x \in X$, there is an R-matrix $(\rho_{i,j})_{i,j=1}^{\infty}$ such that $\left(\sum_{j=1}^{\infty} \rho_{i,j} T^{j-1} x\right)_{i=1}^{\infty}$ converges weakly.

To avoid too much technicalities we skip the definition of Q-ergodicity (see [2], [5]).

Definition 5. A Banach space X has the FPP (fixed point property) (for isometries) iff for each isometry $T : X \rightarrow X$ and for each closed bounded convex $K \subset X$ with $TK \subset K$, there is an $x \in K$ with $Tx = x$.

Definition 6. A Banach space X is flat if there is a function $g : [0,1] \rightarrow \{x \in X : \|x\| = 1\}$ (called girth curve) with $g(0) = -g(1)$ and g is Lipschitz continuous with constant 2.

If $X = \overline{\text{span}} \{g(t) : 0 \leq t \leq 1\}$ then X is called completely flat.

Example 7. $L^1[0,1]$ is completely flat. The function $j : t \mapsto -x_{[0,t]} + x_{[t,1]}$ is a girth curve.

The importance of this example is demonstrated by the following

Theorem 8 [4]. A Banach space X is completely flat iff X is (isometric to) the completion of $L^1[0,1]$ for a norm $\|\cdot\|_X$ with for all $f \in L^1[0,1]$

$$(6) \quad \|f\|_{L^1[0,1]} \geq \|f\|_X \geq \sup \{|\langle j^*(t), f \rangle| : 0 \leq t \leq 1\},$$

(where $j^*(t) = -x_{[0,t]} + x_{[t,1]} \in L^{\infty}[0,1]$), and then the function j (as in Example 7) is a spanning girth curve.

Now we'll give an idea of the proof of Theorem 1.

First note that $L^1[0,1]$ has almost all the properties that Y in Theorem 1 should have. Indeed, $L^1[0,1]$ is completely flat, and the isometry $T : L^1[0,1] \rightarrow L^1[0,1]$ defined by

$$(7) \quad T f(t) = \begin{cases} 2f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

can be used to show that $L^1[0,1]$ is not R-ergodic, not Q-ergodic, and fails to have the FPP.

E.g., if $L^1[0,1]$ would be R-ergodic, there was an R-matrix $(\rho_{i,j})_{i,j=1}^{\infty}$ and an $f_0 \in L^1[0,1]$ with $w\text{-}\lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} \rho_{i,j} T^{j-1} \chi_{[0,1]} = f_0$.

Then $\langle j^*(0), f_0 \rangle = \lim_{i \rightarrow \infty} \langle j^*(0), \sum_{j=1}^{\infty} \rho_{i,j} T^{j-1} \chi_{[0,1]} \rangle = \lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} \rho_{i,j}$.

But also $\langle j^*(0), f_0 \rangle = \lim_{n \rightarrow \infty} \langle j^*(2^{-n}), f_0 \rangle$, and for fixed n a simple calculation, using (5), shows that $\langle j^*(2^{-n}), f_0 \rangle = - \lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} \rho_{i,j}$, a contradiction by (4).

Also, if $f_0 \in K = \overline{\text{co}} \{2^n \chi_{[0, 2^{-n}]} : n = 0, 1, 2, \dots\} \subset L^1[0,1]$ is arbitrary, then for any $\varepsilon > 0$ there is a convex combination f of $2^n \chi_{[0, 2^{-n}]}$ ($n = 0, 1, 2, \dots$) with $\|f_0 - f\| < \varepsilon$, and a $\delta > 0$ with

$\langle j^*(\delta), f \rangle > 1 - \varepsilon$. But for $2^{-m} < \delta$ we have $\langle j^*(\delta), T^m f \rangle = -1$,

so $\|f_0 - T^m f_0\| > 1 - 3\varepsilon$, and therefore $f_0 \neq T f_0$.

So $L^1[0,1]$ doesn't have the FPP.

Now let Y be an arbitrary completely flat space, and identify Y with a completion of $L^1[0,1]$ according to Theorem 8. If T as defined by (7) can be extended to an isometry on Y , exactly the same reasoning as for $L^1[0,1]$ gives that Y is not R-ergodic, not Q-ergodic, and fails to have the FPP.

So for the proof of Theorem 1, if X is not super-reflexive, we want to find a Y as above, with $Y \prec X$. Note that, by (1), we may assume that X is completely flat. (If $Y \prec Z_1 \subset Z \prec X$, with Z_1 completely flat, then $Y \prec X$.)

To construct Y , we'll make a new norm $\|\cdot\|_0$ on the subspace Y_0 of $L^1[0,1]$ defined by $Y_0 = \text{span}_{n=1}^{\infty} L^1[2^{-n}, 2^{-n+1}]$ (We consider $L^1[2^{-n}, 2^{-n+1}]$

as the subspace of $L^1[0,1]$ consisting of the functions with value 0 a.e. outside of $[2^{-n}, 2^{-n+1}]$), and then take for Y the completion of $(Y_0, \|\cdot\|_0)$.

Take $y \in Y_0$. Then there are $n_0 \in \mathbb{N}$ and $y_n \in L^1[2^{-n}, 2^{-n+1}]$ ($n = 1, \dots, n_0$) with $y = \sum_{n=1}^{n_0} y_n$. If $S = \{s_1, \dots, s_{n_0}\}$ is a subset of $\mathbb{N}$ with $s_1 < s_2 < \dots < s_{n_0}$, define $y_S = \sum_{n=1}^{n_0} T_{S,n} y_n$, with $T_{S,n}$ the natural isometry from $L^1[2^{-n}, 2^{-n+1}]$ onto $L^1[2^{-s_n}, 2^{-s_n+1}]$. Now the following holds:

$$(8) \quad \begin{cases} \text{There is a subsequence } N_0 \text{ of } \mathbb{N} \text{ such that for all} \\ y = \sum_{n=1}^{n_0} y_n \in Y_0 \text{ there is a number } \|y\|_0 \in \mathbb{R} \text{ such that for all} \\ \varepsilon > 0 \text{ there is an } n(\varepsilon) \in \mathbb{N} \text{ with } \left| \|y\|_0 - \|y_S\|_X \right| < \varepsilon \text{ when-} \\ \text{ever } S = \{s_1, \dots, s_{n_0}\} \subset N \text{ and } n(\varepsilon) < s_1 < \dots < s_{n_0}. \end{cases}$$

(For the proof of (8), for a fixed $y \in Y_0$ apply Ramsey's theorem countably many times and use a diagonal procedure, and then repeat this for all y in a countable $L^1[0,1]$ -dense subset of Y_0 , and again diagonalize.)

With y and S as above, it is easy to see, using (6), that $\|y_S\|_X \leq \|y\|_0$ and that for any t there is a t_S with

$$| \langle j^*(t), y \rangle | = | \langle j^*(t_S), y_S \rangle | \leq \|y_S\|_X, \text{ so}$$

$$(9) \quad \|y\|_{L^1[0,1]} \geq \|y\|_0 \geq \sup\{ | \langle j^*(t), y \rangle | : 0 \leq t \leq 1 \} \quad (y \in Y_0).$$

Now it is not hard to see that $\|\cdot\|_0$ is a norm on Y_0 , that the completion Y of $(Y_0, \|\cdot\|_0)$ is completely flat by Theorem 8, and that T as defined by (7) can be extended to an isometry on Y .

To complete the proof of Theorem 1, we have to show that

$$(10) \quad Y \prec X.$$

So let F be a finite-dimensional subspace of Y , and take $\varepsilon > 0$. Then there is a subspace F_0 of Y_0 with $d(F, F_0) < 1 + \varepsilon$. Take a finite ε -net for $\|\cdot\|_{L^1[0,1]}$ in $\{y \in F_0 : \|y\|_0 = 1\}$. Then there is an $n(\varepsilon)$ satisfying (8) for all y in this ε -net, and we can take

$S \subset \mathbb{N} \setminus \{1, \dots, n(\varepsilon)\}$ such that with $F_1 = \{y_S : y \in F_0\} \subset X$ we get

$$d(F_0, F_1) < \frac{1 + 2\varepsilon}{1 - 2\varepsilon}.$$

Remark 9. If we replace the requirement 'Y completely flat' in Theorem 1 by 'Y flat', then we can add 'Y doesn't have the Krein-Milman Property'. Indeed, take Y^{**} instead of Y. Then Y^{**} is flat (but not completely flat), so it doesn't have the Radon-Nikodym Property, and as a dual space it doesn't have the KMP.

Question 10. If X is flat, does there exist a Y with $Y \prec X$, Y completely flat, Y doesn't have the KMP?

Or, does flatness already exclude the KMP?

Question 11. Does flatness already exclude R-ergodicity and/or Q-ergodicity and/or the FPP, in other words, does Theorem 1 say anything more than result (1)?

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