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## RAPID ULTRAFILTER NEED NOT BE Q-POINT

## L. Bukovský and E. Copláková

Clasifying points in  $\beta\omega$  mathematicians have introduced several types of ultrafilters on the set  $\omega$  of natural numbers. Some of them are interesting also from other point of view. In this note we shall consider two important types of ultrafilters: rapid and Q-point. Both notions are applicable to filters too.

The existence of rapid ultrafilters and Q-points is undecidable in the set theory. The continuum hypothesis implies the existence of Q-points. Every Q-point is a rapid filter [1]. There exists a model of ZFC in which there is no rapid ultrafilter [5]. Of course, there exists an ultrafilter which is not rapid [1]. This is proved by observing that every rapid filter  $\mathcal U$  has the following property (C): if  $\left\{a_n\right\}_{n=0}^\infty$  is a sequence of positive reals converging to zero then there exists a set  $A \in \mathcal U$  such that  $\sum_{n \in A} a_n < +\infty$ .

This property is characteristic for rapid filters. Proposition. If a filter  $\mathcal U$  possesses the property (C) then  $\mathcal U$  is rapid.

Probably this proposition is known. We did not find it in literature, therefore we present a simple proof of it.

The main goal of this note is the following  $\underline{\text{Theorem A}}$ . Let M be a transitive model of ZFC. Then there exists a generic extension N of M such that

- a) cardinals of N are those of M:
- b)  $(2^{\Re_0})^{M} = (2^{\Re_0})^{N};$
- o) in N there exists a rapid filter  ${\mathcal V}$  such that  ${\mathcal V}$  is not a Q-point.

Moreover we can assume that V is a P-point (and ultrafilter).

By slight modifications of the forcing construction used to prove the theorem A we shall obtain a proof of Theorem B. Assume that there exists a dominating family  $\mathcal{F} \subseteq \omega$ 

of cardinality  $\lambda$ . If MA, holds true for every  $\kappa < \lambda$  then there exists a rapid filter V such that V is not a Q-point. Moreover we can assume that V is an ultrafilter.

§1. Preliminaries. If A is an infinite set of integers then  $\overline{A}$  is the counting function of A, i.e.  $\overline{A}$  is the unique strictly increasing function from  $\omega$  onto A. A family of functions  $\mathfrak{F}\subseteq^\omega\omega$  is dominating iff for every  $f\in^\omega\omega$  there exists a function  $g\in \mathfrak{F}$  and a kew such that  $g(n) \geq f(n)$  for every  $n \geq k$ . A filter  $\mathcal{U}$  on  $\omega$  is rapid iff the family  $\left\{\overline{A}; A \in \mathcal{U}\right\}$  is dominating. Evidently, any extension of a rapid filter is rapid. A filter  $\mathcal{U}$  is a Q-point iff for any partition  $\Omega = \left\{A_n; n \in \omega\right\}$  of  $\omega$ ,  $A_n$  being finite, there exists a set  $A \in \mathcal{U}$  such that  $|A \cap A_n| \leq 1$  for every  $n \in \omega$ . The set A is called a selector for the partition  $\Omega$ .

For the rest of the paper we fix a partition  $\mathbb{R} = \left\{ R_n; n \in \omega \right\}$  of  $\omega$  such that  $\left| R_n \right| = n$ . E.g. we set  $R_n = \left\{ \frac{1}{2} \ n(n-1), \ldots, \frac{1}{2} \ n(n-1) + n-1 \right\}$ . A set  $A \subseteq \omega$  will be called growing iff for every  $n \in \omega$  there exists a  $k \in \omega$  such that  $\left| A \cap R_k \right| \ge n$ . We denote

 $\mathcal{S} = \{ A \subseteq \omega ; \exists n \forall k | R_k - A | \leq n \}.$ 

Evidently,  $\mathcal G$  is a filter on  $\omega$ . If A is a selector for  $\mathbb R$  then  $\omega - A \in \mathcal G$ . A set  $A \subseteq \omega$  is growing if and only if  $\omega - A \notin \mathcal G$ . Moreover if a filter  $\mathcal V$  contains  $\mathcal G$  as a subset then  $\mathcal V$  is not a  $\mathbb Q$ -point and each element of  $\mathcal V$  is a growing set.

Lemma 1. Let  $A \subseteq \omega$  be growing,  $f \in {}^{\omega}\omega$ . Then there exists a growing set  $B \subseteq A$  such that  $\overline{B} > f$ .

PROOF. Let  $b_0 \in A$ ,  $b_0 > f(0)$ . If  $b_0, b_1, \ldots, b_k$ ,  $k = 1 + \ldots + (n-1)$  are already constructed, we choice an integer 1 such that  $|A \cap R_1| > n$  and min  $A \cap R_1 > f(k+n)$ . Now, choice  $b_{k+1}, \ldots, b_{k+n} \in A \cap R_1$ .

Evidently the set  $B = \{b_k, k \in \omega\}$  is growing and  $\overline{B} > f$ .

q.e.d.

We shall use the forcing construction as it is explained in [3]. Thus if M is a transitive model of ZFC, P,  $\leq$  is a partially ordered set in M and G is an M-generic filter on P then M[G] is the corresponding model of ZFC - the generic extension of M. The complete Boolean algebra containing P as a dense subset is denoted by RO(P). The model M[G] is obtained as the range of the G-interpretation i<sub>G</sub> defined on the Boolean-valued model M<sup>RO(P)</sup>. If it is clear which generic filter G is intended, we simply denote the interpretation by i. For a formula  $\varphi$  and Boolean functions  $f_1, \ldots, f_n \in M^{RO(P)}$ , the Boolean value  $\| \varphi(f_1, \ldots, f_n) \|$  is also defined in [3] (see pp. 152-169). If Q is a notion of forcing in M[G],

we denote by P \* Q the iterated forcing (see [3], pp. 232-237).

For a given filter  $\mathcal V$  on  $\omega$  , J.Ciohon [2] has constructed a forcing P(V) as follows. The set P(V) consists of ordered triples  $\langle p, Q, f \rangle$ , where for some integer n, the following holds true: 1)  $p \in {}^{n_2}$ ,  $Q \in [V]^{<\omega}$ ,  $f \in {}^{Q_{\omega}}$ .

2) if  $X \in \mathcal{O}$ ,  $i \notin X$  and  $f(X) \leq i < n$ , then p(i) = 0. The order  $\leq$  on P(V) is defined in the following way:

$$\langle p, a, f \rangle \leq \langle p', a', f' \rangle \equiv p \geq p', a \geq a', f \geq f'.$$

One can easily show that P(V) satisfies the countable chain condition.

Now, let us suppose that M is a transitive model of ZFC,  $V \in M$ is a filter. If G is an M-generic filter on P(V) we denote

$$X(\mathcal{V}) = \{ n \in \omega ; (\exists \langle p, Q, f \rangle \in G) p(n) = 1 \}.$$

In [2] it is shown that for each  $X \in V$ , X(V) - X is finite. Moreover, we obtain

If every element of V is a growing set then X(V) is also Lemma 2. growing.

PROOF. Let us denote

 $\mathcal{E}_{\mathbf{n}} = \left\{ \langle \mathbf{p}, \mathbf{0}, \mathbf{f} \rangle \in \mathbb{P}(\mathcal{V}); (\exists \mathbf{k}) \middle| \left\{ \mathbf{i} \in \mathbb{R}_{\mathbf{k}}; \mathbf{p}(\mathbf{i}) = \mathbf{i} \right\} \right\} \ge \mathbf{n} \right\}.$ It suffices to show that  $\mathcal{E}_n$  is a dense subset of P(V).

Thus, assume  $\langle p, 0, f \rangle \in P(V)$ . Then the set  $Y = \bigcap \hat{0} \in V_{is}$ growing. Let k be such that  $|Y \cap R_k| \ge n$  and  $dom(p) \cap R_k = \emptyset$ . We denote  $p' = p \cup (Y \cap R_k \times \{1\})$ . Then  $\langle p', 0, f \rangle \leq \langle p, 0, f \rangle$  and <p',a,f> ∈ E, .

The Martin axiom MA and MA, is formulated e.g. in [4].

§2. Proof of the theorem A. For to obtain the model N we shall iterate the P(V)-forcing continuum many times. During the iteration we will construct a rapid filter that contains no selector for the partition R.

For the iteration we need a well-known trick of enumerating all possible functions from  $\omega$  into  $\omega$  inside the resulting model N. Similar case of this trick is described with all details e.g. in [6].

Let P be a partially ordered set satisfying the countable chain condition.  $|P| \leq 2^{\aleph_0}$ . Then there exists a function H<sub>p</sub> defined on 2 % such that the range of HD is the set of all Boolean functions  $h \in M^{RO(P)}$  for which  $\|h \in \omega \| = 1$ . The value  $H_p(\xi)$  will be denoted  $H(P, \xi)$ .

Let F be a fixed map of  $2^{\aleph_0}$  onto  $2^{\aleph_0} \times 2^{\aleph_0}$ . Let K,L be maps

of  $2^{\aleph_0}$  onto  $2^{\aleph_0}$  such that  $F(\xi) = \langle K(\xi), L(\xi) \rangle$ . We can assume that  $K(\xi) \leq \xi$  for every  $\xi \in 2^{\mathcal{R}_0}$ .

Now, by the transfinite induction we shall construct sequences  $\{P_{\xi}; \xi \in 2^{\infty}\}, \{\{\xi \in 2^{\infty}\}, \{G_{\xi}; \xi \in 2^{\infty}\}\}$  such

- 3)  $P_{\xi}$  satisfies the countable chain condition,  $|P_{\xi}| \leq 2^{N_0}$ ;
- 4) P = P2 for \ < 2;
- 5) G is an M-generic filter on P; ;
- 6)  $G_{\xi} \subseteq G_{\xi}$  for  $\xi < \chi$ ;
  7)  $M[G_{\xi}] \models "V_{\xi}$  is a filter,  $\mathcal{Y} \subseteq V_{\xi} "$ ;
  8)  $M[G_{\xi}] \models "V_{\xi} \subseteq V_{\xi} "$  for  $\chi < \xi$ ;
- 9)  $M[G_{\xi}] \models "B_{\xi} X$  is finite for each  $X \in \bigcup \{ \mathcal{V}_{\xi}; 2 < \xi \} ";$ 10)  $M[G_{\xi}] \models \overline{B}_{\xi} > i_{G_{\xi}} (H(P_{K(\xi)}, L(\xi)).$

The construction is simple. We set  $P_0 = P(\mathcal{F})$ ,  $G_0$  is any M-generic filter on P . The set  $X(\mathcal{S})$  is growing in  $M[G_0]$  by the lemma 2. By the lemma 1 there exists a growing set  $B_0 \subseteq X(\mathcal{G})$  such that  $\overline{B}_{o} > i(H(P_{K(O)},L(O)))$ . Let  $\mathcal{V}_{o}$  denote the filter generated by  $B_{o}$ and  $\mathcal{G}$  - everything inside the model M[G].

Similarly, if  $P_{\xi}$ ,  $V_{\xi}$ ,  $B_{\xi}$  are already defined, we denote  $P_{\xi+1} = P_{\xi} * P(V_{\xi})$ . Let  $G_{\xi+1}$  be any M-generic filter on  $P_{\xi+1}$  extending  $G_{\xi}$ . By the lemma 1 there exists a growing set  $B_{\xi+1} \subseteq X(V_{\xi})$  such that  $B_{\xi+1} > i(H(P_{K(\xi+1)}, L(\xi+1))$ . Let  $V_{\xi+1}$  be the filter generated by  $B_{\xi+1}$  and  $O_{\xi+1}$ , constructed inside the model  $M[G_{\xi+1}]$ .

For  $\lambda^{\xi}$  limit we denote  $P_{\lambda} = \bigcup \{ P_{\xi}, \xi < \lambda \} * P(\bigcup \{ v_{\xi}^{\xi}, \xi < \lambda \}).$  $G_{\lambda}, B_{\lambda}, V_{\lambda}$  are defined analogously.

Directly from the construction one can see that 4) - 10) are fulfilled. The condition 3) is fulfilled by the well-known lemma

about C.C.C.-iteration (see[3], p. 235 or [6]). Now, we set  $P = \bigcup \{P_{\xi}; \xi \in 2^{\mathcal{R}_0}\}$ ,  $G = \bigcup \{G_{\xi}; \xi \in 2^{\mathcal{R}_0}\}$  and  $\mathcal{V} = \bigcup \{\mathcal{V}_{\xi}; \xi \in 2^{\mathcal{R}_0}\}$ . Since P satisfies the countable chain condition, one can easily see that  $\mathcal{V}$  is a rapid filter. Since each element of V is a growing set, V is not a Q-point.

If we change the construction in such a way, that on every step  $V_{\xi}$  is an ultrafilter containing  $B_{\xi}$  and extending F, then the resulting filter V is an ultrafilter and actually a P-point.

§3. Proof of the theorem B. Let us remind that B is a basis of the filter V iff  $B \subseteq V$  and for every  $A \in V$  there exists a  $B \in B$ such that  $B \subseteq A$ .

Lemma 3. Assume MA, holds true. Let V be a filter such that

- a) each element of V is a growing set:
- b) I has a basis of cardinality at most K.

Then there exists a growing set A such that A - X is finite for each  $X \in \mathcal{V}$ .

PROOF. Let  $B \subseteq V$  be a basis,  $|B| \leq \kappa$ . If  $X \in B$ 

 $\mathcal{C}_X = \left\{ \langle p, Q, f \rangle \in P(\mathcal{V}); X \in Q \right\}.$  Evidently  $\mathcal{C}_X$  is a dense subset of  $P(\mathcal{V})$ . By MA, there exists a filter G on  $P(\mathcal{V})$  such that G is  $\left\{ \mathcal{C}_X; X \in \mathcal{G} \right\} \cup \left\{ \mathcal{E}_n; n \in \omega \right\}$ -generic (the sets  $\mathcal{E}_n$  were defined in the proof of the lemma 2). The set A = X(V) is the desired growing set.

Now, the proof of the theorem B is straightforward. Let  $\mathcal{F} = \{ f_{\xi}; \xi \in \mathcal{A} \}$  be a dominating family. By the lemma 1, there exists a growing set B such that  $\overline{B} > f$ . Let us suppose that B is defined for  $2 < \xi$ . Let  $V_{\xi}$  be the filter generated by  $\{B_{i}, i < \xi\}$ . Then by the lemma 3 and the lemma 1, there exists a growing set B<sub>ξ</sub> such that  $\overline{B}_{\xi} > f_{\xi}$  and  $B_{\xi} - B_{\xi}$  is finite for  $\xi < \xi$ . The filter  $f(\xi) = f(\xi)$  generated by  $f(\xi) = f(\xi)$  is rapid. Since each element of  $f(\xi)$  is a growing set, Vis not a Q-point.

§4. Proof of the proposition. Assume that  ${\mathcal U}$  is a non-rapid filter. Then there exists a function  $f \in {}^{\omega}\omega$  such that for each g > f we have range (g)  $\notin \mathcal{U}$ . We can assume that f is strictly increasing. We set

$$a_0 = a_1 = \dots = a_{f(0)} = 1$$
,  
 $a_{f(0)+1} = \dots = a_{f(1)} = 1/\sqrt{2}$ ,  
 $\vdots$   
 $a_{f(n)+1} = \dots = a_{f(n+1)} = 1/\sqrt{n+2}$ ,  
 $\vdots$ 

If  $\overline{A} > f$  then for each k there exists an  $n \ge k$  such that  $\overline{A}(n) < f(n)$ . Then

$$\sum_{i=0}^{n} a_{\overline{A}(i)} \ge (n+1) \cdot a_{\overline{A}(n)} \ge (n+1) a_{f(n)} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1} .$$

Therefore  $\sum_{n \in A} a_n = \sum_{n=0}^{\infty} a_{\overline{A}(n)} = +\infty$ . Hence, if  $\sum_{n \in A} a_n < +\infty$ then  $\overline{A} > f$  and  $A = range(\overline{A}) \notin \mathcal{U}$ .

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