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Irreducible images of $\beta N - N$

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IRREDUCIBLE IMAGES OF $\beta N-N$

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The space $\beta N-N$ is the remainder of the Čech-Stone compactification of the natural numbers. A mapping $f:X \xrightarrow{onto} Y$ is irreducible if it is continuous and $f(F) \neq Y$ for every closed set $F \subset X$ such that $F \neq X$. Our aim is to investigate irreducible images of $\beta N-N$. Under the assumption of CH (= the continuum hypothesis) we shall show (see Theorem 1) that a zero-dimensional compact space X is an irreducible image of $\beta N-N$ iff weight of X equals 2^{ω} and X has the following property

(P) there are no isolated points in X and non-empty G_{δ}'s in X have non-empty interior.

Spaces in which non-empty G_{δ}'s have non-empty interior are also called almost-P spaces or P'-spaces. Clearly, $\beta N-N$ satisfies condition (P). If X is a compact zero-dimensional space, then $\beta(X \times N) -(X \times N)$ also satisfies condition (P) ; see e.g. Walker [10] . If X and Y satisfy condition (P), then the product $X \times Y$ satisfies (P) as well. Zero-dimensional compact spaces satisfying condition (P) in which every two disjoint open F_{σ}'s have disjoint closures are called by several authors Parovičenko spaces. The well known theorem of Parovičenko [9] says that , under CH , a space is homeomorphic to $\beta N-N$ iff it is a Parovičenko space of weight 2^{ω}. Concerning Parovičenko spaces Broverman and Weiss [1] have shown that a Parovičenko space X has the absolute (= Gleason space) homeomorphic to the absolute of $\beta N-N$ iff π-weight of X equals 2^{ω} . If X is an irreducible image of $\beta N-N$, then X is co-absolute with $\beta N-N$; i.e. the absolute of X is homeomorphic to the absolute of $\beta N-N$. So, our Theorem 1 leads to the following : under CH a compact space X is co-absolute with $\beta N-N$ iff X is dense in itself and has a π-base of power 2^{ω} consisting of non-empty regular-open sets in which every countable chain (with respect to inclusion) has a lower bound (see Theorem 3). This improves the result of Broverman and Weiss [1] as well as the result of Williams [11] who proved, under

CH , that if X is a compact space of π-weight 2^{ω} satisfying condition (P) , then X is co-absolute with $\beta N-N$.

All spaces are assumed to be compact (Hausdorff). Zero-dimensional compact spaces are called Stone spaces. The symbol CO(X) will denote the Boolean algebra of all closed-open subsets of X. If X and Y are Stone spaces, then every continuous mapping from X onto Y is uniquely determined by an embedding of CO(Y) into CO(X). For a space X , w(X) denotes weight and $\pi(X)$ denotes π-weight of X.

§1. Irreducible mappings of $\beta N-N$. Let us note the following

Lemma 1. If f is an irreducible mapping from X onto Y and X is a (compact) space satisfying condition (P) , then Y satisfies (P) as well.

The proof is clear, so can be omited.

One can easily chack that if X satisfies condition (P) , then in X there exists a disjoint family of open sets of size 2^{ω} . In particular $w(X) \geqslant 2^{\omega}$. Thus, by Lemma 1, if X is an irreducible image of $\beta N-N$, then X is a compact space of weight 2^{ω} satisfying condition (P). To obtain the converse we have to prove two lemmas:

Lemma 2. If $U_1, U_2, W \subset CO(\beta N-N)-\{\emptyset\}$ are countable and

(1) for every $i \in \{1,2\}$, every $u_1,\dots,u_n \in U_i$ and every $w \in W$, $w-(u_1 \cup \dots \cup u_n) \neq \emptyset$,

then there exist $z_1, z_2 \in CO(\beta N-N)$ such that

(2) $z_1 \cap z_2 = \emptyset$,

(3) $z_1 \cap u = \emptyset$ for every $u \in U_1$ and $z_2 \cap u = \emptyset$ for every $u \in U_2$,

(4) $z_i \cap w \neq \emptyset$ for $i \in \{1,2\}$ and for every $w \in W$.

Proof. By condition (1), for $i = 1,2$ and for every $w \in W$ there exists $w'_i \in CO(\beta N-N)-\{\emptyset\}$ such that $w'_i \subset w$ and $w'_i \cap u = \emptyset$ for every $u \in U_i$. Since the family $\{w'_i : w \in W \text{ and } i = 1,2\}$ is countable , one can assume that it consists of disjoint elements. We set $F_i = \mathrm{cl}\bigcup\{w'_i : w \in W\}$, $i = 1,2$. Since disjoint open F_σ's in $\beta N-N$ have disjoint closures, we get : $F_1 \cap F_2 = \emptyset$, $F_1 \cap \mathrm{cl}\bigcup U_1 = \emptyset$ and $F_2 \cap \mathrm{cl}\bigcup U_2 = \emptyset$. Then, there exist two disjoint elements $z_1, z_2 \in CO(\beta N-N)$ such that $F_1 \subset z_1$, $F_2 \subset z_2$, $z_1 \cap \bigcup U_1 = \emptyset$ and $z_2 \cap \bigcup U_2 = \emptyset$. It is easy to see that z_1 and z_2 are as required.

The next lemma is well known ; for the proof see e.g. Comfort and Negrepontis [2] , page 36.

Lemma 3. Let A' and B' be subalgebras of Boolean algebras A and B, respectively. Let $h: A' \xrightarrow{\text{onto}} B'$ be an isomorphism and let $a \in A$ and $b \in B$ be such that for every $x \in A'$,

$$x \cap a = 0 \text{ iff } h(x) \cap b = 0 \text{ and}$$
$$x \cap a = 0 \text{ iff } h(x) \cap b = 0.$$

If A'' and B'' are algebras generated by $A' \cup \{a\}$ and $B' \cup \{b\}$, respectively, then there exists an isomorphism $g: A'' \longrightarrow B''$ such that $g|A' = h$ and $g(a) = b$.

Now, we are ready to prove the following

<u>Theorem</u> 1. Assume CH. A Stone space X is an irreducible image of $\beta N-N$ iff X satisfies condition (P) and $w(X) = 2^{\omega}$.

Proof. Assume X is a Stone space satisfying condition (P) such that $w(X) = 2^{\omega} = \omega_1$. To prove the theorem it suffices to show that the algebra $A = CO(X)$ can be embedded as a dense subalgebra in $B = CO(\beta N-N)$. Let $A = \{a_\alpha : \alpha < \omega_1\}$ and $B = \{b_\alpha : \alpha < \omega_1\}$. By transfinite recursion we construct for every $\alpha < \omega_1$ an isomorphism $h_\alpha : A_\alpha \longrightarrow B_\alpha$ such that

(5) A_α and B_α are subalgebras of A and B, respectively,

(6) if $\mu < \alpha$, then $A_\mu \subset A_\alpha$, $B_\mu \subset B_\alpha$ and $h_\alpha | A_\mu = h_\mu$,

(7) $\{a_\mu : \mu \leq \alpha\} \subset A_\alpha$,

(8) there exists $b \in B_\alpha - \{0\}$ such that $b \subset b_\alpha$.

If $h_\alpha : A_\alpha \longrightarrow B_\alpha$, for $\alpha < \omega_1$, are already constructed, we set $h = \bigcup \{h_\alpha : \alpha < \omega_1\}$. Clearly, h is an embedding of A into B and h(A) is dense in B.

Assume, A_α, B_α and h_α are defined for all $\alpha < \gamma$. Thus $h = \bigcup \{h_\alpha : \alpha < \gamma\}$ is an isomorphism of $A' = \bigcup \{A_\alpha : \alpha < \gamma\}$ onto $B' = \bigcup \{B_\alpha : \alpha < \gamma\}$. Suppose $a_\gamma \notin A'$ and denote

$$X_1 = \{x \in A' : x \cap a_\gamma = 0\},$$
$$X_2 = \{x \in A' : x \subset a_\gamma\},$$
$$Y = \{x \in A' : x \cap a_\gamma \neq 0 \text{ and } x - a_\gamma \neq 0\}.$$

For $x \in X_1$ and $y \in X_2$, $h(x) \cap h(y) = 0$. Hence, there exists $u \in B$ such that

(9) $h(x) \cap u = 0$ for all $x \in X_1$ and $h(y) \subset u$ for all $y \in X_2$.

Since X_1, X_2 and Y are countable, by Lemma 2, there exist $z_1, z_2 \in B$ such that $z_1 \cap h(x) = 0$ for every $x \in X_1$, $z_2 \cap h(x) = 0$ for every $x \in X_2$ and $z_1 \cap h(x) \neq 0 \neq z_2 \cap h(x)$ for every $x \in Y$. Now, by (9), it is easy to chack that for $v = (u \cup z_1) - z_2$ we have the following :

$$x \cap a_\gamma = 0 \text{ iff } h(x) \cap v = 0 \text{ and}$$
$$x - a_\gamma = 0 \text{ iff } h(x) - v = 0.$$

So, by Lemma 3, if $A'' \subset A$ is a subalgebra generated by $A' \cup \{a_\gamma\}$ and $B'' \subset B$ is a subalgebra generated by $B' \cup \{v\}$, then there exists an isomorphism $g: A'' \longrightarrow B''$ such that $g|A' = h$ and $g(a_\gamma) = v$. If $a_\gamma \in A'$ we set $g = h$.

Now, since B'' is countable, there exists $w \in B - \{0\}$ such that

$w \subset b_\gamma$ and

(10) for every $b \in B''$, either $b \cap w = 0$ or $w \subset b$.

Let $C = \{x \in A'' : g(x) \cap w = 0\}$ and $D = \{x \in A'' : w \subset g(x)\}$. Clearly, by (10), $A'' = C \cup D$. Since X satisfies condition (P) and $y_1 \cap \ldots \cap y_k - (x_1 \cup \ldots \cup x_n) \neq 0$, for every $x_1, \ldots, x_n \in C$ and $y_1, \ldots, y_k \in D$, there exists $z \in A - \{0\}$ such that

(11) $z \cap x = 0$ for every $x \in C$ and $z \subset y$ for every $y \in D$.

Let $A_\gamma \subset A$ be the algebra generated by $A'' \cup \{z\}$ and $B_\gamma \subset B$ the algebra generated by $B'' \cup \{w\}$. By condition (11) and Lemma 3, there exists an isomorphism $h_\gamma : A_\gamma \longrightarrow B_\gamma$ such that $h_\gamma | A'' = g$ and $h_\gamma(z) = w$. Now, to finish the proof it suffices to see that h_γ, A_γ and B_γ satisfies conditions (5) - (8).

We have already pointed out that $(\beta N - N) \times (\beta N - N)$ satisfies condition (P). Thus, from Theorem 1 we get

Corollary 1. Assume CH. There exists an irreducible mapping $\beta N - N$ onto its square.

However, the following question remains open :

Question. Is it true (in ZFC) that $\beta N - N$ can be mapped onto its square by a continuous mapping ?

Let X be a compact space. The Stone space G(X) of the Boolean algebra of all regular-open subsets of X is called the absolute (= the Gleason space) of X ; see e.g. Comfort and Negrepontis [2], page 57. Compact spaces X and Y are co-absolute iff G(X) and G(Y) are homeomorphic. The following lemma summarize the informations concerning absolutes which will be needed.

Lemma 4. Let X and Y be compact spaces. The following hold :

(a) If X has a dense subspace homeomorphic to a dense subspace of Y, then X is co-absolute with Y.

(b) If Y is an irreducible image of X, then Y is co-absolute with X.

In particular, if X is an irreducible image of $\beta N - N$, then X is co-absolute with $\beta N - N$. The converse implication is not true.

Example. Let F be a closed but not open G_δ-subset of $\beta N - N$ and let X be the quotient space obtained from $\beta N - N$ by collapsing F to a point. Clearly, in X and in $\beta N - N$ there exist π-bases consisting of closed-open subsets homeomorphic to $\beta N - N$. So, by Lemma 4(a), X is co-absolute with $\beta N - N$. By Lemma 1, there does not exist irreducible mapping from $\beta N - N$ onto X. We shall show that also X cannot be mapped onto $\beta N - N$ by an irreducible mapping. Indeed, suppose $f : X \xrightarrow{\text{onto}} \beta N - N$ is irreducible. Then, for every open set $U \subset X$, $\mathrm{Int} f(U) \neq \emptyset$. There exists a point in X with a countable base of neighbourhoods. Then, there exist two countable families $\{H_n : n < \omega\}$ and $\{G_n : n < \omega\}$ of closed-open subsets of βN-N such that $H_n \cap G_k = \emptyset$ for $n \neq k$ and for some $x \in X$, $f(x) \in \mathrm{cl} \bigcup \{H_n : n < \omega\} \cap \mathrm{cl} \bigcup \{G_n : n < \omega\}$. We get a contradiction, because disjoint open F_σ's in βN-N have disjoint closures.

It is known that CH is equivalent to the statement that all Parovičenko spaces of weight 2^ω are homeomorphic ; see Parovičenko [9] , van Douwen and van Mill [4] and Frankiewicz [5] . Broverman and Weiss [1] have shown that CH implies that all Parovičenko spaces of π-weight 2^ω are co-absolute and conjectured that the converse is also true. Recently van Mill and Williams [8] have proved that if $2^\omega = 2^{\omega_1}$, then not all Parovičenko spaces of π-weight 2^ω are co-absolute, whereas Dow [3] has proved that if $cf(2^\omega) = \omega_1$, then all Parovičenko spaces of π- weight 2^ω are co-absolute (note that $cf(2^\omega) > \omega_1$ whenever $2^\omega = 2^{\omega_1}$). But the assertion "X is an irreducible image of βN-N" is stronger than "X is co-absolute with βN-N" ; see the example above. So, the question whether the assertion "every Stone space with the property (P) and weight 2^ω is an irreducible image of βN-N" is equivalent to CH remains open. We only have the following

Theorem 2. It is consistent with ZFC that $cf(2^\omega) = \omega_1 < 2^\omega$ and not every Stone space with the property (P) is an irreducible image of βN-N.

Proof. Let φ denotes the formula asserting that there exists a point $p \in \beta N$-N with $\chi(p, \beta N\text{-}N) = \omega_1$. It is known that there exists a model $\mathcal{M}$ for ZFC such that

$$\mathcal{M} \models \varphi \wedge cf(2^\omega) = \omega_1 < 2^\omega ;$$

see Kunen [6] , page 289. On the other hand one can prove (in ZFC) that if $X = \beta(\omega \times 2^c) - (\omega \times 2^c)$, where 2^c is the Cantor cube of weight 2^ω , then the π-character at every point of X equals 2^ω; see e.g. van Mill [7] , page 41. Now, suppose $f: \beta N\text{-}N \longrightarrow X$ is irreducible and P is a base of neighbourhoods of the point p, $|P|$ is minimal. Then, the family $R = \{X - f(\beta N\text{-}N\text{-}U) : U \in P\}$ is a π-base at the point f(p). Clearly, $2^\omega \leqslant |R| \leqslant |P|$. But in our model $\mathcal{M}$, $|P| = \omega_1 < 2^\omega$; we get a contradiction.

§2. Co-absolutes of βN-N. In this section we shall give a characterization of all compact spaces which are co-absolute with βN-N. Our characterization gives a strenghtening of a result of Williams [11] who has proved that under CH every compact space of π-weight 2^ω satisfying condition (P) is co-absolute with βN-N.

A family R of non-empty sets will be called σ-closed if for every decreasing sequence $\{U_n : n<\omega\}\subset R$ there exists $U\in R$ such that $U\subset U_n$, for all $n<\omega$.

Lemma 5. A compact space X admits a σ-closed π-base consisting of regular-open sets iff the space G(X) admits a σ-closed π-base of the same weight consisting of closed-open sets.

Proof. 1. If P is a σ-closed π-base of X consisting of regular-open sets and $G:G(X)\longrightarrow X$ is the irreducible mapping, then $R=\{\mathrm{cl}G^{-1}(U) : U\in P\}$ is a π-base in G(X) consisting of closed-open sets. Clearly, $|P|=|R|$. In order to show that R is σ-closed it suffices to check only that $\mathrm{cl}G^{-1}(U)\subset \mathrm{cl}G^{-1}(V)$ implies $U\subset V$ (because U and V are regular-open).

2. Assume $R\subset CO(G(X))$ is a σ-closed π-base in G(X). We set $P=\{\mathrm{Int}G(W) : W\in R\}$. Since G is irreducible, $\mathrm{Int}G(W)\neq \mathrm{Int}G(W')$ whenever $W\neq W'$. So, $|R|=|P|$. Clearly, for every $W\in R$, $\mathrm{Int}G(W)$ is regular-open. Hence, it remains to show that P is σ-closed. To do this it suffices to show that

$$\mathrm{cl}G^{-1}(\mathrm{Int}G(W))=W,$$

for every $W\in CO(G(X))$.

To prove that $W\subset \mathrm{cl}G^{-1}(\mathrm{Int}G(W))$ suppose that there exists a closed-open non-empty $U\subset W$ such that $U\cap \mathrm{cl}G^{-1}(\mathrm{Int}G(W))=\emptyset$. Then $G(U)\cap \mathrm{Int}G(W)=\emptyset$, hence $G(U)\subset \mathrm{cl}(X-G(W))\subset G(G(X)-W)$. Thus $G(G(X)-U)=X$; a contradiction, because G is irreducible.

To prove that $\mathrm{cl}G^{-1}(\mathrm{Int}G(W))\subset W$ suppose that there exists a set $U\in CO(G(X))$ such that $U\cap W=\emptyset$ and $\emptyset\neq U\subset \mathrm{cl}G^{-1}(\mathrm{Int}G(W))$. Then $G(U)\subset G(\mathrm{cl}G^{-1}(\mathrm{Int}G(W)))=\mathrm{clInt}G(W)\subset G(W)$. Again, $G(G(X)-U)=X$; a contradiction. The proof is complete.

Clearly, $CO(\beta N-N)$ is a σ-closed π-base of cardinality continuum. Thus, we get

Corollary 2. If X is a compact space which is co-absolute with $\beta N-N$, then X has a σ-closed π-base of cardinality continuuum consisting of regular-open sets.

Lemma 6. Let X be a dense in itself Stone space with a σ-closed π-base $P\subset CO(X)$ of cardinality ω_1. Then X has an irreducible mapping onto a Stone space with the property (P) of weight ω_1.

Proof. Let $P=\{U_\alpha : \alpha<\omega_1\}$. By transfinite recursion one can construct for every $\alpha<\omega_1$ a disjoint family $T_\alpha\subset P$ such that

(12) $\mathrm{cl}\bigcup T_\alpha = X$,

(13) for some $W\in T_\alpha$, $W\subset U_\alpha$,

(14) for every $W\in T_\alpha$, $|\{V\in T_{\alpha+1} : V\subset W\}|=\omega_1$,

(15) if $\alpha<\gamma$ and $V\in T_\gamma$, then $V\subset W$ for some $W\in T_\alpha$.

The construction is possible because P is a σ-closed π-base. In particular, (14) follows from the fact that for every non-empty open set $U \subset X$ there exists a family of size 2^{ω} of disjoint open sets contained in U.

Let $B \subset CO(X)$ be a subalgebra generated by $T = \bigcup\{T_\alpha : \alpha < \omega_1\}$ and let Y be the Stone space of B. By condition (13), B is dense in CO(X). Thus, the mapping from X onto Y appointed by the embedding of B into CO(X) is irreducible. It remains to prove that Y satisfies condition (P). First observe that, by (15),

(16) if $\alpha < \gamma$, $U \in T_\alpha$ and $V \in T_\gamma$, then either $V \subset U$ or $U \cap V = \emptyset$.

This follows that $\overline{B} = \{U-(W_1 \cup \ldots \cup W_k) : U \in T_\gamma \cup \{X\},\ W_i \in T_{\gamma_i} \cup \{\emptyset\},$ $\gamma < \gamma_i < \omega_1$ and $i \leq k < \omega\}$ is a base in Y. Let $\{V_n : n < \omega\}$ be a decreasing sequence of elements of B. By the condition (16), for every $n < \omega$ there exist $\alpha_n < \omega_1$, $U_{\alpha_n} \in T_{\alpha_n}$ and a finite set $R_n \subset B$ such that $V_n = U_{\alpha_n} - \bigcup R_n$ and

(17) if $W \in R_n \cap T_\gamma$, then $\alpha_n < \gamma$.

Clearly, we can assume, that $U_{\alpha_n} \subset U_{\alpha_k}$ whenever $k \leq n$, i.e. if $k \leq n$, then $\alpha_k \leq \alpha_n$. Let $\alpha = \sup\{\alpha_n : n < \omega\}$. If $\alpha = \alpha_n$ for some $n < \omega$, then we can assume that $\alpha_n = \alpha$ for all n. Since $|\bigcup\{R_n : n < \omega\}| \leq \omega$, there exists, by conditions (14) and (17), $U \in T_{\alpha+1}$ such that $U \subset U_\alpha$ and $U \cap W = \emptyset$ for all $W \in \bigcup\{R_n : n < \omega\}$. Hence $U \subset \bigcap\{V_n : n < \omega\}$. So, we can assume that $\alpha_n < \alpha$ for all $n < \omega$. Recall, P is σ-closed. Then, by the condition (12), there exists $U \in T_\alpha$ such that $U \subset U_{\alpha_n}$, for all n. Set $R = \bigcup\{R_n : n < \omega\}$. We claim that

(18) if $W \in R$, then either $W \subset U$ or $W \cap U = \emptyset$.

Indeed, if $W \in T_\gamma$ and $\gamma \geq \alpha$, we apply (16). If $\gamma < \alpha$ and $W \in T_\gamma \cap R_k$, then $\gamma < \alpha_n < \alpha$ for some n such that $k < n < \omega$. The inclusion $U_{\alpha_n} \subset W$ is imposible because $V_n \subset U_{\alpha_n}$, $V_n \subset V_k$ and $V_k \cap W = \emptyset$. Thus, $U_{\alpha_n} \cap W = \emptyset$, which follows $U \cap W = \emptyset$. Now, by conditions (14) and (18), there exists $U' \in T_{\alpha+1}$ such that $U' \subset U$ and $U' \cap W = \emptyset$, for all $W \in R$. Therefore, $U' \subset \bigcap\{V_n : n < \omega\}$, which completes the proof.

Theorem 3. Assume CH. A compact space X is co-absolute with ßN-N iff X is dense in itself and admits a σ-closed π-base of power continuum consisting of regular-open sets.

Proof. By Lemma 5, X is co-absolute with a Stone space of weight ω_1 which admits a σ-closed π-base consisting of closed-open sets. Thus, by Lemma 6, X is co-absolute with a Stone space of weight ω_1 with the property (P). By Lemma 4(b) and Theorem 1, X is co-absolute with ßN-N. Corollary 2 completes the proof.

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