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Abstract: It is well-known that on a metrizable locally convex space any weak Radon probability measure has a strong extension. We show by an example that metrizability is essential. Further, we give a short proof of the classical result using a theorem of R.E. Johnson.

Let E be a separated locally convex vector space with topology τ, topological dual space E' and weak topology $\sigma(E,E')$.
The *weak Borel σ-algebra* on a subset M of E - generated by the weak topology - is denoted by $\mathcal{B}_\sigma(M)$; the *strong Borel σ-algebra* - generated by τ - by $\mathcal{B}_\tau(M)$. A probability measure on $\mathcal{B}_\sigma(M)$ is called a *weak Radon probability measure* (w.R.p.m.) if it is Radon w.r.t. $M \cap \sigma(E,E')$ and a probability measure on $\mathcal{B}_\tau(M)$ is called a *strong Radon probability measure* (s.R.p.m.) if it is Radon w.r.t. $M \cap \tau$.

The following variant of theorems due to Phillips, Dunford-Pettis and Grothendieck is well-known:

1.Theorem: Let E be a metrizable locally convex space. Then any weak Radon probability measure on E has a unique extension to a strong Radon probability measure on E.

A rather lengthy proof is given in [3], p. 162-166. We give a short proof which was indicated to us by J.P.R. Christensen. It is based on a theorem of R.E. Johnson ([2]), which was generalized and supplied with a simpler proof by Christensen ([1]). As far as we know, there is no example in the literature showing that in theorem 1 metrizability is essential. We will present such an example below.

We state Johnsons theorem in a version sufficient for our needs. A proof is given in [1].

2. Theorem: Let X and Y be compact spaces. Assume further that X is the support of some Radon probability measure. Then: if $f : X \times Y \to \mathbb{R}$ is a separately continuous function, the set $\{f(x,\cdot) : x \in X\} \subset C(Y)$ is separable in the supremum norm.

The essential step in the proof of theorem 1 is

3. Proposition: Let E be a Banach space with norm topology τ and p a w.R.p.m. on E with weakly compact support C. Then:
a. the weak and strong Borel σ-algebra coincide on C;
b. the space $(C, C \cap \tau)$ is Polish; in particular p is a s.R.p.m. on C.

Proof: Denote by B' the weak*-compact unit ball of E'. Apply theorem 2 to the evaluation map $f : C \times B' \to \mathbb{R}$, $(x,\varphi) \to \varphi(x)$ to conclude that $\{f(x,\cdot) : x \in C\} \subset C(B')$ is separable in the sup-norm. Since the mapping $C \ni x \to f(x,\cdot) \in C(B')$ is an isometry, C itself is norm separable. Furthermore, C being weakly complete is complete in the norm. As C is Polish, the weak and strong Borel σ-algebra coincide on C ([3], p. 101) and p is a s.R.p.m..

Proof of theorem 1: 1. Observe that a w.R.p.m. is concentrated on a countable union of pairwise disjoint weakly compact sets. Apply proposition 3 to get the conclusion for Banach spaces E.
2. Let now E be metrizable. We may assume that E is complete. Then E is isomorphic with the inverse limit of a sequence of Banach spaces E_i. A w.R.p.m. on E induces a projective system of w.R.p.m. p_i on the spaces E_i. Extend these measures according to part 1 of the present proof to s.R.p.m. q_i. The measures q_i form a projective system . The projective limit is a s.R.p.m. on E which gives us the desired extension.

Let us conclude with the announced example.

Example: Let I be an uncountable index set and for each $i \in I$ let E_i be a copy of $l^2(\mathbb{N})$; denote the norm topology by τ_i. Let further denote E the product of these spaces and τ the product topology. We construct a w.R.p.m. on E which has no strong extension.
The measure $\mu := \sum_{n \in \mathbb{N}} 2^{-n}\varepsilon_n$, where ε_n is the point measure on the n-th unit vector of $l^2(\mathbb{N})$, is concentrated on a weakly compact set, but $\mu(K) < 1$ for every norm compact set K. Let μ_i be a copy of μ on E_i and C_i the weakly compact support of μ_i. For a finite subset J of I consider the product measure μ_J of the measures μ_j, $j \in J$, on $(\prod_{j \in J} E_j, \prod_{j \in J} \sigma(E_j, E_j'))$. Let pr_J be the canonical projection on E which is weakly continuous.

The measures μ_J, $J \subset I$ finite, together with the projections form a projective system of measures; the limit p on $(E, \prod_{i\in I} \sigma(E_i, E_i'))$ exists since the μ_i have weakly compact support (cf. [3], p.75). Because $\sigma(E,E') = \prod_{i\in I} \sigma(E_i,E_i')$, we have constructed a w.R.p.m. p on E.

It cannot be extended to a s.R.p.m., since $p(K) = 0$ for every τ-compact subset of E. In fact:

According to the choice of μ we have

$$\mu_i(pr_{\{i\}}[K]) < 1 \text{ for every } i \in I.$$

Since I is uncountable, at least countably many of these numbers are bounded away from 1. This implies

$$\inf\{\prod_{j\in J} \mu_j(pr_{\{j\}}[K]) : J \text{ finite subset of } I\} = 0,$$

hence

$$\begin{aligned} p(K) &\leq \inf\{p\left(pr_J^{-1}\left[pr_J[K]\right]\right) : J \subset I \text{ finite}\} = \\ &= \inf\{\mu_J(pr_J[K]) : J \subset I \text{ finite}\} \leq \\ &\leq \inf\{\prod_{j\in J} \mu_j(pr_{\{j\}}[K]) : J \subset I \text{ finite}\} = 0. \end{aligned}$$

References:

[1] J.P.R. Christensen: *Remarks on Namioka spaces and R. E. Johnson's theorem on the norm separability of the range of certain mappings.* Math. Scand 52(1983), 112-116

[2] R.E. Johnson: *Separate continuity and measurability.* Proc. Amer. Math. Soc. 20(1969), 420-422

[3] L. Schwartz: *Radon measures on arbitrary topological spaces and cylindrical measures.* Oxford University Press(1973)