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DERIVED ALGEBRA OF THE FRÖLICHER-NIJENHUIS BRACKET ALGEBRA

Jiří Vanžura

All structures appearing in this note are of class C^∞ . Let M be a connected paracompact manifold, $\dim M = m$. Let TM and T^*M denote the tangent and cotangent bundle of M respectively. We denote by $\Lambda^i T^*M$ the i -th exterior power of T^*M . We set

$$L_i = \Gamma(\Lambda^i T^*M \otimes TM), \quad i = 0, 1, \dots, m,$$

where Γ denotes the functor of sections. To complete our notation we set

$$L_i = 0 \quad \text{for } i < 0 \text{ and } i > m,$$

$$L = \sum_{i=-\infty}^{\infty} L_i.$$

We recall that for every $i, j \in \mathbf{Z}$ there is a bilinear mapping

$$[\ , \]: L_i \times L_j \rightarrow L_{i+j}$$

called the Frölicher-Nijenhuis bracket (see e.g. [1]). Endowed with this bracket L is a graded Lie algebra. Similarly we define the Frölicher-Nijenhuis bracket algebra with compact supports L^c . Instead of the functor Γ we use the functor Γ^c of sections with compact supports. Obviously L^c is an ideal in L . We are going to prove the following theorem.

Theorem. *For any $i, j \in \mathbf{Z}$ satisfying $0 \leq i, j, i + j \leq m$ there is*

$$[L_i, L_j] = L_{i+j}, \quad [L_i^c, L_j^c] = L_{i+j}^c.$$

First we shall need the following lemma.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Lemma. *On an m -dimensional paracompact manifold M there exist open coverings*

$$\{U_{k\sigma}\}, \{V_{k\sigma}\}, \quad k = 0, 1, \dots, m, \quad \sigma \in \Gamma_k$$

with the following properties

- (i) $\bar{U}_{k\sigma} \subset V_{k\sigma}$ for every $k = 0, 1, \dots, m$ and every $\sigma \in \Gamma_k$.
- (ii) $V_{k\sigma} \cap V_{k\tau} = \emptyset$ for every $k = 0, 1, \dots, m$ and $\sigma, \tau \in \Gamma_k, \sigma \neq \tau$.
- (iii) Each $V_{k\sigma}$ is a domain of a chart $(V_{k\sigma}, \varphi_{k\sigma})$ such that $\varphi_{k\sigma}(U_{k\sigma}) = I^m \subset \mathbb{R}^m$, where $I = (0, 1)$.

For the proof of this lemma see [2].

Proof of the Theorem: We shall prove the theorem for the algebra L only. The reader will easily find that the proof for the algebra L^c requires only minor modifications.

Let $i, j \in \mathbb{Z}$ be such that $0 \leq i, j, i+j \leq m$, and let $\alpha \in L_{i+j}$ be arbitrary. We set

$$U_k = \cup_{\sigma \in \Gamma_k} U_{k\sigma}, \quad k = 0, 1, \dots, m.$$

Because the covering $\{U_k\}, k = 0, 1, \dots, m$ is locally finite, we can find a partition of unity $\{\rho_k\}, k = 0, 1, \dots, m$ subordinate to this covering. We can write

$$\alpha = \sum_{k=0}^m \rho_k \alpha.$$

Obviously it suffices to prove that for every $k = 0, 1, \dots, m$ there is $\rho_k \alpha \in [L_i, L_j]$.

Let us assume that k is fixed. From now on we shall work in fact on each open set $U_{k\sigma}$ separately. For the sake of simplicity we shall often identify $U_{k\sigma}$ with $\varphi_{k\sigma}(U_{k\sigma}) = I^m$. Let $(x_1^{(k\sigma)}, \dots, x_m^{(k\sigma)})$ be the coordinates on $U_{k\sigma}$ determined by the chart $(U_{k\sigma}, \varphi_{k\sigma})$. On $U_{k\sigma}$ we can write

$$\rho_k \alpha = \sum_{1 \leq r_1 < \dots < r_{i+j} \leq m} \sum_{s=1}^m \sigma f_{r_1 \dots r_{i+j}}^s dx_{r_1}^{(k\sigma)} \wedge \dots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}}.$$

Because $\bar{U}_{k\sigma} \subset V_{k\sigma}$, and $(V_{k\sigma}, \varphi_{k\sigma})$ is a chart it is easy to see that $\text{supp } \sigma f_{r_1 \dots r_{i+j}}^s \subset U_{k\sigma}$ is compact. For every $(i+j)$ -tuple $1 \leq r_1 < \dots < r_{i+j} \leq m$ and every $1 \leq s \leq m$ we can define $\beta_{r_1 \dots r_{i+j}}^s$ by the formula

$$\begin{aligned} \beta_{r_1 \dots r_{i+j}}^s | U_{k\sigma} &= \sigma f_{r_1 \dots r_{i+j}}^s dx_{r_1}^{(k\sigma)} \wedge \dots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}} \\ \beta_{r_1 \dots r_{i+j}}^s | (M \setminus U_k) &= 0. \end{aligned}$$

We have

$$\rho_k \alpha = \sum_{1 \leq r_1 < \dots < r_{i+j} \leq m} \sum_{s=1}^m \beta_{r_1 \dots r_{i+j}}^s,$$

and therefore it suffices to prove that for every $(i+j)$ -tuple $1 \leq r_1 < \dots < r_{i+j} \leq m$ and every $1 \leq s \leq m$ there is $\beta_{r_1 \dots r_{i+j}}^s \in [L_i, L_j]$. Now we shall divide the proof into two parts.

(1) We shall assume here that $s \neq r_1, \dots, r_{i+j}$. We shall abbreviate $\beta = \beta_{r_1 \dots r_{i+j}}^s$, ${}^\sigma f = {}^\sigma f_{r_1 \dots r_{i+j}}^s$. In our considerations we shall use an auxiliary function χ defined on $I = (0, 1)$ which has the following properties

- (i) $\text{supp } \chi$ is compact
- (ii) $\chi \geq 0$ on I
- (iii) $\sqrt{\chi}$ is a C^∞ -function on I
- (iv) $\int_0^1 \chi(t) dt = 1$.

We define ${}^\sigma \chi = \chi \circ \text{pr}^s \circ \varphi_{k\sigma}$, where $\text{pr}^s: I^m \rightarrow I$ denotes the s -th projection. (In the sequel we again identify $U_{k\sigma}$ with $\varphi_{k\sigma}(U_{k\sigma}) = I^m$.) We set

$$\begin{aligned} {}^\sigma \psi(x_1, \dots, \hat{x}_s, \dots, x_m) &= \int_0^1 {}^\sigma f(x_1, \dots, x_m) dx_s \\ {}^\sigma \tilde{g}(x_1, \dots, x_m) &= {}^\sigma \chi(x_s) {}^\sigma \psi(x_1, \dots, \hat{x}_s, \dots, x_m) \\ {}^\sigma g &= {}^\sigma f - {}^\sigma \tilde{g}. \end{aligned}$$

Obviously $\text{supp } {}^\sigma g, \text{supp } {}^\sigma \tilde{g} \subset U_{k\sigma}$ are compact and

$$\int_0^1 {}^\sigma g(x_1, \dots, x_m) dx_s = 0.$$

Furthermore we define.

$${}^\sigma G(x_1, \dots, x_m) = \int_0^{x_s} {}^\sigma g(x_1, \dots, t, \dots, x_m) dt.$$

${}^\sigma G$ is a C^∞ -function on $U_{k\sigma}$, and the vanishing of the above integral implies that $\text{supp } {}^\sigma G \subset U_{k\sigma}$ is compact. We define an element $\gamma \in L_{i+j}$ by the formula

$$\begin{aligned} \gamma|_{U_{k\sigma}} &= {}^\sigma g dx_{r_1}^{(k\sigma)} \wedge \dots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}} \\ \gamma|(M \setminus U_k) &= 0. \end{aligned}$$

Similarly we define $\tilde{\gamma} \in L_{i+j}$. Because $\beta = \gamma + \tilde{\gamma}$ it will be sufficient to prove that both $\gamma, \tilde{\gamma} \in [L_i, L_j]$. Before proceeding further we shall recall one formula for the Frölicher-Nijenhuis bracket (see [1]). If ω and ω' are p -form and q -form on M respectively, and

$X, X' \in L_0$ are vector fields, then

$$[\omega \otimes X, \omega' \otimes X'] = (\omega \wedge \omega') \otimes [X, X'] + (\omega \wedge \mathcal{L}_X \omega') \otimes X' - \\ (\mathcal{L}_{X'} \omega \wedge \omega') \otimes X + (-1)^p ((d\omega \wedge \iota_X \omega') \otimes X' + (\iota_{X'} \omega \wedge d\omega') \otimes X),$$

where $\mathcal{L}_X$ denotes the Lie derivative and ι_X the inner product operator.

Let $\xi \in L_i$ be an element such that

$$\xi|U_{k\sigma} = dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge dx_{r_i}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}}.$$

Further let $\eta \in L_j$ be defined by the formula

$$\eta|U_{k\sigma} = {}^\sigma G dx_{r_{i+1}}^{(k\sigma)} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}} \\ \eta|(M \setminus U_k) = 0.$$

We shall now compute the bracket $[\xi, \eta]$. At any point $x \notin U_k$ we have

$$[\xi, \eta]_x = 0 = \gamma_x.$$

Further we compute $[\xi, \eta]$ on $U_{k\sigma}$. For the sake of simplicity we denote $y_p = x_{r_p}^{(k\sigma)}$, $x_s = x_s^{(k\sigma)}$. The above formula for the Frölicher-Nijenhuis bracket gives

$$[\xi, \eta]|U_{k\sigma} = (dy_1 \wedge \cdots \wedge dy_i \wedge \mathcal{L}_{\frac{\partial}{\partial x_s}}({}^\sigma G dy_{i+1} \wedge \cdots \wedge dy_{i+j})) \otimes \frac{\partial}{\partial x_s} = \\ {}^\sigma G dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = \gamma|U_{k\sigma}.$$

We have thus proved that $\gamma = [\xi, \eta] \in [L_i, L_j]$.

Further let $\lambda \in L_i$ be an element such that

$$\lambda|U_{k\sigma} = \sqrt{{}^\sigma \chi} dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge dx_{r_i}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}}.$$

We define $\mu \in L_j$ by the formula

$$\mu|U_{k\sigma} = x_s^{(k\sigma)} {}^\sigma \psi \sqrt{{}^\sigma \chi} dx_{r_{i+1}}^{(k\sigma)} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}} \\ \mu|(M \setminus U_k) = 0.$$

Considering $[\lambda, \mu]$ we find again that for every $x \notin U_k$ there is

$$[\lambda, \mu]_x = 0 = \tilde{\gamma}_x.$$

It remains to compute $[\lambda, \mu]|U_{k\sigma}$. We write again y_p instead of $x_{r_p}^{(k\sigma)}$, and x_s instead of $x_s^{(k\sigma)}$. The formula for the Frölicher-Nijenhuis bracket gives

$$\begin{aligned} [\lambda, \mu]|U_{k\sigma} &= \\ & (\sqrt{\sigma\chi} dy_1 \wedge \cdots \wedge dy_i \wedge \mathcal{L}_{\frac{\partial}{\partial x_s}}(x_s \sigma\psi \sqrt{\sigma\chi} dy_{i+1} \wedge \cdots \wedge dy_{i+j})) \otimes \frac{\partial}{\partial x_s} - \\ & ((\mathcal{L}_{\frac{\partial}{\partial x_s}}(\sqrt{\sigma\chi} dy_1 \wedge \cdots \wedge dy_i)) \wedge x_s \sigma\psi \sqrt{\sigma\chi} dy_{i+1} \wedge \cdots \wedge dy_{i+j}) \otimes \frac{\partial}{\partial x_s} = \\ & (\sqrt{\sigma\chi} \sigma\psi \sqrt{\sigma\chi} + \sqrt{\sigma\chi} x_s \sigma\psi \frac{\partial \sqrt{\sigma\chi}}{\partial x_s} - \frac{\partial \sqrt{\sigma\chi}}{\partial x_s} x_s \sigma\psi \sqrt{\sigma\chi}) \cdot dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = \\ & \sigma\chi \sigma\psi dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = \tilde{\sigma}g dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = \tilde{\gamma}|U_{k\sigma}. \end{aligned}$$

We have thus proved that $\tilde{\gamma} = [\lambda, \mu] \in [L_i, L_j]$. Consequently $\beta = \gamma + \tilde{\gamma} \in [L_i, L_j]$. This finishes the first part of the proof.

(2) Here we shall assume that there exists $q, 1 \leq q \leq i+j$ such that $s = r_q$. We shall consider only the case $1 \leq q \leq i$. The case $i+1 \leq q \leq i+j$ can be treated similarly. We shall abbreviate $\beta = \beta_{r_1 \dots r_{i+j}}^{r_q}$, $\sigma f = \sigma f_{r_1 \dots r_{i+j}}^{r_q}$. Let $\xi \in L_i$ be an element such that

$$\xi|U_{k\sigma} = dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge dx_{r_i}^{(k\sigma)} \otimes x_{r_q}^{(k\sigma)} \frac{\partial}{\partial x_{r_q}^{(k\sigma)}}.$$

Further let $\eta \in L_j$ be defined by the formula

$$\begin{aligned} \eta|U_{k\sigma} &= -dx_{r_{i+1}}^{(k\sigma)} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \sigma f \frac{\partial}{\partial x_{r_q}^{(k\sigma)}} \\ \eta|(M \setminus U_k) &= 0. \end{aligned}$$

We shall compute the difference $\beta - [\xi, \eta]$. At any point $x \notin U_k$ we have

$$\beta_x - [\xi, \eta]_x = 0.$$

Further we compute $\beta - [\xi, \eta]$ on $U_{k\sigma}$. For the sake of simplicity we denote $y_p = x_{r_p}^{(k\sigma)}$,

$$x_n = x_n^{(k\sigma)}.$$

$$\begin{aligned}
(\beta - [\xi, \eta])|U_{k\sigma} &= \\
& \sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - [dy_1 \wedge \cdots \wedge dy_i \otimes y_q \frac{\partial}{\partial y_q}, -dy_{i+1} \wedge \cdots \wedge dy_{i+j} \otimes \sigma f \frac{\partial}{\partial y_q}] = \\
& \sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} + dy_1 \wedge \cdots \wedge dy_{i+j} \otimes [y_q \frac{\partial}{\partial y_q}, \sigma f \frac{\partial}{\partial y_q}] - \\
& ((\mathcal{L}_{\sigma f \frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}) \otimes y_q \frac{\partial}{\partial y_q} = \\
& \sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} + y_q \frac{\partial \sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - \sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - \\
& y_q ((d\iota_{\sigma f \frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}) \otimes \frac{\partial}{\partial y_q} = \\
& y_q \frac{\partial \sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - \\
& y_q ((d\sigma f \wedge \iota_{\frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}) \otimes \frac{\partial}{\partial y_q} = \\
& y_q \frac{\partial \sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - \\
& y_q ((\frac{\partial \sigma f}{\partial y_q} dy_q \wedge \iota_{\frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}) \otimes \frac{\partial}{\partial y_q} - \\
& y_q \left(\left(\sum_{\substack{n=1 \\ n \neq r_q}}^m \frac{\partial \sigma f}{\partial x_n} dx_n \wedge \iota_{\frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i) \right) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j} \right) \otimes \frac{\partial}{\partial y_q} = \\
& y_q \frac{\partial \sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - y_q \frac{\partial \sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} + \\
& \sum_{\substack{n=1 \\ n \neq r_1, \dots, r_{i+j}}}^m ((-1)^q y_q \frac{\partial \sigma f}{\partial y_q} dx_n \wedge dy_1 \wedge \cdots \wedge \widehat{dy_q} \wedge \cdots \wedge dy_{i+j}) \otimes \frac{\partial}{\partial y_q} = \\
& \sum_{\substack{n=1 \\ n \neq r_1, \dots, r_{i+j}}}^m ((-1)^q y_q \frac{\partial \sigma f}{\partial y_q} dx_n \wedge dy_1 \wedge \cdots \wedge \widehat{dy_q} \wedge \cdots \wedge dy_{i+j}) \otimes \frac{\partial}{\partial y_q}.
\end{aligned}$$

Now we define for $n \neq r_1, \dots, r_{i+j}$ an element $\lambda_n \in L_{i+j}$ by the formula

$$\begin{aligned}
\lambda_n|U_{k\sigma} &= ((-1)^q x_{r_q}^{(k\sigma)} \frac{\partial \sigma f}{\partial x_{r_q}^{(k\sigma)}} dx_n^{(k\sigma)} \wedge dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge \widehat{dx_{r_q}^{(k\sigma)}} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)}) \otimes \frac{\partial}{\partial x_{r_q}^{(k\sigma)}} \\
\lambda_n|(M \setminus U_k) &= 0.
\end{aligned}$$

We can easily see that

$$\beta - [\xi, \eta] = \sum_{\substack{n=1 \\ n \neq r_1, \dots, r_{i+j}}}^m \lambda_n.$$

But by virtue of the first part of the proof $\lambda_n \in [L_i, L_j]$. Thus we have $\beta - [\xi, \eta] \in [L_i, L_j]$, and consequently $\beta \in [L_i, L_j]$. This finishes the proof of the theorem.

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