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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 30. pp. [101]–112.

Persistent URL: http://dml.cz/dmlcz/701510

# NATURAL TRANSFORMATIONS OF AFFINORS INTO FUNCTIONS AND AFFINORS

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presented by Jacek Gancarzewicz

An affinor on a manifold $M$ is a tensor field of type (1,1) on $M$ which can be interpreted as an endomorphism $TM \longrightarrow TM$ of the tangent bundle covering the identity on $M$.

In this paper we give a characterization of the natural transformations of affinors into functions and affinors. In section 2 we prove that all natural transformations of affinors on $n$-dimensional manifolds into functions are of the form $F(a_1(t), ..., a_n(t))$, where $a_1(t), ..., a_n(t)$ denote the coefficients of the characteristic polynomial of $t$ and $F$ is a smooth function on $\mathbf{R}^n$. In section 3 we prove that all natural transformations of affinors (on $n$-dimensional manifolds) into itself are of the form

$$t \longrightarrow \sum_{i=1}^{n} F_i(a_1(t), ..., a_n(t)) \cdot t^{n-i}$$

where $F_1, ..., F_n$ are smooth functions on $\mathbf{R}^n$.

All manifolds and maps are assumed to be infinitely differentiable.

## 1. Natural transformations of tensor fields.

Let $p, q, r, s, n$ be positive integers. Let $M$ be an $n$-dimensional manifold. We denote by $\mathcal{X}^p_q M$ the space of tensor fields of type $(p,q)$ on $M$.

A family of maps $T_M : \mathcal{X}^p_q M \longrightarrow \mathcal{X}^r_s M$ is called a natural transformation of tensor fields if :

[0]This paper is in final form and no version of it will be submitted for publication elsewhere.

(1) for any $M$, any open $U \subset M$ and all $t_1, t_2 \in \mathcal{X}^p_q M$ the following implication

$$t_1|U = t_2|U \Longrightarrow T_M t_1|U = T_M t_2|U$$

is true,

(2) for any two $n$-dimensional manifolds $M, N$ and for every injective immersion $\varphi : M \longrightarrow N$ we have

$$\varphi_* \circ T_M = T_N \circ \varphi_*$$

Using Borel's lemma in a standard way (see [3]) we can easily verify that for tensor fields $t_1, t_2 \in \mathcal{X}^p_q M$ and a point $x \in M$ we have

$$j^\infty_x t_1 = j^\infty_x t_2 \Longrightarrow (T_M t_1)(x) = (T_M t_2)(x)$$

Let $k$ be either a positive integer or $\infty$ and let $L^{k+1}_n$ be the group of $(k+1)$-jets of local diffeomorphisms of $\mathbf{R}^n$ with source and target $0 \in \mathbf{R}^n$.

We denote $V_{p,q} = \bigotimes^p \mathbf{R}^n \otimes \bigotimes^q (\mathbf{R}^n)^*$. The linear group $GL(n, \mathbf{R})$ acts on $V_{p,q}$ in the natural way.

Let $V^k_{p,q} = J^k_0(\mathbf{R}^n, V_{p,q})$. If $X = j^k_0 t$ then $(t^{(0)}, t^{(1)}, t^{(2)}, \ldots)$ are coordinates of $X$, where

$$t^{(s)} = \left\{ \frac{\partial^s t^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial u^{k_1} \ldots \partial u^{k_s}} : i_1, \ldots, i_p, j_1, \ldots, j_q, k_1, \ldots, k_s = 1, \ldots, n \right\}$$

The group $L^{k+1}_n$ acts on $V^k_{p,q}$ in the natural way: if $\xi = j^{k+1}_0 \varphi$, $X = j^k_0 t$ then $\xi \cdot X$ is the $k$-jet at 0 of

$$\mathbf{R}^n \ni u \longrightarrow J_u(\varphi) \cdot t(u) \in V_{p,q}$$

where $J_u(\varphi)$ is the Jacobi matrix of $\varphi$ at $u$.

It is easy to verify that for a homothety

$$\kappa_c(u) = \frac{1}{c} \cdot u$$

where $c \in \mathbf{R} \backslash \{0\}$, the coordinates $(\bar{t}^{(0)}, \bar{t}^{(1)}, \bar{t}^{(2)}, \ldots)$ of $(j^{k+1}_0 \kappa_c) \cdot X$ are given by $\bar{t}^{(s)} = c^s \cdot t^{(s)}$ for $s = 0, 1, 2, \ldots$ .

A map $E : V^k_{p,q} \longrightarrow V_{r,s}$ is called equivariant if

$$E((j^{k+1}_0 \varphi) \cdot X) = J_0(\varphi) \cdot E(X)$$

for $j^{k+1}_0 \varphi \in L^{k+1}_n$, $X \in V^k_{p,q}$.

We have the following:

**Proposition 1.1.** *There is a one-to-one correspondence between natural transformations of tensor fields of type $(p,q)$ into tensor field of type $(r,s)$ and equivariant maps $E : V^{\infty}_{p,q} \longrightarrow V_{r,s}$ which satisfies the condition :*

*(3) for every open subset $\Omega \subset \mathbf{R}^n$ and every smooth $\gamma : \Omega \longrightarrow V_{p,q}$ the map*

$$\Omega \ni x \longrightarrow E(j^{\infty}_0(\gamma \circ \tau_x)) \in V_{r,s}$$

*is smooth, where $\tau_x : \mathbf{R}^n \ni y \longrightarrow y + x \in \mathbf{R}^n$ is the translation by vector $x$.*

*If $T$ is a natural transformation then the corresponding equivariant map $E_T$ is defined by*

$$E_T(j^{\infty}_0 t) = T_{\mathbf{R}^n}(t)(0)$$

*If $E$ is an equivariant map, then the corresponding natural transformation $T^E$ is defined by*

$$(T^E_M t)(x) = (T^r_s \varphi^{-1})(E(j^{\infty}_0(\varphi_* t)))$$

*where $\varphi$ is a local system of coordinates on $M$ such that $\varphi(x) = 0$.*

The one-to-one correspondence between natural transformations and equivariant maps is formulated in Krupka's theorem [2]. We prove only that for a natural transformation $T$ the corresponding equivariant map $E_T$ satisfies the condition (3) and that for every equivariant map $E$ which satisfies the condition (3) and for every tensor fields $t$ of type $(p,q)$ on an $n$-dimensional manifolds $M$ the map $T^E_M(t)$ is smooth.

We have

$$E_T(j^{\infty}_0(\gamma \circ \tau_x)) = E_T(j^{\infty}_0((\tau_{-x})_* \gamma))$$
$$= T_{\mathbf{R}^n}((\tau_{-x})_* \gamma)(0) = (\tau_{-x})_* T_{\mathbf{R}^n}(\gamma)(0) = T_{\mathbf{R}^n}(\gamma)(x)$$

Since $T_{\mathbf{R}^n}(\gamma)$ is smooth, $E_T$ satisfies the condition (3).

Now let us suppose that an equivariant map $E$ satisfied (3) and that $\varphi : U \longrightarrow \mathbf{R}^n$ is a local system of coordinates on $M$ such that $\varphi(x) = 0$. For every $y \in U$ the composition $\tau_{\varphi(y)} \circ \varphi$ is a local system of coordinates on $M$ and $(\tau_{-\varphi(y)} \circ \varphi)(y) = 0$. We have

$$(T^E_M t)(y) = T^r_s(\tau_{-\varphi(y)} \circ \varphi)^{-1}(E(j^{\infty}_0((\tau_{-\varphi(y)} \circ \varphi)_* t))) = T^r_s \varphi^{-1}(E(j^{\infty}_0((\varphi_* t) \circ \tau_{\varphi(y)})))$$

By (3) the map $f(z) = E(j^{\infty}_0(\varphi_* t \circ \tau_z))$ is smooth, hence $T^E_M t = \varphi^{-1}_* f$ is smooth.

A natural transformation $T$ of tensors of type $(p,q)$ into tensors of type $(r,s)$ has order $k$ if for any $n$-dimensional manifold $M$, any $x \in M$ and all $t_1, t_2 \in \mathcal{X}^p_q M$ the following implication

$$j^k_x t_1 = j^k_x t_2 \Longrightarrow (T_M t_1)(x) = (T_M t_2)(x)$$

holds.

From Proposition 1.1 we deduce that $T$ is of order $k$ if and only if the following implication

$$j_0^k s_1 = j_0^k s_2 \Longrightarrow E_T(j_0^\infty s_1) = E_T(j_0^\infty s_2)$$

holds for every smooth $s_1, s_2 : \mathbf{R}^n \longrightarrow V_{p,q}$.

We prove now the following:

**Proposition 1.2.** *If $p = r$ and $r = s$ then every natural transformation of tensors of type $(p, q)$ into tensors of type $(r, s)$ has order zero.*

To prove this proposition we need the following:

**Lemma 1.3.** *Let $f : \mathbf{R}^n \longrightarrow V_{p,q}$ be a smooth map such that support $f$ is compact. Then there is a smooth map $F : \mathbf{R}^n \longrightarrow V_{p,q}$ such that for any $i \in \mathbf{N}$ and for any $\alpha \in \mathbf{N}^n$*

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} F(\frac{1}{i}, 0, ..., 0) = \frac{1}{(2^{i-1})^{|\alpha|}} \cdot \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(0) \tag{4}$$

This lemma implies immediately that $F(0) = f(0)$ and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} F(0) = 0$$

for $|\alpha| > 0$.

*Proof:* Let $\varphi : \mathbf{R} \longrightarrow [0, 1]$ be a smooth function such that $\varphi(x) = 0$ if $x < -1 + \varepsilon$ and $\varphi(x) = 1$ if $x > -\varepsilon$ for some $\varepsilon > 0$. For $i \in \mathbf{N}$, $x \in \mathbf{R}^n$ we denote

$$\varphi_i(x) = \begin{cases} \varphi(i \cdot (i+1) \cdot (x_1 - \frac{1}{i})) & \text{if } x_1 \leq \frac{1}{i} \\ 1 - \varphi((i-1) \cdot i \cdot (x_1 - \frac{1}{i-1})) & \text{if } x_1 \geq \frac{1}{i} \text{ and } i > 1 \\ 1 & \text{if } x_1 \geq 1 \text{ and } i = 1 \end{cases}$$

$$f_i(x) = f(\frac{1}{2^{i-1}} \cdot (x_1 - \frac{1}{i}), \frac{1}{2^{i-1}} \cdot x_2, ..., \frac{1}{2^{i-1}} \cdot x_n)$$

and

$$F(x) = \begin{cases} \sum_{i=1}^\infty \varphi_i(x) \cdot f_i(x) & \text{if } x_1 > 0 \\ f(0) & \text{if } x_1 \leq 0 \end{cases}$$

The proof that $F$ is smooth and satisfies (4) is standard.

*Proof of Proposition 1.2:* Let $T$ be a natural transformation of tensors of type $(p, q)$ into tensors of type (r,s) and let $f$ be a smooth map $\mathbf{R}^n \longrightarrow V_{p,q}$ with compact support. For every $c \in \mathbf{R} \backslash \{0\}$ we have

$$E_T(f^{(0)}, c \cdot f^{(1)}, c^2 \cdot f^{(2)}, ...) = E_T((j_0^\infty \kappa_c) \cdot (j_0^\infty f)) = J_0(\kappa_c) \cdot E_T(j_0^\infty f) = E_T(j_0^\infty f)$$

Let $F$ be the function from Lemma 1.3. Then we have

$$E_T(j_0^\infty(F\circ\tau_{(\frac{1}{i},0,\dots,0)})) = E_T(f^{(0)},\frac{1}{2^{i-1}}\cdot f^{(1)},(\frac{1}{2^{i-1}})^2\cdot f^{(2)},\dots) = E_T(j_0^\infty f)$$

for every $i\in\mathbf{N}$. Since the map $x\longrightarrow E_T(j_0^\infty(F\circ\tau_x))$ is smooth, we obtain that

$$E_T(j_0^\infty f) = \lim_{i\to\infty} E_T(j_0^\infty f)$$

$$= \lim_{i\to\infty} E_T(j_0^\infty(F\circ\tau_{(\frac{1}{i},0,\dots,0)})) = E_T(j_0^\infty F) = E_T(f^{(0)},0,0,\dots)$$

Hence for all smooth $t_1,t_2:\mathbf{R}^n\longrightarrow V_{p,q}$ such that $j_0^0t_1=j_0^0t_2$ we have

$$E_T(j_0^\infty t_1) = E_T(t_1^{(0)},0,0,\dots) = E_T(t_2^{(0)},0,0,\dots) = E_T(j_0^\infty t_2)$$

## 2. Classification of natural transformations of affinors into functions.

If $L:V\longrightarrow V$ is an endomorphism of an $n$-dimensional vector space $V$ then $a_1(L),\dots,a_n(L)$ denote the coefficients of the characteristic polynomial

$$W_L(\lambda) = \det(\lambda\cdot id_V - L) = \lambda^n + a_1(L)\lambda^{n-1}+\dots+a_n(L)id_V$$

**Theorem 2.1** *There is a one-to-one correspondence between natural transformations of affinors into functions and all smooth functions $F:\mathbf{R}^n\longrightarrow\mathbf{R}$. The natural transformation corresponding to a function $F$ is defined by*

$$(T_Mt)(x) = F(a_1(t_x),\dots,a_n(t_x))$$

*for every an $n$-dimensional manifold $M$, $t\in\mathcal{X}_1^1, x\in M$.*

Propositions 1.1 and 1.2 ensure that Theorem 2.1 is equivalent to the following:

**Proposition 2.2.** *There is a one-to-one correspondence between all smooth functions $F:\mathbf{R}^n\longrightarrow\mathbf{R}$ and functions $G:\mathbf{R}^n\otimes(\mathbf{R}^n)^*\longrightarrow\mathbf{R}$ such that*

*(5) for every open set $\Omega\subset\mathbf{R}^n$ and for every smooth map $\gamma:\Omega\longrightarrow\mathbf{R}^n\otimes(\mathbf{R}^n)^*$ the composition $G\circ\gamma$ is smooth,*

*(6) for all matrices $X\in\mathbf{R}^n\otimes(\mathbf{R}^n)^*$, $A\in GL(n,\mathbf{R})$ we have $G(A\cdot X\cdot A^{-1})=G(X)$.*

*The function $G$ corresponding to the function $F$ is defined by*

$$G(X) = F(a_1(X),\dots,a_n(X)) \tag{7}$$

*for every matrix $X\in\mathbf{R}^n\otimes(\mathbf{R}^n)^*$.*

*Proof:* It is clear that for any smooth function $F:\mathbf{R}^n\longrightarrow\mathbf{R}$ the formula (7) defines a function such that the conditions (5) and (6) hold.

We need to show that for a function $G : \mathbf{R}^n \otimes (\mathbf{R}^n)^* \longrightarrow R$ we can construct a function $F$ for which the equality (7) holds. It suffices to prove that for all matrices $X_1, X_2 \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$ the following implication

$$W_{X_1} = W_{X_2} \Longrightarrow G(X_1) = G(X_2) \tag{8}$$

holds. The condition (6) says that the function $G$ is constant on orbits of the group $GL(n, \mathbf{R})$. Let $J_i$ be Jordan's matrices equivalent to $X_i$ for $i = 1, 2$. The matrices $J_i$ are of the form

$$\begin{bmatrix}
\lambda_1 & & & & & & & & & & & & & \\
\varepsilon^i_{11} & \lambda_1 & & & & & & & & & & & & \\
 & \varepsilon^i_{12} & \ddots & & & & & & & & & & & \\
 & & & \lambda_2 & & & & & & & & & & \\
 & & & \varepsilon^i_{21} & \lambda_2 & & & & & & & & & \\
 & & & & \varepsilon^i_{22} & \ddots & & & & & & & & \\
 & & & & & & \ddots & & & & & & & \\
 & & & & & & & A_1 & & & & & & \\
 & & & & & & & E^i_{11} & A_1 & & & & & \\
 & & & & & & & & E^i_{12} & \ddots & & & & \\
 & & & & & & & & & & A_2 & & & \\
 & & & & & & & & & & E^i_{21} & A_2 & & \\
 & & & & & & & & & & & E^i_{22} & \ddots & \\
 & & & & & & & & & & & & & \ddots
\end{bmatrix}$$

where $\varepsilon^i_{11}, \varepsilon^i_{12}, \ldots, \varepsilon^i_{21}, \varepsilon^i_{22}, \ldots, \ldots$ are either 0 or 1 and $E^i_{11}, E^i_{12}, \ldots, E^i_{21}, E^i_{22}, \ldots, \ldots$ are either

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}, A_2 = \begin{bmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & \alpha_2 \end{bmatrix}, \ldots$$

The coefficients $\lambda_1, \lambda_2, \ldots$ and the matrices $A_1, A_2, \ldots$ are the same in both matrices $J_1$ and $J_2$ because $W_{J_1} = W_{X_1} = W_{X_2} = W_{J_2}$ and $\lambda_1, \lambda_2, \ldots, \alpha_1 - i\beta_1, \alpha_1 + i\beta_1, \alpha_2 - i\beta_2, \alpha_2 + i\beta_2, \ldots$ are the eigenvalues. Let us denote

$$P : \mathbf{R}^n \ni t \longrightarrow t_1 \cdot J_1 + (1 - t_1) \cdot J_2 \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$$

For every $t \in \mathbf{R}^n$ the matrix $P(t)$ has the same characteristic polynomial as the matrices $J_1, J_2$. Clearly all matrices having the same characteristic polynomial are included in a finite number of orbits, because every orbit holds Jordan's matrix and there is a finite number of systems $\varepsilon_{11}, \varepsilon_{12}, ..., \varepsilon_{21}, \varepsilon_{22}, ..., ...,$ $E_{11}, E_{12}, ..., E_{21}, E_{22}, ..., ....$ Hence $(G \circ P)(\mathbf{R}^n)$ is a finite set. From (5) the composition $G \circ P$ is the continuous function. Hence $G \circ P$ is a constant function. In particular $(G \circ P)(1, 0, ..., 0) = (G \circ P)(0, 0, ..., 0)$ and the condition (8) is satisfied.

We denote

$$S : \mathbf{R}^n \ni x \longrightarrow \begin{bmatrix} 0 & & & -x_n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -x_2 \\ & & 1 & -x_1 \end{bmatrix} \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$$

It is easily seen that $a_i(S(x)) = x_i$ for $i = 1, ..., n$ and $F = G \circ S$. Hence $F$ is unique and smooth, as the condition (5) is satisfied.

## 3. Classification of natural transformations of affinors into affinors.

**Theorem 3.1.** *There is a one-to-one correspondence between natural transformations of affinors into affinors and all systems of n smooth functions $F_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i = 1, ..., n$. The natural transformation corresponding to functions $F_i$ is defined by*

$$(T_M t)(x) = \sum_{i=1}^{n} F_i(a_1(t_x), ..., a_n(t_x)) \cdot t_x^{n-i}$$

*for every an n-dimensional manifold $M$, $t \in \mathcal{X}_1^1 M$, $x \in M$, where $t_x^k = t_x \circ ... \circ t_x$ (k times).*

Propositions 1.1 and 1.2 ensure that Theorem 3.1 is equivalent to following:

**Proposition 3.2** *There is a one-to-one correspondence between all systems of n smooth functions $F_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i = 1, ..., n$ and maps $G : \mathbf{R} \otimes (\mathbf{R}^n)^* \longrightarrow \mathbf{R}^n \otimes (\mathbf{R}^n)^*$ such that*

*(9) for every open set $\Omega \subset \mathbf{R}^n$ and every smooth map $\gamma : \Omega \longrightarrow \mathbf{R}^n \otimes (\mathbf{R}^n)^*$ the composition $G \circ \gamma$ is smooth,*

*(10) for all matrices $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$, $A \in GL(n, \mathbf{R})$ we have $G(A \cdot X \cdot A^{-1}) = A \cdot G(X) \cdot A^{-1}$.*

*The map $G$ corresponding to functions $F_i$ is defined by*

$$G(X) = \sum_{i=1}^{n} F_i(a_1(X), ..., a_n(X)) \cdot X^{n-i} \tag{11}$$

*for every matrix $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$.*

*Proof:* It is sufficient to show that for every $G$ satisfying (9) and (10) there are the unique smooth functions $F_i$ for $i = 1, ..., n$ such that the equality (11) holds. If $F_i$ satisfy (11) then for $x \in \mathbf{R}^n$ we have

$$G(S(x)) = \sum_{i=1}^{n} F_i(x) \cdot (S(x))^{n-1} = \begin{bmatrix} F_n(x) & \dots \\ \vdots & \\ F_1(x) & \dots \end{bmatrix}$$

because $(S(x))^i(\mathrm{e}_1) = \mathrm{e}_{i+1}$ for $i = 1, ..., n-1$ where $\mathrm{e}_1, ..., \mathrm{e}_n$ denotes the canonical basis in $\mathbf{R}^n$. Hence the functions $F_i$ are unique. Let us define

$$F_i(x) = G(S(x))_1^{n-i+1} \tag{12}$$

for $i = 1, ..., n$. Clearly $F_i$ are smooth from (9). We only need to show that $F_i$ satisfy (11).

At first we prove (11) for a matrix $X$ which has $n$ different eigenvalues. We need following:

**Lemma 3.3.** *Let us suppose that $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$. If the matrix $X$ has $n$ different eigenvalues then $X^{n-i}$ for $i = 1, ..., n$ are linearly independent.*

*Proof:* From Jordan's theorem the matrix

$$Y = \begin{bmatrix} 0 & & & -a_n(X) \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -a_2(X) \\ & & 1 & -a_1(X) \end{bmatrix}$$

is equivalent to the matrix $X$ because $X$ has $n$ different eigenvalues. Assume $Y = A \cdot X \cdot A^{-1}$ where $A \in GL(n, \mathbf{R})$ and $\sum_{i=1}^{n} \lambda_i \cdot X^{n-i} = 0$, then

$$0 = A \cdot (\sum_{i=1}^{n} \lambda_i \cdot X^{n-i}) \cdot A^{-1} = \sum_{i=1}^{n} \lambda_i \cdot (A \cdot X \cdot A^{-1})^{n-i} = \sum_{i=1}^{n} \lambda_i \cdot Y^{n-i} = \begin{bmatrix} \lambda_n & \dots \\ \vdots & \\ \lambda_1 & \dots \end{bmatrix}$$

Hence $\lambda_i = 0$ for $i = 1, ..., n$. This prove the lemma.

If the matrix $X$ has $n$ different eigenvalues then from Jordan's theorem there exists $A \in GL(n, \mathbf{R})$ such that $X = A \cdot J \cdot A^{-1}$ where

$$J = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix} & & \\ & & & & \begin{bmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & \alpha_2 \end{bmatrix} & \\ & & & & & \ddots \end{bmatrix}$$

Let us denote

$$K = \begin{bmatrix} -1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

Clearly $K^{-1} = K$ and $K \cdot J \cdot K^{-1} = J$. From (10) we have $K \cdot G(J) \cdot K^{-1} = G(K \cdot J \cdot K^{-1}) = G(J)$. Multiplying an arbitrary matrix by $K$ on the left is equivalent to multiplying the first row of this matrix by -1. Multiplying an arbitrary matrix by $K$ on the right is equivalent to multiplying the first column of this matrix by -1. Hence the terms of the first row and first column of matrix $G(J)$ are equal to zero except for the term in the (1,1) entry.

Suppose $l$ denotes the integer such that the matrix

$$\begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}$$

is on the $l$th and $(l+1)$th rows and the $l$th and $(l+1)$th columns in the matrix $J$. We denote

$$L = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

where the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

is on the $l$th and $(l+1)$th rows and the $l$th and $(l+1)$th columns in the matrix $L$. Clearly $L^{-1} = L$ and $L \cdot J \cdot L^{-1} = J$. From (10) we have $L \cdot G(J) \cdot L^{-1} = G(L \cdot J \cdot L^{-1}) = G(J)$. Multiplying an arbitrary matrix by $L$ on the left is equivalent to multiplying the $l$th and $(l+1)$th rows of this matrix by -1. Multiplying an arbitrary matrix by $L$ on the right is equivalent to multiplying the $l$th and $(l+1)$th columns of this matrix by -1. Hence the terms of $l$th and $(l+1)$th rows and the $l$th and $(l+1)$th columns of the matrix $G(J)$ are equal to zero except for the terms in the $(l,l)$, $(l,l+1)$, $(l+1,l)$ $(l+1,l+1)$ entries.

Repeated application of the argument above enables us to write

$$G(J) = \begin{bmatrix} \eta_1 & & & & & \\ & \eta_2 & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} \gamma_1 & \delta_1 \\ \varepsilon_1 & \zeta_1 \end{bmatrix} & & \\ & & & & \begin{bmatrix} \gamma_2 & \delta_2 \\ \varepsilon_2 & \zeta_2 \end{bmatrix} & \\ & & & & & \ddots \end{bmatrix} \tag{13}$$

We denote

$$M = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

We see that for all $\alpha, \beta \in \mathbf{R}$ we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}\cdot\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}$$

It follows that $M\cdot J\cdot M^{-1}=N\cdot J\cdot N^{-1}$. From (10) we have $M\cdot G(J)\cdot M^{-1}=G(M\cdot J\cdot M^{-1})=G(N\cdot J\cdot N^{-1})=N\cdot G(J)\cdot N^{-1}$. We see that for all $\gamma,\delta,\varepsilon,\zeta\in\mathbf{R}$ we have

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}\gamma & \delta\\ \varepsilon & \zeta\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}^{-1}=\begin{bmatrix}\zeta & \varepsilon\\ \delta & \gamma\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\gamma & \delta\\ \varepsilon & \zeta\end{bmatrix}\cdot\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}\gamma & -\delta\\ -\varepsilon & \zeta\end{bmatrix}$$

From the equality $M\cdot G(J)\cdot M^{-1}=N\cdot G(J)\cdot N^{-1}$ and from (13) we conclude that

$$\begin{bmatrix}\zeta_1 & \varepsilon_1\\ \delta_1 & \gamma_1\end{bmatrix}=\begin{bmatrix}\gamma_1 & -\delta_1\\ -\varepsilon_1 & \zeta_1\end{bmatrix}$$

Hence $\zeta_1=\gamma_1$ and $\varepsilon_1=-\delta_1$. Repeated application of the above arguments enables us to write

$$G(J)=\begin{bmatrix}\eta_1 & & & & & \\ & \eta_2 & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix}\gamma_1 & -\delta_1\\ \delta_1 & \gamma_1\end{bmatrix} & & \\ & & & & \begin{bmatrix}\gamma_2 & -\delta_2\\ \delta_2 & \gamma_2\end{bmatrix} & \\ & & & & & \ddots\end{bmatrix} \tag{14}$$

Let $V$ be a set consisting of all matrices of the form (14) for $\eta_1,\eta_2,\dots,\gamma_1,\delta_1,\gamma_2,\delta_2,\dots\in\mathbf{R}$. The set $V$ is an $n$-dimensional linear space. By induction $J^k\in V$ for every $k\in\mathbf{N}$. By Lemma 3.3 the matrices $J^{n-i}$ for $i=1,\dots,n$ form a basis of $V$. Hence there is $\lambda_i$ such that $G(J)=\sum_{i=1}^{n}\lambda_i\cdot J^{n-i}$ as $G(J)\in V$. By (10) we have

$$G(X)=G(A\cdot J\cdot A^{-1})=A\cdot G(J)\cdot A^{-1}$$

$$=A\cdot(\sum_{i=1}^{n}\lambda_i\cdot J^{n-i})\cdot A^{-1}=\sum_{i=1}^{n}\lambda_i\cdot(A\cdot J\cdot A^{-1})^{n-i}=\sum_{i=1}^{n}\lambda_i\cdot X^{n-i}$$

We need only show that $\lambda_i=F_i(a_1(X),\dots,a_n(X))$ for $i=1,\dots,n$. From Jordan's theorem there exists $B\in GL(n,\mathbf{R})$ such that $S(a_1(X),\dots,a_n(X))=B\cdot X\cdot B^{-1}$ as matrix $X$ has $n$ different eigenvalues. Hence

$$G(S(a_1(X),\dots,a_n(X)))=B\cdot G(X)\cdot B^{-1}=B\cdot(\sum_{i=1}^{n}\lambda_i\cdot X^{n-i})\cdot B^{-1}$$

$$= \sum_{i=1}^{n} \lambda_i \cdot (B \cdot X \cdot B^{-1})^{n-i} = \sum_{i=1}^{n} \lambda_i \cdot (S(a_1(X), ..., a_n(X)))^{n-i} = \begin{bmatrix} \lambda_n & \cdots \\ \vdots & \\ \lambda_1 & \cdots \end{bmatrix}$$

From (12) we have that $\lambda_i = F_i(a_1(X), ..., a_n(X))$ for $i = 1, ..., n$. This proves (11) for the case of $X$ with $n$ different eigenvalues.

We next show (11) in general case. Let $X$ be an arbitrary matrix and let $Y$ be a matrix which has $n$ different eigenvalues. Let $P$ be an $n$-dimensional affine subspace in the $\mathbf{R}^n \otimes (\mathbf{R}^n)^*$ such that $X, Y \in P$. Suppose that W(Z) denotes the discriminant of the characteristic polynomial of a matrix $Z \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$. Then $W$ is a polynomial and $W(Z) \neq 0$ if and only if $Z$ has $n$ different eigenvalues. We have $W|P \neq 0$ because $W(Y) \neq 0$. Hence $Q = \{Z \in P | W(Z) \neq 0\}$ is a dense subset of $P$. We know that $G|Q = C|Q$ where $C(Z) = \sum_{i=1}^{n} F_i(a_1(Z), ..., a_n(Z)) \cdot Z^{n-i}$ for $Z \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$. Suppose $D$ denotes an affine parametrization of $P$. From (9) $G \circ D$ is smooth and $G|P = G \circ D \circ D^{-1}$ is smooth too. If two smooth maps are equal on a dense set then these maps are equal. In particular $G(X) = C(X)$. This ends the proof.

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