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NATURAL TRANSFORMATIONS OF AFFINORS INTO LINEAR FORMS

Jacek Dębecki

An affinor on a manifold M is a tensor field of type $(1,1)$ on M and a linear form on M is a tensor field of type $(0,1)$ on M .

In this paper we give a characterisation of the natural transformations of affinors into linear forms satisfying the regularity condition. In Section 2 we prove that these natural transformations are of the form

$$T_M(t) = \sum_{i=1}^n \sum_{j=1}^n f_{ij}(a_1(t), \dots, a_n(t)) \cdot d(a_i(t)) \circ t^{n-j}$$

where f_{ij} are smooth functions on $\mathbf{R}^n$ and $a_1(t), \dots, a_n(t)$ denote the coefficients of the characteristic polynomial of the linear endomorphism t .

In the proof of this theorem we will use a classification of the natural transformations of affinors into tensor fields of type $(2,2)$ which we give in Section 1.

All manifolds and maps are assumed to be infinitely differentiable.

1. Natural transformations of affinors into tensor fields of type $(2,2)$.

Let n, p, q, r, s be nonnegative integers. Let M be an n -dimensional manifold. We denote by $\mathcal{X}_q^p M$ the space of tensor fields of type (p, q) on M .

Definition 1.1. A family of maps $T_M : \mathcal{X}_q^p M \longrightarrow \mathcal{X}_s^r M$ is called a natural transformation of tensor fields of type (p, q) into tensor fields of type (r, s) if for any n -dimensional manifolds M, N , for $t \in \mathcal{X}_q^p M$, $u \in \mathcal{X}_q^p N$ and for any injective immersion $\varphi : M \longrightarrow N$ the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow \iota & & \downarrow u \\ T_q^p M & \xrightarrow{T_q^p \varphi} & T_q^p N \end{array}$$

⁰This paper is in final form and no version of it will be submitted for publication elsewhere.

implies the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ T_M(t) \downarrow & & \downarrow T_N(u) \\ T_t^* M & \xrightarrow{T_t^* \varphi} & T_t^* N \end{array}$$

If t is an affinor on M then $a_1(t), \dots, a_n(t) : M \longrightarrow \mathbf{R}$ denote the coefficients of the characteristic polynomial $\det(\lambda \cdot \text{id}_M - t) = \lambda^n + \sum_{i=1}^n a_i(t) \cdot \lambda^{n-i}$ and $t^k = t \circ \dots \circ t$ (k times) where t is interpreted as a linear endomorphism of $\mathcal{X}_0^1 M$.

If t is tensor of type (2,2) then we define the following four operations by the formulas:

$$\begin{aligned} (t_S^S)_{kl}^{ij} &= \frac{1}{4}(t_{kl}^{ij} + t_{kl}^{ji} + t_{lk}^{ij} + t_{lk}^{ji}) \\ (t_A^S)_{kl}^{ij} &= \frac{1}{4}(t_{kl}^{ij} + t_{kl}^{ji} - t_{lk}^{ij} - t_{lk}^{ji}) \\ (t_S^A)_{kl}^{ij} &= \frac{1}{4}(t_{kl}^{ij} - t_{kl}^{ji} + t_{lk}^{ij} - t_{lk}^{ji}) \\ (t_A^A)_{kl}^{ij} &= \frac{1}{4}(t_{kl}^{ij} - t_{kl}^{ji} - t_{lk}^{ij} + t_{lk}^{ji}) \end{aligned}$$

Theorem 1.2. *There is a one-to-one correspondence between natural transformations of affinors into tensor fields of type (2,2) and all system of $2n^2 - n$ smooth functions*

$$(1) \quad \begin{aligned} \alpha_{ij} : \mathbf{R}^n &\longrightarrow \mathbf{R} \text{ for } 0 \leq i \leq j \leq n-1 \\ \beta_{ij} : \mathbf{R}^n &\longrightarrow \mathbf{R} \text{ for } 0 \leq i < j \leq n-1 \\ \gamma_{ij} : \mathbf{R}^n &\longrightarrow \mathbf{R} \text{ for } 0 \leq i < j \leq n-1 \\ \delta_{ij} : \mathbf{R}^n &\longrightarrow \mathbf{R} \text{ for } 0 \leq i \leq j \leq n-2 \end{aligned}$$

The natural transformation T corresponding to the system of functions (1) is defined by

$$(2) \quad \begin{aligned} T_M(t) &= \sum_{0 \leq i \leq j \leq n-1} \alpha_{ij}(a_1(t), \dots, a_n(t)) \cdot (t^i \otimes t^j)_S^S \\ &+ \sum_{0 \leq i < j \leq n-1} \beta_{ij}(a_1(t), \dots, a_n(t)) \cdot (t^i \otimes t^j)_A^S \\ &+ \sum_{0 \leq i < j \leq n-1} \gamma_{ij}(a_1(t), \dots, a_n(t)) \cdot (t^i \otimes t^j)_S^A \\ &+ \sum_{0 \leq i \leq j \leq n-2} \delta_{ij}(a_1(t), \dots, a_n(t)) \cdot (t^i \otimes t^j)_A^A \end{aligned}$$

for any n -dimensional manifold M , $t \in \mathcal{X}_1^1 M$.

The group $GL(n, \mathbb{R})$ acts on $\bigotimes^p \mathbb{R}^n \otimes \bigotimes^q \mathbb{R}^{n*}$ on the right in the standard way.

In the paper [1] it is shown that the above theorem is equivalent to the following:

Lemma 1.3. *There is a one-to-one correspondence between all systems of $2n^2 - n$ smooth functions (1) and maps $E : \mathbb{R}^n \otimes \mathbb{R}^{n*} \longrightarrow \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ such that*

(3) $E(t \cdot A) = E(t) \cdot A$ for $t \in \mathbb{R}^n \otimes \mathbb{R}^{n*}$, $A \in GL(n, \mathbb{R})$,

(4) for every smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \otimes \mathbb{R}^{n*}$ the composition $E \circ f$ is smooth.

If the map E corresponds to the system of functions (1) then $E(t)$ is equal to the right hand side of the equality (2) for every $t \in \mathbb{R}^n \otimes \mathbb{R}^{n*}$.

Proof: It is clear that for every system of smooth functions (1) the right hand side of (2) defines a map E such that the conditions (3) and (4) hold. At first we prove that for every map E there exists at most one system of functions (1) such that $E(t)$ is equal to the right hand side of (2).

Let $E(t)$ is equal to the right hand side of (2). We denote

$$R : \mathbb{R}^n \ni x \longrightarrow \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -x_n & \dots & -x_2 & -x_1 \end{bmatrix} \in \mathbb{R}^n \otimes \mathbb{R}^{n*}$$

If we compute $(E(R(x)))_{pq}^s{}^{11}$ for p, q such that $1 \leq p \leq q \leq n$ then we obtain

$$(5) \quad \alpha_{rs}(x) = \begin{cases} (E(R(x)))_{s}^s{}^{11}{}_{r+1s+1} & \text{if } r = s \\ 2(E(R(x)))_{s}^s{}^{11}{}_{r+1s+1} & \text{if } r < s \end{cases}$$

for r, s such that $0 \leq r \leq s \leq n-1$.

If we compute $(E(R(x)))_{pq}^s{}^{11}$ for p, q such that $1 \leq p < q \leq n$ then we obtain

$$(6) \quad \beta_{rs}(x) = 2(E(R(x)))_{s}^s{}^{11}{}_{r+1s+1}$$

for r, s such that $0 \leq r < s \leq n-1$. Analogously, for

$$S : \mathbb{R}^n \ni x \longrightarrow \begin{bmatrix} 0 & & & -x_n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -x_2 \\ & & 1 & -x_1 \end{bmatrix} \in \mathbb{R}^n \otimes \mathbb{R}^{n*}$$

we obtain

$$(7) \quad \gamma_{rs}(x) = 2(E(S(x)))_{s}^s{}^{11}{}_{r+1s+1}$$

for r, s such that $0 \leq r < s \leq n-1$. If we compute $(E(R(x))_A^A)_{pq}^{12}$ for p, q such that $1 \leq p < q \leq n$ then we obtain

$$(8) \quad \delta_{0s}(x) = \begin{cases} 2(E(R(x))_A^A)_{1s+2}^{12} & \text{if } s = 0 \\ 4(E(R(x))_A^A)_{1s+2}^{12} & \text{if } s \neq 0 \end{cases}$$

for $s = 1, \dots, n-2$,

$$(9) \quad \delta_{rn-2}(x) = \begin{cases} 2(E(R(x))_A^A)_{r+1n}^{12} & \text{if } r = n-2 \\ 4(E(R(x))_A^A)_{r+1n}^{12} & \text{if } r \neq n-2 \end{cases}$$

for $r = 1, \dots, n-2$,

$$(10) \quad \delta_{rs}(x) = \begin{cases} 2(E(R(x))_A^A)_{r+1s+2}^{12} + \frac{1}{2}\delta_{r-1s+1}(x) & \text{if } r = s \\ 4(E(R(x))_A^A)_{r+1s+2}^{12} + \delta_{r-1s+1}(x) & \text{if } r \neq s \end{cases}$$

for r, s such that $1 \leq r \leq s \leq n-3$. We can compute the functions δ_{rs} for r, s such that $0 \leq r \leq s \leq n-2$ from the formulas (8)–(10) by induction on $\min\{r, n-s-2\}$. From the formulas (5)–(10) we conclude that for every map E there exists at most one system of functions (1) such that $E(t)$ is equal to the right hand side of (2).

Let now E be a map satisfying (3) and (4). We define the system of functions (1) by the formulas (5)–(10). These functions are smooth because if R and S are smooth then from (4) the compositions $E \circ R$ and $E \circ S$ are smooth. It is sufficient to show that for these functions the right hand side of equality (2) is equal to $E(t)$.

At first we consider the case of a matrix $t \in \mathbb{R}^n \otimes \mathbb{R}^{n*}$ with n different complex eigenvalues. In this case Jordan's theorem ensures that there exists $A \in GL(n, \mathbb{R})$ such that $t = J \cdot A$, where

$$J = \begin{bmatrix} \pi_1 & & & & \\ & \ddots & & & \\ & & \pi_p & & \\ & & & \begin{bmatrix} \varrho_1 & -\sigma_1 \\ \sigma_1 & \varrho_1 \end{bmatrix} & \\ & & & & \ddots \\ & & & & & \begin{bmatrix} \varrho_q & -\sigma_q \\ \sigma_q & \varrho_q \end{bmatrix} \end{bmatrix}$$

Let $Y \subset \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ be the vector space consisting of all tensors $F(J)$ for $F: \mathbb{R}^n \otimes \mathbb{R}^{n*} \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ which satisfy the conditions (3) and (4).

We define the map $\varepsilon: \{1, \dots, n\} \rightarrow \{1, \dots, p+q\}$ by the formula

$$\varepsilon(i) = \begin{cases} i & \text{if } i \leq p \\ \left[\frac{i+p+1}{2} \right] & \text{if } i > p \end{cases}$$

Let $a_r = p + 2r - 1$, $b_r = p + 2r$. We define two maps $\zeta_r : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ by the formula

$$\zeta_r(i) = \begin{cases} b_r & \text{if } i = a_r \\ a_r & \text{if } i = b_r \\ i & \text{if } i \neq a_r, i \neq b_r \end{cases}$$

and $\eta_r : \{1, \dots, n\}^4 \longrightarrow \{0, \dots, 4\}$ such that $\eta_r(i, j, k, l)$ is the number of terms of the sequence (i, j, k, l) which are equal to a_r .

A standard computation shows that $Y \subset Z$ where $Z \subset \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ is a vector space consisting of all tensors u such that for $i, j, k, l = 1, \dots, n$, $r = 1, \dots, q$

$$\begin{aligned} u_{kl}^{ij} &= 0 \text{ if } \{e(i), e(j)\} \neq \{e(k), e(l)\} \\ u_{\zeta_r(k)\zeta_r(l)}^{(i)\zeta_r(j)} &= (-1)^{\eta_r(i,j,k,l)} \cdot u_{kl}^{ij} \\ (u_S^S)_{a_r b_r}^{a_r b_r} &= \frac{(u_S^S)_{a_r a_r}^{a_r a_r} - (u_S^S)_{b_r b_r}^{a_r a_r}}{2} \\ (u_S^S)_{a_r b_r}^{a_r a_r} &= -(u_S^S)_{a_r a_r}^{a_r b_r} \end{aligned}$$

It is easy to compute that $\dim Z = 2n^2 - n$.

Repeated the argument demonstrating that the functions (1) are unique gives that the tensors

$$(11) \quad \begin{aligned} &(J^i \otimes J^j)_S^S \text{ for } 0 \leq i \leq j \leq n-1 \\ &(J^i \otimes J^j)_A^S \text{ for } 0 \leq i < j \leq n-1 \\ &(J^i \otimes J^j)_S^A \text{ for } 0 \leq i < j \leq n-1 \\ &(J^i \otimes J^j)_A^A \text{ for } 0 \leq i \leq j \leq n-2 \end{aligned}$$

are linearly independent. Hence the tensors (11) form a basis of the vector space Z and $E(J)$ is a linear combination of the tensors (11). From this we conclude that $E(t)$ is equal to the right hand side of (2).

We now turn to the general case. Let u be an arbitrary matrix and let v be a matrix which has n different complex eigenvalues. Let P be an n -dimensional affine subspace in the $\mathbf{R}^n \otimes \mathbf{R}^{n*}$ such that $u, v \in P$. Suppose that $D(w)$ denotes the discriminant of characteristic polynomial of matrix $w \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$. Then D is a polynomial and $D(w) \neq 0$ if and only if w has n different complex eigenvalues. We have $D|P \neq 0$ because $D(v) \neq 0$. Hence $Q = \{w \in P | D(w) \neq 0\}$ is a dense subset of P . We know that $E|Q = F|Q$ where $F(t)$ is equal to the right hand side of equality (2). Suppose G denotes an affine parametrization of P . From (4) $E \circ G$ is smooth and $E|P = E \circ G \circ G^{-1}$ is smooth too. If two smooth maps are equal on a dense set then these maps are equal. In particular $E(u) = F(u)$. This ends the proof.

2. Natural transformations of affinors into linear forms.

Definition 2.1. A natural transformation T of tensor fields of type (p, q) into tensor fields of type (r, s) satisfies the regularity condition if for a manifold M , an n -dimensional manifold N and smooth map $M \times N \ni (\alpha, x) \longrightarrow t_\alpha(x) \in T_q^p N$ such that $t_\alpha \in \mathcal{X}_q^p N$ for every $\alpha \in M$, the map $M \times N \ni (\alpha, x) \longrightarrow T_N(t_\alpha)(x) \in T_s^r N$ is smooth.

We can now formulate our main result.

Theorem 2.2. There is a one-to-one correspondence between natural transformations of affinors into linear forms satisfying the regularity condition and all systems of n^2 smooth functions $f_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i, j = 1, \dots, n$.

The natural transformations T corresponding to the functions f_{ij} is defined by

$$T_M(t) = \sum_{i=1}^n \sum_{j=1}^n f_{ij}(a_1(t), \dots, a_n(t)) \cdot d(a_i(t)) \circ t^{n-j}$$

for any n -dimensional manifold M , $t \in \mathcal{X}_1^1 M$.

Let L_n^∞ be the group of infinite jets of local diffeomorphism of $\mathbf{R}^n$ with source and target $0 \in \mathbf{R}^n$. The group L_n^∞ acts on $J_0^\infty(\mathbf{R}^n, \otimes^p \mathbf{R}^n \otimes \otimes^q \mathbf{R}^{n*})$ on the right in the following way: if $J_x \varphi$ denotes the Jacobi matrix of φ at x then $j_0^\infty t \cdot j_0^\infty \varphi$ is equal to the infinite jet at 0 of $\mathbf{R}^n \ni x \longrightarrow t(x) \cdot J_x \varphi \in \otimes^p \mathbf{R}^n \otimes \otimes^q \mathbf{R}^{n*}$.

From Krupka's theorem (see [3]) we conclude that Theorem 2.2. is equivalent to the following:

Lemma 2.3. There is a one-to-one correspondence between all systems of n^2 smooth functions $f_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i, j = 1, \dots, n$ and maps $E : J_0^\infty(\mathbf{R}^n, \mathbf{R}^n \otimes \mathbf{R}^{n*}) \longrightarrow \mathbf{R}^{n*}$ such that

$$(12) \ E(j_0^\infty t \cdot j_0^\infty \varphi) = E(j_0^\infty t) \cdot J_0 \varphi \text{ for } j_0^\infty t \in J_0^\infty(\mathbf{R}^n, \mathbf{R}^n \otimes \mathbf{R}^{n*}), j_0^\infty \varphi \in L_n^\infty,$$

(13) for smooth map $\mathbf{R}^k \times \mathbf{R}^n \ni (\alpha, x) \longrightarrow t_\alpha(x) \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$ the map $\mathbf{R}^k \ni \alpha \longrightarrow E(j_0^\infty t_\alpha) \in \mathbf{R}^{n*}$ is smooth.

The map corresponding to the system of functions f_{ij} is defined by

$$(14) \quad E(j_0^\infty t) = \sum_{i=1}^n \sum_{j=1}^n f_{ij}(a_1(t), \dots, a_n(t)) \cdot d_0(a_i(t)) \circ t_0^{n-j}$$

Proof: It is clear that for every system of smooth functions f_{ij} the formula (14) defines a map E such that the conditions (12)-and (13) hold.

At first we prove that for every map E there exists at most one system of functions

f_{ij} such that the equality (14) holds. We denote for $i = 1, \dots, n$

$$R_i : \mathbf{R}^n \times \mathbf{R}^n \ni (\alpha, x) \longrightarrow \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \beta_n & \dots & \beta_2 & \beta_1 \end{bmatrix} \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$$

where

$$\beta_j = \begin{cases} -\alpha_j - x_1 & \text{if } j = i \\ -\alpha_j & \text{if } j \neq i \end{cases}$$

If we compute $E(j_0^\infty R_{p\alpha})_q$ for $p, q = 1, \dots, n$ then we obtain

$$(15) \quad f_{rs}(\alpha) = E(j_0^\infty R_{r\alpha})_{n-s+1}$$

for $r, s = 1, \dots, n$. From the formula (15) we conclude that for every map E there exists at most one system of functions f_{ij} such that the equality (14) is true.

Let now E be a map satisfying (12) and (13). We define the functions f_{ij} by the formula (15). From (13) it follows that these functions are smooth. It is sufficient to show that for these functions the equality (14) holds.

For $c \in \mathbf{R} \setminus \{0\}$ we define the homothety $\kappa_c : \mathbf{R}^n \ni x \longrightarrow \frac{1}{c}x \in \mathbf{R}^n$. A trivial verification shows that for a tensor field t of type (1,1) on $\mathbf{R}^n$

$$\frac{\partial^k (\kappa_{c*} t)_j^i}{\partial x^{k_1} \dots \partial x^{k_n}}(0) = c^k \cdot \frac{\partial^k t_j^i}{\partial x^{k_1} \dots \partial x^{k_n}}(0)$$

where $k = k_1 + \dots + k_n$ and for a tensor t of type (0,1) on $\mathbf{R}^n$ $t \cdot J_0 \kappa_c = c \cdot t$. From (12) we have $E(j_0^\infty (\kappa_{c*} t)) = c \cdot E(j_0^\infty t)$ and we know that the condition (13) is satisfied. Hence the theorem about homogeneous functions ensures that there exist smooth functions $G_{kl}^{ij} : \mathbf{R}^n \otimes \mathbf{R}^{n*} \longrightarrow \mathbf{R}$ for $i, j, k, l = 1, \dots, n$ such that

$$(16) \quad E(j_0^\infty t)_k = G_{k\gamma}^{\alpha\beta}(t(0)) \frac{\partial t_\alpha^\gamma}{\partial x^\beta}(0)$$

An easy computation shows that

$$(17) \quad G_{kl}^{ij}(t(0)) \cdot J_0 \varphi = \frac{\partial \varphi^i}{\partial x^\alpha}(0) \frac{\partial \varphi^j}{\partial x^\beta}(0) G_{\gamma\delta}^{\alpha\beta}(t(0)) \frac{\partial \varphi^{-1\gamma}}{\partial x^k}(0) \frac{\partial \varphi^{-1\delta}}{\partial x^l}(0)$$

$$(18) \quad -G_{kl}^{\alpha j}(t) t_\alpha^i - G_{kl}^{\alpha i}(t) t_\alpha^j + G_{k\alpha}^{ij}(t) t_l^\alpha + G_{\alpha k}^{ij}(t) t_l^\alpha = 0$$

Since the equality (17) is true, Lemma 1.3. now shows that there exist a system of $2n^2 - n$ smooth functions (1) such that $G(t)$ is equal to the right hand side of (2) for $t \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$.

It is easy to show that if we put either $G(t) = (t^p \otimes t^q)_S^S$ or $G(t) = (t^p \otimes t^q)_A^A$ then the equality (18) has a form $((t^p \otimes t^{q+1})_A^S - (t^{p+1} \otimes t^q)_A^S)_{kl}^{ij} = 0$. If we put either $G(t) = (t^p \otimes t^q)_A^S$ or $G(t) = (t^p \otimes t^q)_S^A$ then the equality (18) has a form $((t^p \otimes t^{q+1})_S^S - (t^{p+1} \otimes t^q)_S^S)_{kl}^{ij} = 0$. Using these formulas and the equalities $t^n = -\sum_{i=1}^n a_i(t) \cdot t^{n-i}$, $(t^q \otimes t^p)_S^S = (t^p \otimes t^q)_S^S$, $(t^q \otimes t^p)_A^S = -(t^p \otimes t^q)_A^S$ we can represent the left hand side of the equality (18) for G such that $G(t)$ is equal to the right hand side of (2) as a linear combination of

$$\begin{aligned} & (t^i \otimes t^j)_S^S \text{ for } 0 \leq i \leq j \leq n-1 \\ & (t^i \otimes t^j)_A^S \text{ for } 0 \leq i < j \leq n-1 \\ & (t^i \otimes t^j)_S^A \text{ for } 0 \leq i < j \leq n-1 \\ & (t^i \otimes t^j)_A^A \text{ for } 0 \leq i \leq j \leq n-2 \end{aligned}$$

We know that the coefficients of this linear combination are unique. Therefore these coefficients are equal to zero. If we compute these coefficients and denote

$$\begin{aligned} \bar{\alpha}_{ij}(t) &= \alpha_{ij}(a_1(t), \dots, a_n(t)) \\ \bar{\beta}_{ij}(t) &= \beta_{ij}(a_1(t), \dots, a_n(t)) \\ \bar{\gamma}_{ij}(t) &= \gamma_{ij}(a_1(t), \dots, a_n(t)) \\ \bar{\delta}_{ij}(t) &= \delta_{ij}(a_1(t), \dots, a_n(t)) \end{aligned}$$

then we obtain the following system of linear equations:

$$\begin{aligned} & \text{for } n = 2 \\ & \quad 2\bar{\alpha}_{00} - a_1\bar{\alpha}_{01} + 2a_2\bar{\alpha}_{11} + 2\bar{\delta}_{00} = 0 \\ & \quad -a_2(\bar{\beta}_{01} + \bar{\gamma}_{01}) = 0 \\ & \quad -a_1(\bar{\beta}_{01} + \bar{\gamma}_{01}) = 0 \\ & \quad -\bar{\beta}_{01} - \bar{\gamma}_{01} = 0, \\ & \text{for } n = 3 \\ & \quad 2\bar{\alpha}_{00} - a_2\bar{\alpha}_{01} + a_3\bar{\alpha}_{12} + 2\bar{\delta}_{00} = 0 \\ & \quad \bar{\alpha}_{01} - a_1\bar{\alpha}_{02} + \bar{\delta}_{01} = 0 \\ & \quad -\bar{\alpha}_{02} + 2\bar{\alpha}_{11} - a_1\bar{\alpha}_{12} + 2a_2\bar{\alpha}_{22} + 2\bar{\delta}_{11} = 0 \\ & \quad -a_3(\bar{\beta}_{02} + \bar{\gamma}_{02}) = 0 \\ & \quad -a_2\bar{\beta}_{02} - a_3\bar{\beta}_{12} - a_2\bar{\gamma}_{02} - a_3\bar{\gamma}_{12} = 0 \\ & \quad \bar{\beta}_{01} - a_1\bar{\beta}_{02} + \bar{\gamma}_{01} - a_1\bar{\gamma}_{02} = 0 \\ & \quad -\bar{\beta}_{12} - \bar{\gamma}_{12} = 0 \\ & \quad -\bar{\beta}_{01} - a_2\bar{\beta}_{12} - \bar{\gamma}_{01} - a_2\bar{\gamma}_{12} = 0 \\ & \quad -\bar{\beta}_{02} - a_1\bar{\beta}_{12} - \bar{\gamma}_{02} - a_1\bar{\gamma}_{12} = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} & \quad 2\bar{\alpha}_{00} - a_2\bar{\alpha}_{01} + a_3\bar{\alpha}_{12} + 2\bar{\delta}_{00} = 0 \\ & \quad \bar{\alpha}_{01} - a_1\bar{\alpha}_{02} + \bar{\delta}_{01} = 0 \\ & \quad -\bar{\alpha}_{02} + 2\bar{\alpha}_{11} - a_1\bar{\alpha}_{12} + 2a_2\bar{\alpha}_{22} + 2\bar{\delta}_{11} = 0 \\ & \quad -a_3(\bar{\beta}_{02} + \bar{\gamma}_{02}) = 0 \\ & \quad -a_2\bar{\beta}_{02} - a_3\bar{\beta}_{12} - a_2\bar{\gamma}_{02} - a_3\bar{\gamma}_{12} = 0 \\ & \quad \bar{\beta}_{01} - a_1\bar{\beta}_{02} + \bar{\gamma}_{01} - a_1\bar{\gamma}_{02} = 0 \\ & \quad -\bar{\beta}_{12} - \bar{\gamma}_{12} = 0 \\ & \quad -\bar{\beta}_{01} - a_2\bar{\beta}_{12} - \bar{\gamma}_{01} - a_2\bar{\gamma}_{12} = 0 \\ & \quad -\bar{\beta}_{02} - a_1\bar{\beta}_{12} - \bar{\gamma}_{02} - a_1\bar{\gamma}_{12} = 0, \end{aligned} \tag{20}$$

for $n = 4$

$$\begin{aligned}
 (21) \quad & 2\bar{\alpha}_{00} - a_3\bar{\alpha}_{02} + a_4\bar{\alpha}_{13} + 2\bar{\delta}_{00} = 0 \\
 & \bar{\alpha}_{01} - a_2\bar{\alpha}_{03} + a_4\bar{\alpha}_{23} + \bar{\delta}_{01} = 0 \\
 & \bar{\alpha}_{02} - a_1\bar{\alpha}_{03} + 2a_4\bar{\alpha}_{33} + \bar{\delta}_{02} = 0 \\
 & -\bar{\alpha}_{13} + 2\bar{\alpha}_{22} - a_1\bar{\alpha}_{23} + 2a_2\bar{\alpha}_{33} + 2\bar{\delta}_{22} = 0 \\
 & -\bar{\alpha}_{02} + 2\bar{\alpha}_{11} - a_2\bar{\alpha}_{13} + a_3\bar{\alpha}_{23} - \bar{\delta}_{02} + 2\bar{\delta}_{11} = 0 \\
 & \bar{\alpha}_{12} - a_3\bar{\alpha}_{03} - a_1\bar{\alpha}_{13} + 2a_3\bar{\alpha}_{33} + \bar{\delta}_{12} = 0 \\
 & -a_4(\bar{\beta}_{03} - \bar{\gamma}_{03}) = 0 \\
 & -a_3\bar{\beta}_{03} - a_4\bar{\beta}_{13} - a_3\bar{\gamma}_{03} - a_4\bar{\gamma}_{13} = 0 \\
 & \bar{\beta}_{02} - a_1\bar{\beta}_{03} + \bar{\gamma}_{02} - a_1\bar{\gamma}_{03} = 0 \\
 & \bar{\beta}_{01} - a_2\bar{\beta}_{03} - a_4\bar{\beta}_{23} + \bar{\gamma}_{01} - a_2\bar{\gamma}_{03} - a_4\bar{\gamma}_{23} = 0 \\
 & -\bar{\beta}_{23} - \bar{\gamma}_{23} = 0 \\
 & -\bar{\beta}_{01} - a_3\bar{\beta}_{13} - \bar{\gamma}_{01} - a_3\bar{\gamma}_{13} = 0 \\
 & -\bar{\beta}_{12} - a_2\bar{\beta}_{23} - \bar{\gamma}_{12} - a_2\bar{\gamma}_{23} = 0 \\
 & -\bar{\beta}_{13} - a_1\bar{\beta}_{23} - \bar{\gamma}_{13} - a_1\bar{\gamma}_{23} = 0 \\
 & -\bar{\beta}_{02} - a_2\bar{\beta}_{13} - a_3\bar{\beta}_{23} - \bar{\gamma}_{02} - a_2\bar{\gamma}_{13} - a_3\bar{\gamma}_{23} = 0 \\
 & \bar{\beta}_{12} - \bar{\beta}_{03} - a_1\bar{\beta}_{13} + \bar{\gamma}_{12} - \bar{\gamma}_{03} - a_1\bar{\gamma}_{13} = 0,
 \end{aligned}$$

for $5 \leq n$

$$\begin{aligned}
 (22) \quad & 2\bar{\alpha}_{00} - a_{n-1}\bar{\alpha}_{0n-2} + a_n\bar{\alpha}_{1n-1} + 2\bar{\delta}_{00} = 0 \\
 (23) \quad & \bar{\alpha}_{0n-2} - a_1\bar{\alpha}_{0n-1} + 2a_n\bar{\alpha}_{n-1n-1} + \bar{\delta}_{0n-2} = 0 \\
 (24) \quad & \bar{\alpha}_{0p-1} - a_{n-p}\bar{\alpha}_{0n-1} + a_n\bar{\alpha}_{pn-1} + \bar{\delta}_{0p-1} = 0 \text{ for } 2 \leq p \leq n-2 \\
 (25) \quad & -\bar{\alpha}_{n-3n-1} + 2\bar{\alpha}_{n-2n-2} - a_1\bar{\alpha}_{n-2n-1} + 2a_2\bar{\alpha}_{n-1n-1} \\
 & + 2\bar{\delta}_{n-2n-2} = 0 \\
 (26) \quad & -\bar{\alpha}_{p-1p+1} + 2\bar{\alpha}_{pp} - a_{n-p-1}\bar{\alpha}_{pn-1} + a_{n-p}\bar{\alpha}_{p+1n-1} \\
 & -\bar{\delta}_{p-1p+1} + 2\bar{\delta}_{pp} = 0 \text{ for } 1 \leq p \leq n-3 \\
 (27) \quad & \bar{\alpha}_{pn-2} - \bar{\alpha}_{p-1n-1} - a_1\bar{\alpha}_{pn-1} + 2a_{n-p}\bar{\alpha}_{n-1n-1} \\
 & + \bar{\delta}_{pn-2} = 0 \text{ for } 1 \leq p \leq n-3 \\
 (28) \quad & -\bar{\alpha}_{p-1q} + \bar{\alpha}_{pq-1} - a_{n-q}\bar{\alpha}_{pn-1} + a_{n-p}\bar{\alpha}_{qn-1} \\
 & -\bar{\delta}_{p-1q} + \bar{\delta}_{pq-1} = 0 \text{ for } 1 \leq p, p+2 \leq q \leq n-2 \\
 (29) \quad & -a_n(\bar{\beta}_{0n-1} + \bar{\gamma}_{0n-1}) = 0 \\
 (30) \quad & -a_{n-1}\bar{\beta}_{0n-1} - a_n\bar{\beta}_{1n-1} - a_{n-1}\bar{\gamma}_{0n-1} - a_n\bar{\gamma}_{1n-1} = 0
 \end{aligned}$$

- $$\begin{aligned}
(31) \quad & \bar{\beta}_{0n-2} - a_1 \bar{\beta}_{0n-1} + \bar{\gamma}_{0n-2} - a_1 \bar{\gamma}_{0n-1} = 0 \\
(32) \quad & \bar{\beta}_{0p-1} - a_{n-p} \bar{\beta}_{0n-1} - a_n \bar{\beta}_{pn-1} \\
& + \bar{\gamma}_{0p-1} - a_{n-p} \bar{\gamma}_{0n-1} - a_n \bar{\gamma}_{pn-1} = 0 \text{ for } 2 \leq p \leq n-2 \\
(33) \quad & -\bar{\beta}_{n-2n-1} - \bar{\gamma}_{n-2n-1} = 0 \\
(34) \quad & \bar{\beta}_{p-1p} - a_{n-p} \bar{\beta}_{pn-1} - \bar{\gamma}_{p-1p} - a_{n-p} \bar{\gamma}_{pn-1} = 0 \text{ for } 1 \leq p \leq n-2 \\
(35) \quad & -\bar{\beta}_{n-3n-1} - a_1 \bar{\beta}_{n-2n-1} - \bar{\gamma}_{n-3n-1} - a_1 \bar{\gamma}_{n-2n-1} = 0 \\
& -\bar{\beta}_{p-1p+1} - a_{n-p-1} \bar{\beta}_{pn-1} - a_{n-p} \bar{\beta}_{p+1n-1} \\
(36) \quad & -\bar{\gamma}_{p-1p+1} - a_{n-p-1} \bar{\gamma}_{pn-1} - a_{n-p} \bar{\gamma}_{p+1n-1} = 0 \text{ for } 1 \leq p \leq n-3 \\
(37) \quad & \bar{\beta}_{pn-2} - \bar{\beta}_{p-1n-1} - a_1 \bar{\beta}_{pn-1} \\
& + \bar{\gamma}_{pn-2} - \bar{\gamma}_{p-1n-1} - a_1 \bar{\gamma}_{pn-1} = 0 \text{ for } 1 \leq p \leq n-3 \\
(38) \quad & -\bar{\beta}_{p-1q} + \bar{\beta}_{pq-1} - a_{n-q} \bar{\beta}_{pn-1} - a_{n-q} \bar{\beta}_{qn-1} - \bar{\gamma}_{p-1q} + \bar{\gamma}_{pq-1} \\
& - a_{n-q} \bar{\gamma}_{pn-1} - a_{n-p} \bar{\gamma}_{qn-1} = 0 \text{ for } 1 \leq p, p+2 \leq q \leq n-2
\end{aligned}$$

An trivial verification shows that there are $n^2 - n$ linearly independent equations in each of the systems (19), (20), (21). We now prove that there are $n^2 - n$ linearly independent equations in the system of linear equations (22)–(38).

We can compute $\bar{\delta}_{00}$ from (22), $\bar{\delta}_{0n-2}$ from (23), $\bar{\delta}_{0s}$ for $1 \leq s \leq n-3$ from (24) for $p = s+1$, $\bar{\delta}_{n-2n-2}$ from (25), $\bar{\delta}_{rn-2}$ for $1 \leq r \leq n-3$ from (27) for $p = r$. We next can compute $\bar{\delta}_{rs}$ for $1 \leq r \leq s \leq n-3$ by induction on r . Namely if $r = s$ then we compute $\bar{\delta}_{rs}$ from (26) for $p = r$, if $r+1 \leq s$ then we compute $\bar{\delta}_{rs}$ from (28) for $p = r, q = s+1$.

We can compute $\bar{\beta}_{n-2n-1}$ from (33), $\bar{\beta}_{n-3n-2}$ from (34) for $p = n-2$, $\bar{\beta}_{n-3n-1}$ from (35), $\bar{\beta}_{n-4n-3}$ from (34) for $p = n-3$, $\bar{\beta}_{n-4n-2}$ from (36) for $p = n-3$, $\bar{\beta}_{n-4n-1}$ from (37) for $p = n-3$. We next can compute $\bar{\beta}_{rs}$ for $r \leq n-5$ and $0 \leq r < s \leq n-1$ by induction on $n-r-4$. Namely if $r+1 = s$ then we compute $\bar{\beta}_{rs}$ from (34) for $p = r+1$, if $r+2 = s$ then we compute $\bar{\beta}_{rs}$ from (36) for $p = r+1$, if $s = n-1$ then we compute $\bar{\beta}_{rs}$ from (37) for $p = r+1$, if $r+3 \leq s$ and $s \leq n-2$ then we compute $\bar{\beta}_{rs}$ from (38) for $p = r+1, q = s$.

Therefore the equations (22)–(28) and (33)–(38) are linearly independent.

Let M be a set consisting of all smooth maps $H : \mathbf{R}^n \otimes \mathbf{R}^{n*} \rightarrow \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ such that $H(t \cdot A) = H(t) \cdot A$ for $t \in \mathbf{R}^n \otimes \mathbf{R}^{n*}, A \in GL(n, \mathbf{R})$. It is obvious that M is an R -module where R is a ring consisting of all smooth functions $F : \mathbf{R}^n \otimes \mathbf{R}^{n*} \rightarrow \mathbf{R}$ such that $F(t \cdot A) = F(t)$. Lemma 1.3. ensures that $\dim M = 2n^2 - n$. Let N be a

submodule of M consisting of all maps of the form

$$H(t) = \sum_{0 \leq i \leq j \leq n-1} \bar{\alpha}_{ij}(t) \cdot (t^i \otimes t^j)_S^S + \sum_{0 \leq i < j \leq n-1} \bar{\beta}_{ij}(t) \cdot (t^i \otimes t^j)_A^S \\ + \sum_{0 \leq i < j \leq n-1} \bar{\gamma}_{ij}(t) \cdot (t^i \otimes t^j)_S^A + \sum_{0 \leq i \leq j \leq n-2} \bar{\delta}_{ij}(t) \cdot (t^i \otimes t^j)_A^A$$

where $\bar{\alpha}_{ij}$, $\bar{\beta}_{ij}$, $\bar{\gamma}_{ij}$, $\bar{\delta}_{ij}$ denote arbitrary elements of R such that the system of linear equations (19) if $n = 2$, (20) if $n = 3$, (21) if $n = 4$, (22)-(38) if $5 \leq n$ holds. We see that $\dim N = (2n^2 - n) - (n^2 - n) = n^2$. Let $K_{ij} : \mathbf{R}^n \otimes \mathbf{R}^n \longrightarrow \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ be maps such that

$$(d_0(a_i \circ t) \circ t_0^{n-j})_k = K_{ij}(t(0))_{k\gamma}^{\alpha\beta} \frac{\partial t_\alpha^\gamma}{\partial x^\beta}(0)$$

for $j_0^\infty t \in J_0^\infty(\mathbf{R}^n, \mathbf{R}^n \otimes \mathbf{R}^{n*})$. It is evident that the maps K_{ij} are elements of N . Repeating the argument proving that the functions f_{ij} are unique give that the maps K_{ij} , for $i, j = 1, \dots, n$, are linearly independent. Hence these maps form a basis of the R -module N . Consequently $G \in N$ is a linear combination of K_{ij} and it follows immediately that the equality (14) is true. This finished the proof.

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