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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Topology". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 32. pp. [69]–79.

Persistent URL: http://dml.cz/dmlcz/701527

Γ-FOLIATIONS AND WEIL PROLONGATIONS

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In the paper we present a construction of the prolongation of a Γ- foliation on a manifold X to X^A - the Weil prolongation (the A- prolongation of the manifold X). Moreover, using the construction of Bott- Haefliger of the characteristic classes for Γ-foliations, we study relationships between the characteristic classes of Γ-foliations on X and the characteristic classes of Weil prolongations.

1. Basic remarks about Weil prolongations.

Let A be an algebra with 1 over $\mathbf{R}$. We say that A is loal if it is associative, commutative and of finite dimension over $\mathbf{R}$. Furthemore, in A there exists the unique maximal ideal m such that:

a) $\dim A/m = 1$,

b) there exists a number $h \in \mathbf{N}$ for which $m^{h+1} = 0$.

The smallest such h will be called the height of A. One can prove ([6]) that any local algebra is of the form $\mathbf{R}[p]/a$, where $\mathbf{R}[p] = \mathbf{R}[X_1, ..., X_p]$ is the algebra of all formal power series of p indeterminantes and a an ideal of $\mathbf{R}[p]$ such that

$$dim\mathbf{R}[p]/a < \infty$$

Let A be a local algebra with the maximal ideal m and $C^\infty(M)$ be the space of C^∞ functions on a manifold M. Let φ and ψ be two C^∞ maps from $\mathbf{R}^p$ to M. We say that these maps are A-equivalent if

$$\xi_A(\tau(f \circ \varphi)) = \xi_A(\tau(f \circ \psi)) \qquad \text{for any } f \in C^\infty(M)$$

[0]This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\tau = \tau_p$ is a map of the form

$$\tau : C^\infty(\mathbf{R}^p) \longrightarrow \mathbf{R}[p]$$

$$\tau(g) = \sum_{\nu \in \mathbf{N}^p} \frac{1}{\nu!}[D^\nu g](0)X^\nu$$

and ξ_A is the canonical projection of $\mathbf{R}[p]$ on A. The equivalence class of φ in this relation we denote by $[\varphi]_A$. By M^A we denote the set of all equivalence classes of C^∞ maps $\varphi : \mathbf{R}^p \to M$. We have the natural projection $\pi : M^A \to M$ defined by

$$\pi_A([\varphi]_A) = \varphi(0)$$

The structure of a manifold on a M^A we introduce in a natural way ([5], [6]). If $F : M \to N$ is a C^∞-map,, then we define $F^A : M^A \to N^A$ by the formula

$$F^A([\rho]_A) = [F \circ \rho]_A \qquad \text{for } [\rho]_A \in M^A$$

The correspondence $M \to M^A$ is a functor which has many important properties (see [5]. [6]).

The following proposition gives a topological relation between a manifold M and its A-prolongation M^A.

Proposition. 1. *If A is a local algebra, then M and M^A have the same homotopy type.*

Proof. Denote by i the canonical imbedding of M in M^A defined by the formula $i(x) = [\gamma_x]_A$ where $\gamma_x : \mathbf{R}^p \to M$, $\gamma_x(t) = x$ for each $t \in \mathbf{R}^p$. Now we define a map

$$F : M^A \times \mathbf{R} \longrightarrow M^A$$

$$F([\varphi]_A, s) = [\varphi_s]_A$$

where $[\varphi_s] \in M^A$ is represented by a map φ_s and

$$\varphi_s(t) = \varphi((1-s)t) \qquad \text{for } t \in \mathbf{R}^p$$

The map F is, ofcourse, continuous, and

$$F|_{M^A \times \{0\}} = id_{M^A} \qquad F|_{M^A \times \{1\}} = i \circ \pi_A$$

Q.E.D.

Immediately, we have the folloving

Corollary. 1. *If A is a local algebra, then the de Rham cohomology complexes $H^*(M)$ and $H^*(M^A)$ are canonically isomorphic.*

2. A-prolongations of pseudogroups and foliations.

Let Γ be a pseudogroup of local diffeomorphisms of a manifold M. For any $g \in \Gamma$ we denote by $\mathcal{O}_g$ a family of local diffeomorphisms of M^A, which cover g. Then the set

$$^A\Gamma = \bigcup_{g \in \Gamma} \mathcal{O}_g$$

ia a pseudogroup of local diffeomorphisms of M^A.

Before we define the A-prolongation of a foliation, we recall, a definition of a foliation, which we shall use. Let M be a differentiable manifold and Γ a pseudogroup of diffeomorphisms acting transitively on M. Suppose, that $^A\Gamma$ is a transitive Lie pseudogroup.

Actually we shall consider $M = \mathbf{R}$ and Γ a pseudogroup of local diffeomorphisms of $\mathbf{R}^n$.

To define a Γ-foliation on a manifold X we need the following data:

1) an open covering $\{U_i\}_{i \in I}$ of X,

2) a family $\mathcal{F}$ of submersions ("local projections") $f_i : U_i \to M$,

3) a family of local diffeomorphisms $g_{ij} \in \Gamma$ such that

$$g_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$$

and

$$g_{ij} \circ f_j|_{U_i \cap U_j} = f_i|_{U_i \cap U_j}$$

A map $f : X' \to X$ is transverse to $\mathcal{F}$ if the maps $f_i \circ f$ are submersions. In this case the maps $f_i \circ f$ are local projections of a Γ-foliation on X'. This foliation is called the inverse image $f^{-1}\mathcal{F}$ of $\mathcal{F}$ via f. The map f is a morphism from f^{-1} to $\mathcal{F}$. We can say that Γ-foliations form a category $Fol(\Gamma)$.

Now we shall construct the A-prolongation of a Γ-foliation $\mathcal{F}$.

Proposition. 2. *Let $\mathcal{F}$ be a Γ-foliation on X. There exists canonically defined a $^A\Gamma$-foliation $\mathcal{F}^A$ such that the correspondence $\mathcal{F} \to \mathcal{F}^A$ is a contravariant functor from $Fol(\Gamma)$ to $Fol(^A\Gamma)$.*

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of X, and $\{f_i\}_{i \in I}$ a family of submersions defining the foliation $\mathcal{F}$.

The family

$$\{U_i^A = \pi_A^{-1}(U_i) : U_i \in \{U_i\}_{i\in I}\}$$

is an open covering of X^A. Now we shall define the prolongation $\mathcal{F}^A$ of $\mathcal{F}$. As the family of submersions for $\mathcal{F}^A$ we can take the family $\{f_i^A\}$ where $f_i^A : U_i^A \to M^A$ The compatibility condition is fulfilled. **Q.E.D.**

If $f : X' \to X$ is a regular map transversal to $\mathcal{F}$, then $f^A : X'^A \to X^A$ is transversal to $\mathcal{F}^A$ and

$$(f^{-1}\mathcal{F})^A = f^{A^{-1}}(\mathcal{F}^A)$$

Thus we can give the following definition:

Definition. 1. *Let $\mathcal{F}$ be a Γ-foliation on X. The ${}^A\Gamma$-foliation $\mathcal{F}^A$ on X^A we call the A-prolongation or the Weil prolongation of $\mathcal{F}$.*

Now we shall define a homotopy of foliations. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two Γ-foliations on X. We denote by

$$i_t : X \longrightarrow X \times \mathbf{R}$$

$$x \mapsto (x, t)$$

the canonical inclusion. Two Γ-foliations are homotopic if there exists a Γ-foliation $\mathcal{F}$ on $X \times \mathbf{R}$ such that i_0 and i_1 are transversal to $\mathcal{F}$ and

$$i_0^{-1}\mathcal{F} = \mathcal{F}_0 \qquad i_1^{-1}\mathcal{F} = \mathcal{F}_1$$

The homotopy relation is in natural way, an equivalence relation. Denote by $Htp_\Gamma(X)$ the set of homotopy classes of Γ-foliations on X. If $f : X' \to X$ is a morphism in $Fol(\Gamma)$, then we obtain the induced map

$$Htp(f) : Htp_\Gamma(X) \longrightarrow Htp_\Gamma(X')$$

It is easy to prove that $Htp_\Gamma(\bullet)$ is a contravariant functor.

We still have some remarks about homotopy of foliations.

Proposition. 3. *Let A be a local algebra. If $\mathcal{F}_0$ and $\mathcal{F}_1$ are two homotopic Γ-foliations on X, then $\mathcal{F}_0^A$ and $\mathcal{F}_1^A$ are homotopic.*

The proof is easy consequence of definitions and properties of the Weil functor.

On the other hand we can define

Definition. 2. *Let A be a local algebra. Two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ are A-homotopic if their A-prolongations $\mathcal{F}_0^A$ and $\mathcal{F}_1^A$ are homotopic.*

This relation is an equivalence relation. Let $Htp_\Gamma^A(X)$ be the family of A-homotopy classes. As previously, $Htp_\Gamma^A(\bullet)$ is a contravariant functor. The following simple proposition is true.

Proposition. 4. *Let f_0, $f_1 : X' \to X$ be two homotopic maps. Then for a local algebra A, the maps f_0^A and f_1^A are homotopic.*

3. Characteristic classes of Γ-foliations and their prolongations.

Now we recall briefly the Bott-Haefliger construction of characteristic classes of Γ-foliations ([2], [4]). Let Γ be a Lie pseudogroup acting transitively on M. A vector field on M is called a Γ-field, if its local one parameter group consists of elements of Γ. Let $o \in M$ be a fixed point in M. The set of k-jets at o of Γ-fields is a vector space denoted by $\underline{\Gamma}^k$ i.e.

$$\underline{\Gamma}^k = \{j_0^k v : v \in \mathcal{X}_\Gamma(M)\}$$

where $\mathcal{X}_\Gamma(M)$ is the space of Γ-fields on M.

Now $\underline{\Gamma} = \varprojlim \underline{\Gamma}^k$ is a Lie algebra called the Lie algebra of formal Γ-fields.

Let us denote by $\mathcal{A}(\underline{\Gamma})$ the inductive limit of the algebras $\mathcal{A}(\underline{\Gamma}^k)$ of multilinear antysymetric forms on $\underline{\Gamma}^k$. The bracket on $\underline{\Gamma}$ induces a differential on $\mathcal{A}(\underline{\Gamma})$, and we obtain the cohomology groups $H^*(\underline{\Gamma})$.

Let

$$J_0^k(\Gamma) = \{j_0^k\varphi : \varphi \in \Gamma\}$$

and

$$\Gamma_0^k = \{j_0^k \in J_0^k(\Gamma) : \varphi(0) = 0\}$$

Γ_0^k acts on the right on $J_0^k(\Gamma)$, and $J_0^k(\Gamma)$ is a principal fibre bundle with base M and structure group Γ_0^k. Take

$$J_0^\infty(\Gamma) = \varprojlim J_0^k(\Gamma)$$

On $J_0^\infty(\Gamma)$ we can introduce a structure of a differentiable manifold: the map $f : X \to J_0^\infty(\Gamma)$ is regular i.e. C^∞ if for any k, $\pi_k \circ f$ is regular, where

$$\pi_k : J_0^\infty(\Gamma) \to J_0^k(\Gamma)$$

is the canonical projection.

$J_0^\infty(\Gamma)$ has a structure of a principal fibre bundle with the structure group Γ_0^∞. Let $\mathcal{A}(J_0^\infty(\Gamma)$ be an algebra of differential forms on $J_0^\infty(\Gamma)$ defined as

$$\varprojlim \mathcal{A}(J_0^k(\Gamma))$$

Then we have the following

Proposition. 5. *$\mathcal{A}(\underline{\Gamma})$ is canonically isomorphic to the algebra of differential forms on $J_0^\infty(\Gamma)$, which are invariant the action of Γ. This isomorphism comutes with the differential operator ([4]).*

Now, let K^r be a maximal compact subgroup in Γ_0^r and let

$$K = \varprojlim K^s$$

Then $\mathcal{A}(\underline{\Gamma}, K)$ is a subcomplex of K-basic forms in $\mathcal{A}(\underline{\Gamma})$, and its cohomology group we denote by $H^*(\underline{\Gamma}, K)$.

The following theorem is true

Theorem. 1. *Let $\mathcal{F}$ be a Γ-foliation on X. There exists a homomorphism of algebras $\varphi_{\mathcal{F}} : H^*(\underline{\Gamma}, K) \to H^*(X)$ such that if $f : X' \to X$ is transversal to $\mathcal{F}$ then*

$$f^* \bullet \varphi_{\mathcal{F}} = \varphi_{f^{-1}\mathcal{F}}$$

Definition. 3. *The set $im\varphi_{\mathcal{F}}$ is called the set of characteristic classes of a foliation $\mathcal{F}$.*

Proposition. 6. *If $\mathcal{F}_0$ and $\mathcal{F}_1$ are homotopic Γ-foliations on X, then*

$$im\varphi_{\mathcal{F}_0} = im\varphi_{\mathcal{F}_1}$$

Now we can formulate the main theorem of this paper.

Theorem. 2. *Let A be a local algebra and $\mathcal{F}_0$, $\mathcal{F}_1$ two Γ-foliations on X. If $\mathcal{F}_0$ and $\mathcal{F}_1$ are A-homotopic, then*

$$im\varphi_{\mathcal{F}_0} = im\varphi_{\mathcal{F}_1}$$

This theorem is the generalisation of the analogous theorem due to L. A. Cordero in [3]. It is a consequence of the following theorem

Theorem. 3. *Let $\mathcal{F}$ be a* Γ*-foliation on* X, *and* A *a local algebra, then*

$$im\varphi_{\mathcal{F}} = i^{*} im\varphi_{\mathcal{F}^A}$$

wher $i^* = i^*_X$ *is the isomorhism induced by the inclusion*

$$i : X \to X^A$$

To prove this theorem we use the following technical Lemma:

Lemma. 1. *Let* A *be a local algebra,* $\mathcal{F}$ *a* Γ*-foliation on* X *and* $\mathcal{F}^A$ *that of its* A*-prolongation, then*

a) there exists a canonical homomorphism

$$\sigma : H^*(\underline{\Gamma}, K) \to H^*({}^A\underline{\Gamma}, {}^A K)$$

such that the diagram

$$\begin{array}{ccc} H^*({}^A\underline{\Gamma}, {}^A K) & \xrightarrow{\varphi_{\mathcal{F}^A}} & H^*(X^A) \\ \sigma \uparrow & & \downarrow i^*_X \\ H^*(\underline{\Gamma}, K) & \xrightarrow{\varphi_{\mathcal{F}}} & H^*(X) \end{array}$$

is commutative,

b) there exists a canonical homomorphism

$$\tau : H^*({}^A\underline{\Gamma}, {}^A K) \to H^*(\underline{\Gamma}, K)$$

such that the diagram

$$\begin{array}{ccc} H^*({}^A\underline{\Gamma}, {}^A K) & \xrightarrow{\varphi_{\mathcal{F}^A}} & H^*(X^A) \\ \tau \downarrow & & \uparrow (\pi_A)^* \\ H^*(\underline{\Gamma}, K) & \xrightarrow{\varphi_{\mathcal{F}}} & H^*(X) \end{array}$$

is commutative

c)

$$\tau \circ \sigma = id_{H^*(\underline{\Gamma}, K)}$$

Proof. Let $i_M(o) = \widetilde{o} \in M^A$. For any $k \geq 0$ take

$$\sigma_k : J^k_{\widetilde{o}}({}^A\Gamma) \to J^k_o(\Gamma)$$

defined in the following way: if $j^k_{\underset{o}{-}}({}^Af) \in J^k_{\underset{o}{-}}({}^A\Gamma)$, where ${}^Af$ is an element of ${}^A\Gamma$, which cover one f, then we put

$$\sigma_k(j^k_{\underset{o}{-}}({}^Af) = j^k_o(f)$$

. It is easy to prove, that σ_k is well defined. This map induces a homomorphism of Lie groups

$$\sigma_k : {}^A\Gamma^k_{\underset{o}{-}} \to \Gamma^k_o$$

and further we have the morphism of fibre bundles

$$\begin{array}{ccc} J^k_{\underset{o}{-}}({}^A\Gamma) & \xrightarrow{\sigma_k} & J^k_o(\Gamma) \\ \downarrow & & \downarrow \\ M^A & \xrightarrow{\pi_A} & M \end{array}$$

For any ${}^Af \in {}^A\Gamma$ such that ${}^Af \in \mathcal{O}_f$, let $\lambda_{{}^Af}$ and λ_f be differential transformations of $J^k_{\underset{o}{-}}({}^A\Gamma)$ and $J^k_o(\Gamma)$ respectively, defined by the left action of ${}^Af$ and f respectively.
The following equality is true

$$\lambda_f \circ \sigma_k = \sigma_k \circ \lambda_{{}^Af}$$

Further, the induced homomorphism of algebras of differential forms we denote also by σ_k

$$\sigma_k : \mathcal{A}(J^k_o(\Gamma)) \to \mathcal{A}(J^k_{\underset{o}{-}}({}^A\Gamma))$$

which invariant forms under the action Γ sends to forms invariant under the action ${}^A\Gamma$, and consequently we have got

$$\sigma : \mathcal{A}(J^\infty_o(\Gamma)) \to \mathcal{A}(J^\infty_{\underset{o}{-}}({}^A\Gamma))$$

which induces (by proposition 5)

$$\sigma : \mathcal{A}(\underline{\Gamma}) \to \mathcal{A}({}^A\underline{\Gamma})$$

The mapping σ defines two new homomorphisms, denoted also by σ.

$$\sigma : \mathcal{A}(\underline{\Gamma}, K) \to \mathcal{A}({}^A\underline{\Gamma}, {}^AK)$$

and

$$\sigma : H^*(\underline{\Gamma}, K) \to H^*({}^A\underline{\Gamma}, {}^AK)$$

For the proof of the commutativity of hte diagram, it suffice to prove commutativity of the following diagram

$$\begin{array}{ccccc}
\mathcal{A}(J^k_{\underset{o}{\sim}}(^A\Gamma) & \xrightarrow{^A\eta} & \mathcal{A}(P^k(\mathcal{F}^A)|_{U^A} & \xrightarrow{^Ap} & \mathcal{A}(U^A) \\
\sigma_k \uparrow & & & & \downarrow i^*_U \\
\mathcal{A}(J^k_o(\Gamma)) & \xrightarrow{\eta} & \mathcal{A}(P^k(\mathcal{F})|_U) & \xrightarrow{p} & \mathcal{A}(U)
\end{array}$$

where U is an open set in X, $P^k(\mathcal{F})|_U$ and $P^k(\mathcal{F}^A)|_{U^A}$ are restrictions to U and U^A, respectively the fibre bundles of k-jets of lokal projections of $\mathcal{F}$ and $\mathcal{F}^A$, respectively, p and ^Ap are the homomorphisms induced by local inclusions, and, at last, η and $^A\eta$ are the maps induced by the identification of $J^k_o(\Gamma)$ and $J^k_{\underset{o}{\sim}}(^A\Gamma)$ with $P^k(\mathcal{F})|_U$ and $P^k_{\underset{o}{\sim}}(\mathcal{F}^A)|_{U^A}$ respectively.

The inclusion

$$j_U : U \to P^k(\mathcal{F})|_U$$

we can define in the following way: if $f_U : U \to M$ is a local submersion of $\mathcal{F}$, then for each $\in U$

$$j_U(x) = j^k_o(g^{-1} \circ f_U)$$

where $g \in \Gamma$ and $g(o) = f_U(x)$. The map j_{U^A} we define in the analogous way.

Let $\omega \in J^k_o(\Gamma)$, thus

$$p(\eta(\omega))|_x = \eta(\omega)|_{j^k_o(g^{-1} \circ f_U)} = \omega|_{j^k_o(g)}$$

If $\tilde{x} = i_U(x)$ then

$$i^{*}_U(^Ap(^A\eta(\sigma_k(\omega))))_x = ^A p(^A\eta(\sigma_k(\omega)))_{\tilde{x}} =$$

$$= ^A \eta(\sigma_k(\omega))|_{j^k_{\underset{o}{\sim}}((g^A)^{-1} \circ f^A_U} =$$

$$= \sigma_k(\omega)|_{j^k_{\underset{o}{\sim}}(g^A)} = \omega_{j^k_o(g)}$$

This finishes the proof of the point a).

b) In this case we construct the map τ. For $k \geq 0$ the map

$$\tau_k : J^{k+r}_o(\Gamma) \to J^k_{\underset{o}{\sim}}(^A\Gamma)$$

is defined by the equality

$$\tau_k(j^{k+r}_o(f) = j^k_{\underset{o}{\sim}}(f^A)$$

for $f \in \Gamma$, where r is the order of the natural bundle $X \to X^A$. It is easy to prove that τ_k is well defined. This τ_k induces a homomorphism denoted also by τ_k:

$$\tau_k : \mathcal{A}(J^k_{\underset{o}{\sim}}(^A\Gamma)) \to \mathcal{A}(J^{k+r}_o(\Gamma))$$

Passing to limit, we get

$$\tau : \mathcal{A}(J^{\infty}_{\tilde{o}}({}^{A}\Gamma)) \to \mathcal{A}(J^{\infty}_{o}(\Gamma))$$

Analogously as previously τ sends forms invariant under the action of ${}^{A}\Gamma$ into forms invariant under the action of Γ because

$$\lambda_{f^A} \circ \tau_k = \tau_k \circ \lambda_f$$

The map τ defines a homomorphism

$$\mathcal{A}({}^{A}\underline{\Gamma}) \to \mathcal{A}(\underline{\Gamma})$$

denoted for convenience also by τ and τ_k defines a morphism of principal fibre bundles

$$\begin{array}{ccc} J^k_o(\Gamma) & \xrightarrow{\tau_k} & J^k_{\tilde{o}}({}^{A}\Gamma) \\ \downarrow & & \downarrow \\ M & \xrightarrow{i_M} & M^A. \end{array}$$

Finally we take

$$\tau : H^*({}^{A}\underline{\Gamma}, K) \to H^*(\underline{\Gamma}, K)$$

The proof of comutativity of the diagram is analogous as in the case of the morphism σ.

c) To prove that

$$\tau \circ \sigma = id_{H^{\bullet}({}^{A}\underline{\Gamma}, K)}$$

it suffices to remark that the map $\mu_k = \tau_k \circ \sigma_k$ induces the identity if $k \to \infty$. This is the consequence of definitions of τ_k and σ_k. **Q.E.D.**

Now we can prove the Theorem 3. From the first diagram of the lemma we have

$$i^*_X(im\varphi_{\mathcal{F}^A}) \supset im\varphi_{\mathcal{F}}$$

From the second

$$im\varphi_{\mathcal{F}^A} \subset (\pi_A)^*(im\varphi_{\mathcal{F}^A}$$

because τ is a surjection. Since

$$i^*_X \circ \pi^*_A = id_{H^{\bullet}(X)}$$

we have

$$im\varphi_{\mathcal{F}} = i^*_X im\varphi_{\mathcal{F}^A}$$

Q.E.D.

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