Martin Markl; James D. Stasheff

Deformation theory via deviations

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Topology". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 32. pp. [97]–124.

Persistent URL: http://dml.cz/dmlcz/701530

DEFORMATION THEORY VIA DEVIATIONS

Martin Markl, James D. Stasheff

Introduction

The present paper was inspired by Drinfel'd's introduction and study of quasi-Hopf algebras in [2] and [3].

His proof of the existence of a quasi-triangular quasi-Hopf algebra starting with initial 'classical' data was reminiscent of arguments in the ordinary deformation theory of algebras, but there was noticeably lacking an appropriate complex controlling an appropriate deformation theory.

Given an algebra (in the most general sense for which it makes sense to talk about structure constants), there always exists a cohomology theory which controls the deformations, which is canonical (in some sense) and defined in all degrees. To construct it, take the affine coordinate ring $\mathrm{k}[M]$ of the variety M of structure constants and construct a resolution $(\mathcal{A}_*, d) \longrightarrow (\mathrm{k}[M], d{=}0)$ with $\mathcal{A}_*$ a graded commutative algebra $\wedge(X)_*$ on a graded vector space $X = \bigoplus_{i\geq 0} X_i$ and a differential satisfying $d(X_i) \subset \wedge(X)_{i-1}$ and $H_i(\wedge(X), d) = 0$, $i \geq 1$. Let $L^* = \mathrm{Der}(\mathcal{A}_*, \mathrm{k})$ and let δ be the differential on L^* induced by d. Then $H^*(L, \delta)$ "captures the deformations".

The above is too general to be useful for practical computation. The relevant part of (L^*, δ) is $L^0 \xrightarrow{\delta} L^1 \xrightarrow{\delta} L^2$. The first two pieces are easy to describe: L^0 is related to coordinates for the variety of structure constants and L^1 is related to relations among the coordinates related with the axioms of our algebra. But L^2 reflects the "relations among relations" – the 2nd syzygy, an ultimate mystery.

In the case of quasi-Hopf algebras, there is a natural and fairly obvious proto-complex, but it fails to be a complex: $\delta^2 \neq 0$. Our approach to the problem is to realize that the failure is a result of a relevant non-linearity. Our deviation calculus gives a way to capture these "relations among relations" using two-dimensional diagrams or at least to understand where these relations came from.

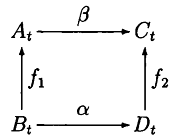
1. Basic principles

In this section we introduce the notion of a deviation and prove the main principle – the additivity of deviations. Everything will be formulated for square diagrams only, but it will be obvious how to generalize our results and definitions for diagrams of more general forms.

Let us introduce first some notation. For a fixed field $\mathbf{k}$, denote as usual by $\mathbf{k}[[t]]$ the ring of formal power series over $\mathbf{k}$. For a $\mathbf{k}$-module A, denote by A_t the $\mathbf{k}[[t]]$-module $A \otimes_{\mathbf{k}} \mathbf{k}[[t]]$. Notice that every t-adically complete flat $\mathbf{k}[[t]]$-module is of the form A_t for some $\mathbf{k}$-module A and clearly $A \cong A_t/(tA_t)$.

Definition 1.1 *Let A, B, C and D be $\mathbf{k}$-modules and consider the following diagram of $\mathbf{k}[[t]]$-modules and their maps:*

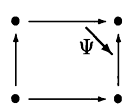
(1)



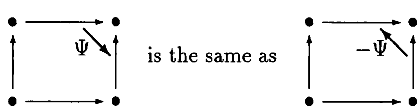
Suppose that this diagram is commutative modulo t^{n+1}. The deviation of (1) is then the map $\Psi : B \to C$ defined by

$$t^{n+1}\Psi = f_2\alpha - \beta f_1 \quad mod\ t^{n+2}.$$

The fact that Ψ is the deviation of (1) will be sometimes expressed as

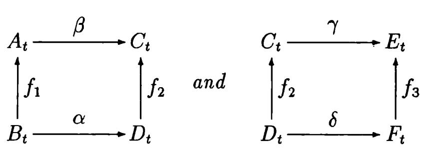


We hope that it is easy to understand what we mean by



For a $\mathbf{k}[[t]]$-linear map $g : U_t \to V_t$ let $(g)_0 : U \to V$, $U := U_t/(tU_t)$ and $V := V_t/(tV_t)$, denote the map defined by $(g)_0(u \bmod tU_t) = g(u) \bmod tV_t$ (the "absolute" part of g). We need this notation in the formulation of the following main principle of our "deviation calculus". It is as simple as:

Proposition 1.2 (Additivity principle) *Suppose that the diagrams*



commute mod t^{n+1} and let $\Psi_1 : B \to C$ resp. $\Psi_2 : D \to E$ be the corresponding deviations. Then the "big" diagram

$$\begin{array}{ccc} A_t & \xrightarrow{\gamma\circ\beta} & E_t \\ \uparrow{\scriptstyle f_1} & & \uparrow{\scriptstyle f_3} \\ B_t & \xrightarrow{\delta\circ\alpha} & F_t \end{array}$$

commutes again mod t^{n+1} (this is trivial) and the corresponding deviation Ψ can be computed as

$$\Psi = (\gamma)_0\Psi_1 + \Psi_2(\alpha)_0.$$

Proof. By definition, we have the relations

$$f_2\alpha = \beta f_1 + t^{n+1}\Psi_1 \bmod t^{n+1} \text{ and } f_3\delta = \gamma f_2 + t^{n+1}\Psi_2 \bmod t^{n+1}.$$

Using these, we get the following equality mod t^{n+2}:

$$f_3\delta\alpha = \gamma f_2\alpha + t^{n+1}\Psi_2\alpha = \gamma\beta f_1 + t^{n+1}\gamma\Psi_1 + t^{n+1}\Psi_2\alpha,$$

i.e. $f_3\delta\alpha - \gamma\beta f_1 = t^{n+1}(\gamma\Psi_1 + \Psi_2\alpha)$ mod t^{n+2} which means exactly $\Psi = (\gamma)_0\Psi_1 + \Psi_2(\alpha)_0$. ■

2. Classical examples

Associative algebras. By an *associative algebra* we mean a couple $A = (V, \mu)$, where V is a $\mathbf{k}$-linear space and $\mu : V \otimes V \to V$ a bilinear map satisfying the associativity relation ($\mathbb{1}$ = the identity map)

(2) $$\mu(\mu \otimes \mathbb{1}) - \mu(\mathbb{1} \otimes \mu) = 0.$$

Before reviewing some classical results about deformations of these objects, recall briefly the definition of Hochschild cohomology of A with coefficients in an A-bimodule M.

For two $\mathbf{k}$-vector spaces X and Y, let $\mathrm{Hom}(X,Y)$ be the space of $\mathbf{k}$-linear maps $f : X \to Y$. The differential $d_{\mathrm{Hoch}} : \mathrm{Hom}(V^{\otimes n}, M) \to \mathrm{Hom}(V^{\otimes n+1}, M)$ is, for $f \in \mathrm{Hom}(V^{\otimes n}, M)$, defined as

(3) $$d_{\mathrm{Hoch}}(f) = \nu_1(\mathbb{1}\otimes f) + \sum_{0\leq i\leq n-1}(-1)^{i+1} f(\mathbb{1}^i \otimes \mu \otimes \mathbb{1}^{n-i-1}) + (-1)^{n+1}\nu_2(f\otimes\mathbb{1}),$$

where $\nu_1 : A \otimes M \to M$ and $\nu_2 : M \otimes A \to M$ are the left and right actions, respectively. The *Hochshild cohomology* $H^*_{\mathrm{Hoch}}(A; M)$ is then defined as the cohomology of the complex $(\mathrm{Hom}(V^{\otimes *}, M), d_{\mathrm{Hoch}})$.

Deformations of the algebra A as above are related with the Hochshild cohomology of A with coefficients in A considered in an obvious way as an A-bimodule (i.e. $\nu_1 = \nu_2 = \mu$). For the convenience of the reader, we write explicitly the formulas for d_{Hoch} in relevant degrees:

$$\begin{aligned} d_{\mathrm{Hoch}}(f) &= \mu(\mathbb{1} \otimes f) - f(\mu) + \mu(f \otimes \mathbb{1}), \\ d_{\mathrm{Hoch}}(g) &= \mu(\mathbb{1} \otimes g) - g(\mu \otimes \mathbb{1}) + g(\mathbb{1} \otimes \mu) - \mu(g \otimes \mathbb{1}) \quad \text{and} \\ d_{\mathrm{Hoch}}(\psi) &= \mu(\mathbb{1} \otimes \psi) - \psi(\mu \otimes \mathbb{1}^2) + \psi(\mathbb{1} \otimes \mu \otimes \mathbb{1}) - \psi(\mathbb{1}^2 \otimes \mu) + \mu(\psi \otimes \mathbb{1}), \end{aligned}$$

where $f \in \mathrm{Hom}(V,V)$, $g \in \mathrm{Hom}(V^{\otimes 2}, V)$ and $\psi \in \mathrm{Hom}(V^{\otimes 3}, V)$.

By a *deformation* of $A = (V,\mu)$ we mean an associative $\mathbf{k}[[t]]$-algebra $A_t = (V_t, \mu_t)$, where $V_t = V \otimes \mathbf{k}[[t]]$ and $A_t/(tA_t) \cong A$. In other words, a deformation is given by a sequence of maps $\mu_i : V \otimes V \to V$, $i \geq 1$, such that $\mu_t := \mu + t\mu_1 + t^2\mu_2 + \cdots : V_t \otimes V_t \to V_t$, satisfy (2) over $\mathbf{k}[[t]]$.

One of the basic problems of deformation theory is the following *integrability problem*: given a "partial" deformation $\overline{\mu} = \mu + t\mu_1 + \cdots + t^n\mu_n$ satisfying (2) mod t^{n+1}, is it possible to construct an honest deformation μ_t of μ with $\mu_t = \overline{\mu}$ mod t^{n+1}?

Recall now the classical approach to the construction of an obstruction theory related with the possibility of a step-by-step integration of μ_t as above. First, suppose we had some $\mu_{n+1} : V \otimes V \to V$ such that $\tilde{\mu} := \mu + t\mu_1 + \cdots + t^n\mu_n + t^{n+1}\mu_{n+1}$ $(= \overline{\mu} + t^{n+1}\mu_{n+1})$ satisfy (2) mod t^{n+2}, i.e. that

$$(\overline{\mu} + t^{n+1}\mu_{n+1})((\overline{\mu} + t^{n+1}\mu_{n+1}) \otimes \mathbb{1}) - (\overline{\mu} + t^{n+1}\mu_{n+1})(\mathbb{1} \otimes (\overline{\mu} + t^{n+1}\mu_{n+1})) = 0 \bmod t^{n+2}.$$

An easy degree check shows that the last equation is equivalent to

$$\mu(\mathbb{1} \otimes \mu_{n+1}) - \mu_{n+1}(\mu \otimes \mathbb{1}) + \mu_{n+1}(\mathbb{1} \otimes \mu) - \mu(\mu_{n+1} \otimes \mathbb{1}) = \psi,$$

where $\psi : V \otimes V \otimes V \to V$ is defined by $\overline{\mu}(\overline{\mu} \otimes \mathbb{1}) - \overline{\mu}(\mathbb{1} \otimes \overline{\mu}) = t^{n+1}\psi$ mod t^{n+2}, which can easily be rewritten as

$$\psi = d_{\mathrm{Hoch}}(\mu_{n+1}).$$

Now ψ is defined from $\bar{\mu}$ without using μ_{n+1}. Suppose for the moment that we already know

$$d_{\mathrm{Hoch}}(\psi) = 0. \tag{4}$$

Then we have the following theorem (see, for example [4]).

Theorem 2.1 *The primary obstruction to the integrability of a partial deformation $\overline{\mu}$ is an element $[\psi] \in H^3_{\mathrm{Hoch}}(A; A)$.*

The equality (4) is indeed always true. The classical proof of this statement [4, Proposition 3, page 69] uses the graded pre-Lie ring structure on the Hochschild cochain complex, invoking an inductive argument. We show that this formula (and, more generally, formulas of the same type) is a consequence of some combinatorial property of a 2-dimensional polyhedron which is formally described by what we call the *deviation calculus.*

The first step is to interpret ψ as the deviation of some diagram. This is very easy; ψ is, by definition, the deviation of

(5)
$$\begin{array}{ccc} V_t & \xrightarrow{\mathbb{1}} & V_t \\ \uparrow{\scriptstyle \overline{\mu}(\mathbb{1}\otimes\overline{\mu})} & & \uparrow{\scriptstyle \overline{\mu}(\overline{\mu}\otimes\mathbb{1})} \\ V_t\otimes V_t\otimes V_t & \xrightarrow{\mathbb{1}^3} & V_t\otimes V_t\otimes V_t \end{array}$$

The next step is to apply to (5) all operations which occur in the formula for $d_{\mathrm{Hoch}}(\psi)$. We get in turn:

Applying $\overline{\mu}(\mathbb{1}\otimes *)$:

$$\begin{array}{ccc} V_t & \xrightarrow{\mathbb{1}} & V_t \\ \uparrow{\scriptstyle \overline{\mu}(\mathbb{1}\otimes\overline{\mu})(\mathbb{1}^2\otimes\overline{\mu})} & \Psi_1\searrow & \uparrow{\scriptstyle \overline{\mu}(\mathbb{1}\otimes\overline{\mu})(\mathbb{1}\otimes\overline{\mu}\otimes\mathbb{1})} \\ V_t^{\otimes 4} & \xrightarrow{\mathbb{1}^4} & V_t^{\otimes 4} \end{array}$$

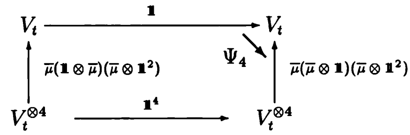
where $\Psi_1 = \mu(\mathbb{1}\otimes\psi)$. The composition of (5) with $(\mathbb{1}\otimes\overline{\mu}\otimes\mathbb{1})$ is

$$\begin{array}{ccc} V_t & \xrightarrow{\mathbb{1}} & V_t \\ \uparrow{\scriptstyle \overline{\mu}(\mathbb{1}\otimes\overline{\mu})(\mathbb{1}\otimes\overline{\mu}\otimes\mathbb{1})} & \Psi_2\searrow & \uparrow{\scriptstyle \overline{\mu}(\overline{\mu}\otimes\mathbb{1})(\mathbb{1}\otimes\overline{\mu}\otimes\mathbb{1})} \\ V_t^{\otimes 4} & \xrightarrow{\mathbb{1}^4} & V_t^{\otimes 4} \end{array}$$

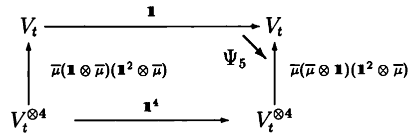
where $\Psi_2 = \psi(\mathbb{1}\otimes\mu\otimes\mathbb{1})$. The application of $\overline{\mu}(*\otimes\mathbb{1})$ gives

$$\begin{array}{ccc} V_t & \xrightarrow{\mathbb{1}} & V_t \\ \uparrow{\scriptstyle \overline{\mu}(\overline{\mu}\otimes\mathbb{1})(\mathbb{1}\otimes\overline{\mu}\otimes\mathbb{1})} & \Psi_3\searrow & \uparrow{\scriptstyle \overline{\mu}(\overline{\mu}\otimes\mathbb{1})(\overline{\mu}\otimes\mathbb{1}^2)} \\ V_t^{\otimes 4} & \xrightarrow{\mathbb{1}^4} & V_t^{\otimes 4} \end{array}$$

with $\Psi_3 = \mu(\psi\otimes\mathbb{1})$. Composing (5) with $(\overline{\mu}\otimes\mathbb{1}^2)$ we get



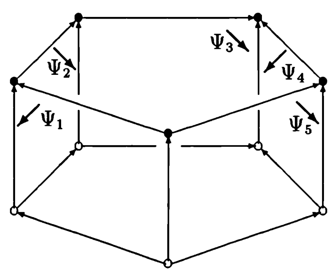
where $\Psi^4 = \psi(\mu \otimes \mathbb{1}^2)$ and, finally, composing (5) with $(\mathbb{1}^2 \otimes \overline{\mu})$ we obtain



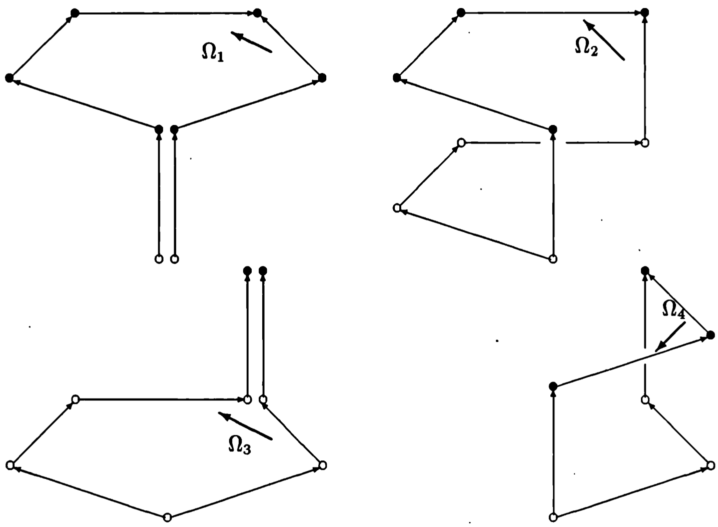
with $\Psi_5 = \psi(\mathbb{1} \otimes \mu)$.

Noticing that $\overline{\mu}(\overline{\mu} \otimes \mathbb{1})(\mathbb{1}^2 \otimes \overline{\mu}) = \overline{\mu}(\overline{\mu} \otimes \overline{\mu}) = \overline{\mu}(\mathbb{1} \otimes \overline{\mu})(\overline{\mu} \otimes \mathbb{1}^2)$, we can piece the diagrams above together into the following object:

(6)



where o denotes V_t and • denotes $V_t^{\otimes 4}$. Consider the following subdiagrams of (6):



Clearly $\Omega_1 = \Omega_3 = 0$. By Proposition 1.2, $\Omega_2 = -(\Psi_1+\Psi_2+\Psi_3)$ and $\Omega_4 = \Psi_4+\Psi_5$ (notice that all horizontal maps in (6) are identities). Again by Proposition 1.2 we get also that $\Omega_1 = \Omega_2+\Omega_3+\Omega_4$. Combining these equations we get $\Psi_1 + \Psi_2 + \Psi_3 = \Psi_4 + \Psi_5$ which is exactly $d_{\text{Hoch}}(\psi) = 0$.

Loosely speaking, our arguments above were based on the following principle. The "deviation diagram" (6) is topologically a 2-sphere which has no boundary. This means that the sum of the "partial deviations" must be zero, which is exactly the equation $d_{\text{Hoch}}(\psi) = 0$. But, as we will see in the case of quasi-Hopf algebras, we must be very careful when applying this principle.

Bialgebras. By an (associative and coassociative) *bialgebra* we mean an object $A = (V, \mu, \Delta)$ where V is a $\mathbf{k}$-vector space, $\mu : V \otimes V \to V$ and $\Delta : V \to V \otimes V$ (the product resp. coproduct) are linear maps and the following conditions are satisfied:

(7) $$\mu(\mu \otimes 1\!\!1) - \mu(1\!\!1 \otimes \mu) = 0, \qquad \text{(associativity)}$$

(8) $$(\Delta \otimes 1\!\!1)\Delta - (1\!\!1 \otimes \Delta)\Delta = 0, \qquad \text{(coassociativity)}$$

(9) $$(\mu \otimes \mu)(S)(\Delta \otimes \Delta) = \Delta\mu, \qquad \text{(compatibility)}$$

where S is defined by $S(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_1 \otimes x_3 \otimes x_2 \otimes x_4$. Deformations of these objects are related with *bialgebra cohomology*, introduced in [6] (see also [5]). Recall the necessary definitions. For any $q \geq 0$, $V^{\otimes q}$ has a natural structure of (V, μ)-bimodule induced from μ. This means that the formula (3) defines, for any $p \geq 1$, the differential

$$d_{\text{Hoch}} : \text{Hom}(V^{\otimes p}, V^{\otimes q}) \to \text{Hom}(V^{\otimes p+1}, V^{\otimes q}).$$

Similarly, $V^{\otimes q}$ has, for any $p \geq 0$, a natural structure of (V, Δ)-bicomodule (induced from Δ). This enables one to define dually the *coHochschild* differential

$$d_{\text{coH}} : \text{Hom}(V^{\otimes p}, V^{\otimes q}) \to \text{Hom}(V^{\otimes p}, V^{\otimes q+1}).$$

Consider the following hypercomplex

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}(V^{\otimes 4},V) & \longrightarrow & \mathrm{Hom}(V^{\otimes 4},V^{\otimes 2}) & \longrightarrow & \mathrm{Hom}(V^{\otimes 4},V^{\otimes 3}) & \longrightarrow & \mathrm{Hom}(V^{\otimes 4},V^{\otimes 4}) \longrightarrow \\
\uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow \\
\mathrm{Hom}(V^{\otimes 3},V) & \longrightarrow & \mathrm{Hom}(V^{\otimes 3},V^{\otimes 2}) & \longrightarrow & \mathrm{Hom}(V^{\otimes 3},V^{\otimes 3}) & \longrightarrow & \mathrm{Hom}(V^{\otimes 3},V^{\otimes 4}) \longrightarrow \\
\uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow \\
\mathrm{Hom}(V^{\otimes 2},V) & \longrightarrow & \mathrm{Hom}(V^{\otimes 2},V^{\otimes 2}) & \longrightarrow & \mathrm{Hom}(V^{\otimes 2},V^{\otimes 3}) & \longrightarrow & \mathrm{Hom}(V^{\otimes 2},V^{\otimes 4}) \longrightarrow \\
\uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow d_{\mathrm{Hoch}} & d_{\mathrm{coH}} & \uparrow \\
\mathrm{Hom}(V,V) & \longrightarrow & \mathrm{Hom}(V,V^{\otimes 2}) & \longrightarrow & \mathrm{Hom}(V,V^{\otimes 3}) & \longrightarrow & \mathrm{Hom}(V,V^{\otimes 4}) \longrightarrow
\end{array}$$

and let $(C^*_{\mathrm{b}}(A;A),D)$ be the associated total complex with the degree convention that

$$C^n_{\mathrm{b}}(A;A) = \mathrm{Hom}(V,V^{\otimes n}) \oplus \mathrm{Hom}(V^{\otimes 2},V^{\otimes n-1}) \oplus \cdots \oplus \mathrm{Hom}(V^{\otimes n-1},V^{\otimes 2}) \oplus \mathrm{Hom}(V^{\otimes n},V).$$

The (restricted) bialgebra cohomology of A with coefficients in A is then defined as $\hat{H}^*_{\mathrm{b}}(A;A) = H^*(C^*_{\mathrm{b}}(A;A),D)$. For the convenience of the reader we again write down explicitly the differentials of the bicomplex above in degrees relevant for our discussion: for $\psi_1 \in \mathrm{Hom}(V^{\otimes 3},V)$, $\psi_2 \in \mathrm{Hom}(V^{\otimes 2},V^{\otimes 2})$ and $\psi_3 \in \mathrm{Hom}(V,V^{\otimes 3})$ we have

$$\begin{aligned}
d_{\mathrm{Hoch}}(\psi_1) &= \mu(\mathbb{1}\otimes\psi_1) - \psi_1(\mu\otimes\mathbb{1}^2) + \psi_1(\mathbb{1}\otimes\mu\otimes\mathbb{1}) - \psi_1(\mathbb{1}^2\otimes\mu) + \mu(\psi_1\otimes\mathbb{1}),\\
d_{\mathrm{coH}}(\psi_1) &= (\mu(\mathbb{1}\otimes\mu)\otimes\psi_1)(X)(\Delta\otimes\Delta\otimes\Delta) - \Delta\psi_1 + (\psi_1\otimes\mu(\mu\otimes\mathbb{1}))(X)(\Delta\otimes\Delta\otimes\Delta),\\
d_{\mathrm{Hoch}}(\psi_2) &= (\mu\otimes\mu)(Z)(\Delta\otimes\psi_2) - \psi_2(\mu\otimes\mathbb{1}) + \psi_2(\mathbb{1}\otimes\mu) - (\mu\otimes\mu)(Z)(\psi_2\otimes\Delta),\\
d_{\mathrm{coH}}(\psi_2) &= (\mu\otimes\psi_2)(Z)(\Delta\otimes\Delta) - (\Delta\otimes\mathbb{1})(\psi_2) + (\mathbb{1}\otimes\Delta)(\psi_2) - (\psi_2\otimes\mu)(Z)(\Delta\otimes\Delta),\\
d_{\mathrm{Hoch}}(\psi_3) &= (\mu\otimes\mu\otimes\mu)(Y)(\Delta(\mathbb{1}\otimes\Delta)\otimes\psi_3) - \psi_3(\mu) + (\mu\otimes\mu\otimes\mu)(Y)(\psi_3\otimes\Delta(\Delta\otimes\mathbb{1})),\\
d_{\mathrm{coH}}(\psi_3) &= (\mathbb{1}\otimes\psi_3)\Delta - (\Delta\otimes\mathbb{1}^2)(\psi_3) + (\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\psi_3) - (\mathbb{1}^2\otimes\Delta)(\psi_3) + (\psi_3\otimes\mathbb{1})\Delta,
\end{aligned}$$

where $Z(x_1\otimes x_2\otimes x_3\otimes x_4) = x_1\otimes x_3\otimes x_2\otimes x_4$, $X(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5\otimes x_6) = x_1\otimes x_3\otimes x_5\otimes x_2\otimes x_4\otimes x_6$ and $Y(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5\otimes x_6) = x_1\otimes x_4\otimes x_2\otimes x_5\otimes x_3\otimes x_6$. Notice also the "self-duality" of the conditions above.

Suppose we have a "partial" deformation $(\overline{\mu},\overline{\Delta}) = (\mu+\cdots+t^n\mu_n, \Delta+\cdots t^n\Delta_n)$ satisfying (7), (8) and (9) modulo t^{n+1} and look for some $\mu_{n+1} \in \mathrm{Hom}(V^{\otimes 2},V)$ and $\Delta_{n+1} \in \mathrm{Hom}(V,V^{\otimes 2})$ such that $(\tilde{\mu},\tilde{\Delta}) := (\overline{\mu}+t^{n+1}\mu_{n+1}, \overline{\Delta}+t^{n+1}\Delta_{n+1})$ would satisfy (7), (8) and (9) modulo t^{n+2}. Define $\psi_1 \in \mathrm{Hom}(V^{\otimes 3},V)$, $\psi_2 \in \mathrm{Hom}(V^{\otimes 2},V^{\otimes 2})$ and $\psi_3 \in \mathrm{Hom}(V,V^{\otimes 3})$ by the following equations:

$$\overline{\mu}(\overline{\mu}\otimes\mathbb{1}) - \overline{\mu}(\mathbb{1}\otimes\overline{\mu}) \;=\; t^{n+1}\psi_1 \quad \text{modulo } t^{n+2},$$

$$\begin{aligned}\overline{\Delta}\overline{\mu} - (\overline{\mu}\otimes\overline{\mu})(Z)(\overline{\Delta}\otimes\overline{\Delta}) &= t^{n+1}\psi_2 \quad \text{modulo } t^{n+2},\\ (\overline{\Delta}\otimes \mathbb{1})\overline{\Delta} - (\mathbb{1}\otimes\overline{\Delta})\overline{\Delta} &= t^{n+1}\psi_3 \quad \text{modulo } t^{n+2}.\end{aligned}$$

As in the case of associative algebras, we can show that $(\mu_{n+1}, \Delta_{n+1})$ and (ψ_1, ψ_2, ψ_3) would be related by

$$D(\mu_{n+1}, \Delta_{n+1}) = (\psi_1, \psi_2, \psi_3),$$

where we consider $(\mu_{n+1}, \Delta_{n+1})$ in the evident sense as an element of $C^2_{\mathrm{b}}(A;A)$ and (ψ_1, ψ_2, ψ_3) as an element of $C^3_{\mathrm{b}}(A;A)$. Then the condition

(10) $$D(\psi_1, \psi_2, \psi_3) = 0$$

would imply the following theorem.

Theorem 2.2 *The primary obstruction to the integrability of a partial deformation* $(\overline{\mu}, \overline{\Delta})$ *is an element* $[(\psi_1, \psi_2, \psi_3)] \in \hat{H}^3_{\mathrm{b}}(A;A)$.

We prove (10) using our "deviation calculus". Notice that Theorem 2.2 is already known (see [6]) but the authors have never seen an explicit proof of (10) anywhere.

Notice first that (10) is, by the definition of the differential of the total complex, equivalent to the following four conditions:

(11) $$d_{\mathrm{Hoch}}(\psi_1) = 0,$$

(12) $$d_{\mathrm{coH}}(\psi_1) = d_{\mathrm{Hoch}}(\psi_2),$$

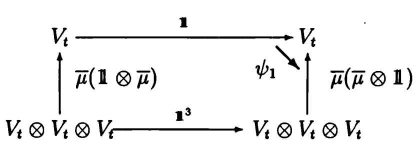
(13) $$d_{\mathrm{coH}}(\psi_2) = d_{\mathrm{Hoch}}(\psi_3) \quad \text{and}$$

(14) $$d_{\mathrm{coH}}(\psi_3) = 0.$$

The equation (11) has already been proven in the first part of this paragraph while (14) is just its dual. It is enough to prove (12) only, because (13) is again only the dual of (12).

As usual, start with interpreting ψ_1 and ψ_2 as deviations of some diagrams (for ψ_1, it was already done in the first part of this paragraph):

(15)



$$
(16)\qquad
\begin{array}{ccc}
V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^2\ } & V_t \otimes V_t \\
\big\uparrow{\scriptstyle (\overline{\mu}\otimes\overline{\mu})(Z)(\overline{\Delta}\otimes\overline{\Delta})} & {\scriptstyle \psi_2}\searrow & \big\uparrow{\scriptstyle \overline{\Delta}\overline{\mu}} \\
V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^2\ } & V_t \otimes V_t
\end{array}
$$

The next step will again be to apply to (15) and (16) operations which occur in $d_{\mathrm{coH}}(\psi_1)$ and $d_{\mathrm{Hoch}}(\psi_2)$. Applying $(\overline{\mu}(\mathbb{1}\otimes\overline{\mu})\otimes *)(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta})$ to (15), we get

$$
\begin{array}{ccc}
V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^2\ } & V_t \otimes V_t \\
\big\uparrow{\scriptstyle (\overline{\mu}\otimes\overline{\mu})(\mathbb{1}\otimes\overline{\mu}\otimes\mathbb{1}\otimes\overline{\mu})(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta})} & \Omega_1\searrow & \big\uparrow{\scriptstyle (\overline{\mu}\otimes\overline{\mu})(\mathbb{1}\otimes\overline{\mu}\otimes\overline{\mu}\otimes\mathbb{1})(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta})} \\
V_t \otimes V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^3\ } & V_t \otimes V_t \otimes V_t
\end{array}
$$

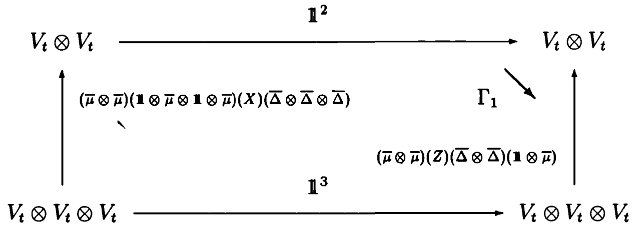
with $\Omega_1 = (\mu(\mathbb{1}\otimes\mu)\otimes\psi_1)(X)(\Delta\otimes\Delta\otimes\Delta) = (\mu(\mu\otimes\mathbb{1})\otimes\psi_1)(X)(\Delta\otimes\Delta\otimes\Delta)$ (μ is associative but $\overline{\mu}$ need not be). The application of $\overline{\Delta}$ to (15) gives

$$
\begin{array}{ccc}
V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^2\ } & V_t \otimes V_t \\
\big\uparrow{\scriptstyle \overline{\Delta}\overline{\mu}(\mathbb{1}\otimes\overline{\mu})} & \Omega_2\searrow & \big\uparrow{\scriptstyle \overline{\Delta}\overline{\mu}(\overline{\mu}\otimes\mathbb{1})} \\
V_t \otimes V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^3\ } & V_t \otimes V_t \otimes V_t
\end{array}
$$

with $\Omega_2 = \Delta\psi_1$. The composition of (15) with $(*\otimes\overline{\mu}(\overline{\mu}\otimes\mathbb{1}))(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta})$ is

$$
\begin{array}{ccc}
V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^2\ } & V_t \otimes V_t \\
\big\uparrow{\scriptstyle (\overline{\mu}\otimes\overline{\mu})(\mathbb{1}\otimes\overline{\mu}\otimes\overline{\mu}\otimes\mathbb{1})(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta})} & \Omega_3\searrow & \big\uparrow{\scriptstyle (\overline{\mu}\otimes\overline{\mu})(\overline{\mu}\otimes\mathbb{1}\otimes\overline{\mu}\otimes\mathbb{1})(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta})} \\
V_t \otimes V_t \otimes V_t & \xrightarrow{\ \mathbb{1}^3\ } & V_t \otimes V_t \otimes V_t
\end{array}
$$

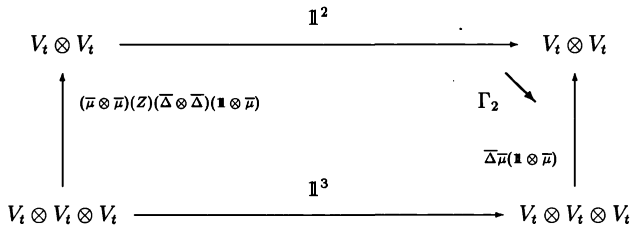
with $\Omega_3 = (\psi_1\otimes\mu(\mu\otimes\mathbb{1}))(X)(\Delta\otimes\Delta\otimes\Delta)$. Applying $(\overline{\mu}\otimes\overline{\mu})(Z)(\overline{\Delta}\otimes *)$ to (16), we get



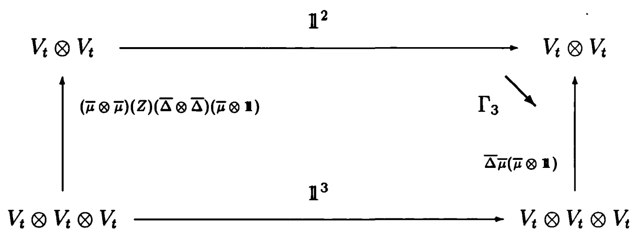
where $\Gamma_1 = (\mu \otimes \mu)(Z)(\Delta \otimes \psi_2)$; here we use the relation

$$(\overline{\mu}\otimes\overline{\mu})(Z)(\overline{\Delta}\otimes(\overline{\mu}\otimes\overline{\mu})(Z)(\overline{\Delta}\otimes\overline{\Delta})) = (\overline{\mu}\otimes\overline{\mu})(1\!\!1\otimes\overline{\mu}\otimes 1\!\!1\otimes\overline{\mu})(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta}).$$

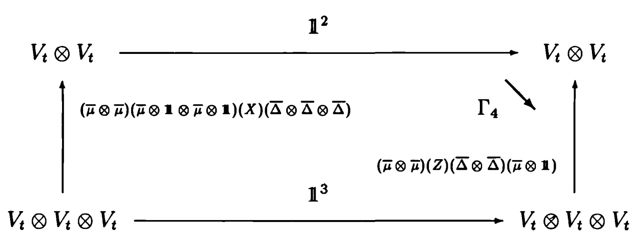
Composing (16) with $(1\!\!1 \otimes \overline{\mu})$ from the right we get



with $\Gamma_2 = \psi_2(1\!\!1 \otimes \mu)$. The composition of (16) with $(\overline{\mu} \otimes 1\!\!1)$ from the right is



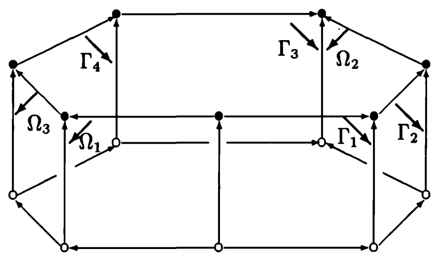
with $\Gamma_3 = \psi_2(\mu \otimes 1\!\!1)$ and, finally, applying $(\overline{\mu}\otimes\overline{\mu})(Z)(* \otimes \overline{\Delta})$ to (16) we get



where $\Gamma_4 = (\mu \otimes \mu)(Z)(\psi_2 \otimes \Delta)$ and we use the relation

$$(\overline{\mu}\otimes\overline{\mu})(Z)((\overline{\mu}\otimes\overline{\mu})(Z)(\overline{\Delta}\otimes\overline{\Delta})\otimes\overline{\Delta}) = (\overline{\mu}\otimes\overline{\mu})(\overline{\mu}\otimes 1\!\!1\otimes\overline{\mu}\otimes 1\!\!1)(X)(\overline{\Delta}\otimes\overline{\Delta}\otimes\overline{\Delta}).$$

Now we can easily form the following object:



Topologically, this is again a 2-sphere and by the same arguments as in the case of associative algebras, we can infer from this that the (oriented) sum of deviations must be zero, i.e. that

$$\Omega_1 + \Omega_3 + \Gamma_4 + \Gamma_3 = \Omega_2 + \Gamma_2 + \Gamma_1,$$

which is exactly (12).

3. Drinfel'd algebras

Following [2] and [3], by a *Drinfel'd algebra* (or a *quasi-bialgebra* in the terminology of [5]) we mean an object of the form $A = (V, \mu, \Delta, \Phi)$, where V is a k-linear space, $\mu : V \otimes V \to V$ (the product) and $\Delta : V \to V \otimes V$ (the coproduct) are linear maps and $\Phi \in V \otimes V \otimes V$ is an invertible (in the natural product structure induced on $V \otimes V \otimes V$ by μ) element. Moreover, μ is supposed to satisfy the associativity condition (7), μ and Δ should satisfy the compatibility condition (9) and we also assume that the product μ has an unit $1 \in V$ and that $\Delta(1) = 1 \otimes 1$. The coassociativity condition on Δ is in the Drinfel'd case replaced by

$$(\mathbb{1} \otimes \Delta)\Delta \cdot \Phi = \Phi \cdot (\Delta \otimes \mathbb{1})\Delta \tag{17}$$

and, moreover, the validity of the following "pentagon" condition is supposed:

$$(\mathbb{1}^2 \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1}^2)(\Phi) = (1 \otimes \Phi) \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \cdot (\Phi \otimes 1). \tag{18}$$

In both equations above $\cdot$ denotes the multiplication induced by μ. Notice that our definition of a Drinfel'd algebra is the same as the definition of a quasi-Hopf algebra given in [2] and [3], except that we do not require the existence of an antipode. Notice also that an (associative and coassociative) bialgebra $A = (V, \mu, \Delta)$ can be in a canonical way considered as a Drinfel'd algebra - put $\Phi = 1$.

A suitable cohomology theory which captures deformations of (associative and coassociative) *bialgebras* in the category of Drinfel'd algebras is already known – see [8], [11], [5] or [13].

As the first step toward a cohomology theory capturing deformations of a general Drinfel'd algebra, we describe a cohomology theory related to Drinfel'd deformations of $A = (V, \mu, \Delta, \Phi)$ (no additional restrictions on Φ or Δ) leaving Δ and μ fixed. The program is then completed in [9].

It can be expected that, as in [2], the cobar construction over the coalgebra (V, Δ) will play the central rôle in our computation. The standard definitions still make sense even for noncoassociative Δ, but the condition $d^2_{\rm cob} = 0$ may be violated. We aim to give a suitable generalization of the cobar construction for coalgebras for which the coassociativity is replaced by the pentagon condition (18). The construction is rather sophisticated and involves the following objects.

Let $F^* = \bigoplus_{n\geq 0} F^n$ be the free unitary nonassociative algebra on the vector space V and let $\star$ denote the product in F^*. As a vector space, F^n is isomorphic with the direct sum of copies of $V^{\otimes n}$ indexed by various bracketings of n indeterminates, $F^0 = \mathbf{k}$, $F^1 = V$, $F^2 = V^{\otimes 2}_{\bullet\bullet}$, $F^3 = V^{\otimes 3}_{\bullet(\bullet\bullet)} \oplus V^{\otimes 3}_{(\bullet\bullet)\bullet}$, $F^4 = V^{\otimes 4}_{\bullet(\bullet(\bullet\bullet))} \oplus V^{\otimes 4}_{((\bullet\bullet)\bullet)\bullet} \oplus V^{\otimes 4}_{(\bullet\bullet)(\bullet\bullet)} \oplus V^{\otimes 4}_{\bullet((\bullet\bullet)\bullet)} \oplus V^{\otimes 4}_{(\bullet(\bullet\bullet))\bullet}, \ldots$.

Notice first that F^* admits a natural left action of the algebra (V,μ), $(a,f) \mapsto a \bullet f \in F^*$, given by the following two rules:

1. for $f \in F^1 = V$, $a \bullet f = \mu(a,f)$,
2. $a \bullet (f \star g) = \sum(\Delta'(a) \bullet f) \star (\Delta''(a) \bullet g)$ where we use the standard notation $\Delta(a) = \sum \Delta'(a) \otimes \Delta''(a)$.

The right action $(f,b) \mapsto f \bullet b$ is defined by similar rules.

It is easy to verify that these operations define on F^* the structure of a (V,μ)-bimodule, i.e. that $a \bullet (b \bullet f) = (a \cdot b) \bullet f$, $a \bullet (f \bullet b) = (a \bullet f) \bullet b$ and $(f \bullet a) \bullet b = f \bullet (a \cdot b)$ for $a, b \in V$ and $f \in F^*$.

Having in mind future applications, we give a more explicit description of the $\bullet$-action. To this end, let $\mathcal{B}_n$ denotes, for $n \geq 1$, the set of all bracketings of n nonassociative indeterminates (the disjoint union $\coprod \mathcal{B}_n$ with the evident multiplication is exactly the free magma [10, I.4.§1] on one element). For $b \in \mathcal{B}_n$ and $1 \leq i \leq n$, let $b_{[i]} \in \mathcal{B}_{n+1}$ denote the bracketing obtained from b by the replacement $\bullet \mapsto (\bullet\bullet)$ at the i-th place. Notice that any bracketing can be obtained from the "trivial bracketing" $(\bullet) \in \mathcal{B}_1$ by a successive application of this operation (for example, $(\bullet\bullet)\bullet = ((\bullet)_{[1]})_{[1]}$, $\bullet(\bullet\bullet) = ((\bullet)_{1]}))_{[2]}, \ldots$). The following two conditions clearly define, for any $b \in \mathcal{B}_n$, a map $\Delta^{\{b\}} : V \to V^{\otimes n}$:

1. $\Delta^{\{\bullet\}} = \mathbb{1}$ and
2. $\Delta^{\{b_{[i]}\}} = (\mathbb{1}^{i-1} \otimes \Delta \otimes \mathbb{1}^{n-i})(\Delta^{\{b\}})$ for $b \in \mathcal{B}_n$ and $1 \leq i \leq n$.

We have the following lemma.

Lemma 3.1 *For $a \in V$ and $f \in V_b$, $b \in \mathcal{B}_n$, we have*

$$a \bullet f = \Delta^{\{b\}}(a) \cdot f,$$

where $\cdot$ denotes the usual multiplication induced by μ. A similar formula holds also for the right $\bullet$-action.

Proof. The proof is based on the formula

$$\Delta^{\{bb'\}} = \left(\Delta^{\{b\}} \otimes \Delta^{\{b'\}}\right) \Delta, \ b \in \mathcal{B}_i, \ b' \in \mathcal{B}_j,$$

(where the "multiplication" $bb' \in \mathcal{B}_{i+j}$ has an obvious meaning) which immediately follows from an easy inductive argument. The rest is a consequence of the very definition of the operation $\bullet$. ■

Let $\sim$ be the relation, $\star$-multiplicatively generated on F^* by expressions of the form

$$\Phi \bullet (x \star (y \star z)) = ((x \star y) \star z) \bullet \Phi, \quad x, y, z \in F^*, \tag{19}$$

where $\Phi \bullet (x \star (y \star z))$ means, expanding Φ as $\sum(\Phi_1 \otimes \Phi_2 \otimes \Phi_3)$, simply $\sum(\Phi_1 \bullet x) \star ((\Phi_2 \bullet y) \star (\Phi_3 \bullet z))$, the meaning of the right-hand expression being similar. Finally, let M^* be the graded k-module $F^*/\sim$.

Proposition 3.2

1. *The left and right actions $\bullet$ of (V, μ) on F^* induce on M^* the structure (denoted again by $\bullet$) of a graded (V, μ)-bimodule.*

2. *The (nonassociative) multiplication $\star$ on F^* induces on M^* a (nonassociative) multiplication $\odot$ satisfying*

$$\Phi \bullet [p \odot (q \odot r)] = [(p \odot q) \odot r] \bullet \Phi, \tag{20}$$

where again $\Phi \bullet [p \odot (q \odot r)]$ abbreviates $\sum(\Phi_1 \bullet p) \odot ((\Phi_2 \bullet q) \odot (\Phi_3 \bullet r))$ and similarly for the second expression.

Proof a). We shall show that the left $\bullet$-action is compatible with (19), i.e that

$$a \bullet [\Phi \bullet (x \star (y \star z))] = a \bullet [((x \star y) \star z) \bullet \Phi]. \tag{21}$$

We have, by definition, $a \bullet [\Phi \bullet (x \star (y \star z))] = \sum a \bullet [(\Phi_1 \bullet x) \star ((\Phi_2 \bullet y) \star (\Phi_3 \bullet z))] = \sum(\Delta'(a) \bullet (\Phi_1 \bullet x)) \star (\Delta''(a) \bullet ((\Phi_2 \bullet y) \star (\Phi_3 \bullet z))) = \sum(\Delta'(a) \bullet (\Phi_1 \bullet x)) \star ((\Delta'\Delta''(a) \bullet (\Phi_2 \bullet y)) \star (\Delta''\Delta''(a) \bullet (\Phi_3 \bullet z))) = \sum((\Delta'(a) \cdot \Phi_1) \bullet x) \star (((\Delta'\Delta''(a) \cdot \Phi_2) \bullet y) \star ((\Delta''\Delta''(a) \cdot \Phi_3) \bullet z))) = [(\mathbb{1} \otimes \Delta)\Delta(a) \cdot \Phi] \bullet (x \star (y \star z)) = [\Phi \cdot (\Delta \otimes \mathbb{1})\Delta(a)] \bullet (x \star (y \star z)) = \Phi \bullet [(\Delta'\Delta'(a) \bullet x) \star ((\Delta''\Delta'(a) \bullet y) \star (\Delta''(a) \bullet z))]$ so, summing up,

$$a \bullet [\Phi \bullet (x \star (y \star z))] = \Phi \bullet [(\Delta'\Delta'(a) \bullet x) \star ((\Delta''\Delta'(a) \bullet y) \star (\Delta''(a) \bullet z))]. \tag{22}$$

On the other hand, $a \bullet [((x \star y) \star z) \bullet \Phi] = [a \bullet ((x \star y) \star z)] \bullet \Phi$ and from the definition of the $\bullet$-action we get

$$a \bullet [((x \star y) \star z) \bullet \Phi] = [((\Delta'\Delta'(a) \bullet x) \star (\Delta''\Delta'(a) \bullet y)) \star (\Delta''(a) \bullet z)] \bullet \Phi \tag{23}$$

and (21) is an easy consequence of (22), (23) and (19). The argument for the right action is similar.

Proof b). The fact that $\odot$ is well defined follows from the very definition of $\sim$. The relation (20) is then simply the projection of (19). ■

Proposition 3.3 *There exists a unique isomorphism* $J : M^* \to V^{\otimes *}$ *satisfying the following condition:*

Let $x \in M^n$ *and let* $v \in V^{\otimes n}_{((\cdots(\bullet\bullet)\cdots\bullet)\bullet)\bullet}$ *and* $w \in V^{\otimes n}_{\bullet(\bullet(\bullet\cdots(\bullet\bullet)\cdots))}$ *be two representatives of* x. *Then*

$$J(x) = A \cdot v = w \cdot B \quad \textit{for some invertible } A, B \in V^{\otimes n}. \tag{24}$$

Here $\cdot$ *denotes, as usual, the multiplication induced by* μ.

Proof. Recall that we denoted by $\mathcal{B}_n$ the set of all bracketings of n nonassociative variables. The existence of an isomorphism $M^* \cong V^{\otimes *}$ follows from the following conditions:

$$\text{if } b, b' \in \mathcal{B}_n \text{ and } v \in V^{\otimes n}_b \quad \text{then} \quad v \sim v' \text{ for some } v' \in V^{\otimes n}_{b'}, \tag{25}$$

$$\text{if } v, v' \in V^{\otimes n}_b \quad \text{then} \quad v \sim v' \text{ if an only if } v = v'. \tag{26}$$

The condition (25) is an easy consequence of the definition of $\sim$. The condition (26) is trivial for $n \leq 3$. For $n = 4$, it is a consequence (in fact, it is equivalent) to the pentagon condition (18) (a nice exercise). For $n > 4$, it follows from the celebrated Mac Lane coherence theorem [7] which, loosely speaking, says that "there are no unexpected relations provided the pentagonal condition is satisfied".

We show that there exists a unique J satisfying (24). By (25) and (26), $J|_{M^n}$ is uniquely determined by a choice of an isomorphism $V^{\otimes n}_{\bullet(\bullet(\bullet\cdots(\bullet\bullet)\cdots))} \cong V^{\otimes n}$, so suppose directly that $J(x) = A \cdot v$ for some invertible element A of $V^{\otimes n}$ and for x and v as in (24). If w is a representative of x in $V^{\otimes n}_{((\cdots(\bullet\bullet)\cdots\bullet)\bullet)\bullet}$ then clearly $v = X \cdot w \cdot Y$ for some invertible $X, Y \in V^{\otimes n}$ and, of course, $J(x) = A \cdot X \cdot w \cdot Y$. We see now that (24) is fulfilled with $A = X^{-1}$ and $B = Y$. ∎

We state without proof (as we will not need it) the following lemma.

Lemma 3.4 *If* J *is the map from Proposition 3.3,* $b \in \mathcal{B}_n$ *and* $v \in V^{\otimes n}_b$ *is a representative of some* $x \in M^n$, *then* $J(x) = K \cdot v \cdot L$, *where* K *and* L *are invertible elements of* $V^{\otimes n}$ *created from* Φ *by applications of* Δ *and tensoring by the identity (no inverses involved).*

Example 3.5 We would like to write down explicit formulas for $J : M^n \to V^{\otimes n}$ for small n. To this end, it is good to have in mind the following picture (which also illustrates the omnipresence of the associahedra K_n, introduced in [12]).

For $n \geq 2$ and $0 \leq i \leq n-2$, let $\mathcal{B}_{n,i}$ be the set of all (meaningful) insertions of i pairs of brackets between n nonassociative indeterminates. Clearly $\mathcal{B}_{n,n-2} = \mathcal{B}_n$, the set of all (full) bracketings introduced above. Denote also, for $n \geq 3$, $\mathcal{E}_n = \mathcal{B}_{n,n-3}$ and for $n = 2$ put $\mathcal{E}_2 = \emptyset$. Let L_n be, for $n \geq 2$, the graph whose vertices are in one-to-one correspondence with the elements of $\mathcal{B}_n$ and whose edges are indexed by the elements of $\mathcal{E}_n$. The incidence relations in L_n are defined by the rule that $b \in \mathcal{B}_n$ is an endpoint of $e \in \mathcal{E}_n$ if and only if b can be obtained from e by inserting one more pair of brackets. It is clear from this description that L_n is the 1-skeleton of the associahedron K_n (see [12] or [1, 1.2] for the definition of K_n). Another way to describe the edges of L_n is the following:

(27) Let n_1, n_2, n_3 and k be positive natural numbers, $k < n$ and $n_1+n_2+n_3+k = n+1$. Let $b \in \mathcal{B}_k$ and $b_i \in \mathcal{B}_{n_i}$, $i = 1,2,3$. Let b' be the bracketing obtained from b by the replacement $\bullet \mapsto (b_1b_2)b_3$ at the j-th place and let $b'' \in \mathcal{B}_n$ be obtained from b by the replacement $\bullet \mapsto b_1(b_2b_3)$ at the same j-th place, $1 \leq j \leq k$. Then the vertices b' and b'' are endpoints of an edge of L_n.

It is easily seen that the relation $b' < b''$ induces on L_n the structure of an *oriented graph*.

By definition, the components of the space F^n are in one-to-one correspondence with the vertices of L_n while the edges of L_n correspond to the defining relations of M^n. More precisely, let b' and b'' be as in (27) and let $e \in \mathcal{E}_n$ denote the edge of L_n corresponding to the pair (b', b''). Then we identify $v \in V_{b'}^{\otimes n}$ with $C_{\Omega_e}(v) \in V_{b''}^{\otimes n}$, where $C_{\Omega_e} : v \mapsto (\Omega_e)\cdot v \cdot (\Omega_e)^{-1}$ is the conjugation by $\Omega_e = 1^{j-1} \otimes \Delta^{\{b_1\}}(\Phi_1) \otimes \Delta^{\{b_2\}}(\Phi_2) \otimes \Delta^{\{b_3\}}(\Phi_3) \otimes 1^{k-j}$. Here $\Phi = \sum \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ and we use the description of the $\bullet$-action given in Lemma 3.1.

The computation of the map J is based on the following scheme. Let $v \in V^{\otimes n}_{((\cdots(\bullet\bullet)\cdots\bullet)\bullet)\bullet}$ be a representative for $x \in M^n$, let $b \in \mathcal{B}_n$ and let $w \in V_b^{\otimes n}$ be another representative of x. Choose a path, say $e = e_1e_2\cdots e_m$, joining $((\cdots(\bullet\bullet)\cdots\bullet)\bullet)\bullet$ with b in L_n. Then $w = \Omega_e \cdot v \cdot \Omega_e^{-1}$ with $\Omega_e = \Omega_{e_m} \cdot \Omega_{e_{m-1}} \cdot \cdots \cdot \Omega_{e_1}$ and we moreover know that $J(x) = A \cdot v = X \cdot w \cdot Y$ for some invertible elements $A, X, Y \in V^{\otimes n}$ (see Proposition 3.1 and its proof). From this we obtain easily

$$X^{-1} \cdot A = \Omega_e = Y. \tag{28}$$

In the special case when $b = \bullet(\bullet(\bullet\cdots(\bullet\bullet)\cdots))$, Proposition 3.1 says that $X = 1^n$, therefore

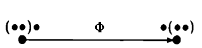
(29) $A = \Omega_f$ for a path f joining $((\cdots(\bullet\bullet)\cdots\bullet)\bullet)\bullet$ and $\bullet(\bullet(\bullet\cdots(\bullet\bullet)\cdots))$ in L_n.

Plugging this value back into (28) enables us to carry out the computation for an arbitrary $b \in \mathcal{B}_n$.

We give the explicit formulas for $J : M^n \to V^{\otimes n}$, $n \leq 5$. To simplify the notation, we index the edges of L_n's directly by the corresponding elements $\Omega_e \in V^{\otimes n}$. For $b \in \mathcal{B}_n$, $v_b \in V_b^{\otimes n}$ will denote a representative of $x \in M^n$.

$\boxed{n = 1,2}$ J is the identity (trivial).

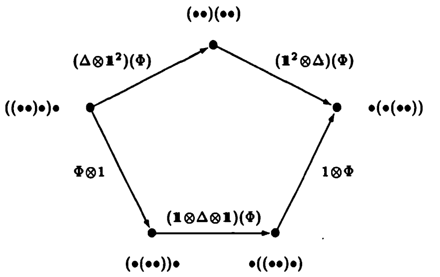
$\boxed{n = 3}$ L_3 is the arrow



and

$$J(x) = v_{\bullet(\bullet\bullet)} \cdot \Phi = \Phi \cdot v_{(\bullet\bullet)\bullet}.$$

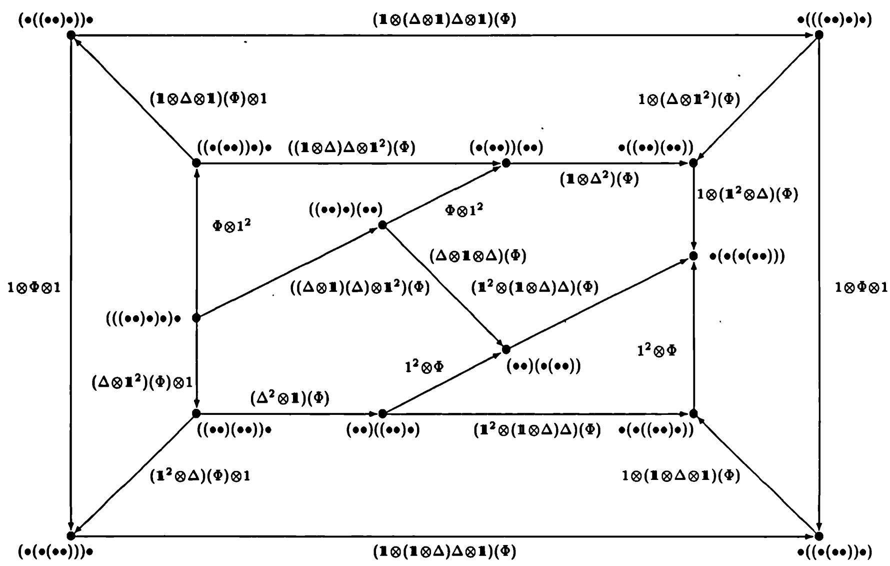
$\boxed{n = 4}$ L_4 is the pentagon



We obtain the following description of J:

$$\begin{aligned} J(x) &= (\mathbb{1}^2 \otimes \Delta)(\Phi) \cdot v_{(\bullet\bullet)(\bullet\bullet)} \cdot (\Delta \otimes \mathbb{1}^2)(\Phi) = (\mathbb{1}^2 \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1}^2)(\Phi) \cdot v_{(((\bullet\bullet)\bullet)\bullet)} \\ &= v_{(\bullet(\bullet(\bullet\bullet)))} \cdot (\mathbb{1}^2 \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1}^2)(\Phi) = (1 \otimes \Phi) \cdot v_{(\bullet((\bullet\bullet)\bullet))} \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \cdot (\Phi \otimes \mathbb{1}) \\ &= (1 \otimes \Phi) \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \cdot v_{((\bullet(\bullet\bullet))\bullet)} \cdot (\Phi \otimes 1). \end{aligned}$$

$\boxed{n = 5}$ We use the following projection of L_5 (borrowed from [2], for another projection of this graph see Diagram 38):



The formulas for J are already rather complicated, but we will need them for the description of the relevant part of the differential in our cohomology. The computation is still straightforward. We have

$$\begin{aligned} J(x) &= (\mathbb{1}^2 \otimes (\mathbb{1} \otimes \Delta)\Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1} \otimes \Delta)(\Phi) \cdot ((\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1}^2)(\Phi) \cdot v_{(((\bullet\bullet)\bullet)\bullet)\bullet} \\ &= v_{\bullet(\bullet(\bullet(\bullet\bullet)))} \cdot (\mathbb{1}^2 \otimes (\mathbb{1} \otimes \Delta)\Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1} \otimes \Delta)(\Phi) \cdot ((\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1}^2)(\Phi) \end{aligned}$$

$$
\begin{aligned}
&=(1\otimes(\mathbb{1}^2\otimes\Delta)(\Phi))\cdot(\mathbb{1}\otimes\Delta\otimes\Delta)(\Phi)\cdot((\mathbb{1}\otimes\Delta)\Delta\otimes\mathbb{1}^2)(\Phi)\cdot v_{((\bullet(\bullet\bullet))\bullet)\bullet}\cdot(\Phi\otimes 1^2)\\
&=(1^2\otimes\Phi)\cdot v_{\bullet(\bullet((\bullet\bullet)\bullet))}\cdot(\mathbb{1}^2\otimes(\Delta\otimes\mathbb{1})\Delta)(\Phi)\cdot(\Delta\otimes\Delta\otimes\mathbb{1})(\Phi)\cdot((\Delta\otimes\mathbb{1}^2)(\Phi)\otimes 1)\\
&=(1\otimes(\mathbb{1}^2\otimes\Delta)(\Phi))\cdot v_{\bullet((\bullet\bullet)(\bullet\bullet))}\cdot(\mathbb{1}\otimes\Delta\otimes\Delta)(\Phi)\cdot((\mathbb{1}\otimes\Delta)\Delta\otimes\mathbb{1}^2)(\Phi)\cdot(\Phi\otimes 1^2)\\
&=(1^2\otimes\Phi)\cdot(\mathbb{1}^2\otimes(\Delta\otimes\mathbb{1})\Delta)(\Phi)\cdot(\Delta\otimes\Delta\otimes\mathbb{1})(\Phi)\cdot v_{((\bullet\bullet)(\bullet\bullet)\bullet}\cdot((\Delta\otimes\mathbb{1}^2)(\Phi)\otimes 1)\\
&=(1^2\otimes\Phi)\cdot(1\otimes(\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi))\cdot v_{\bullet((\bullet(\bullet\bullet))\bullet)}\cdot(\mathbb{1}\otimes(\mathbb{1}\otimes\Delta)\Delta\otimes\mathbb{1})(\Phi)\cdot\\
&\quad\cdot((\mathbb{1}^2\otimes\Delta)(\Phi)\otimes 1)\cdot((\Delta\otimes\mathbb{1}^2)(\Phi)\otimes 1)\\
&=(1\otimes(\mathbb{1}^2\otimes\Delta)(\Phi))\cdot(1\otimes(\Delta\otimes\mathbb{1}^2)(\Phi))\cdot(\mathbb{1}\otimes(\Delta\otimes\mathbb{1})\Delta\otimes\mathbb{1})(\Phi)\cdot\\
&\quad\cdot v_{(\bullet((\bullet\bullet)\bullet))\bullet}\cdot((\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi)\otimes 1)\cdot(\Phi\otimes 1^2)\\
&=(\mathbb{1}^2\otimes(\mathbb{1}\otimes\Delta)\Delta)(\Phi)\cdot v_{(\bullet\bullet)(\bullet(\bullet\bullet))}\cdot(\Delta\otimes\mathbb{1}\otimes\Delta)(\Phi)\cdot((\Delta\otimes\mathbb{1})\Delta\otimes\mathbb{1}^2)(\Phi)\\
&=(\mathbb{1}^2\otimes(\mathbb{1}\otimes\Delta))(\Phi)\cdot(\Delta\otimes\mathbb{1}\otimes\Delta)(\Phi)\cdot v_{((\bullet\bullet)\bullet)(\bullet\bullet)}\cdot((\Delta\otimes\mathbb{1})\Delta\otimes\mathbb{1}^2)(\Phi)\\
&=(1\otimes(\mathbb{1}^2\otimes\Delta)(\Phi))\cdot(1\otimes(\Delta\otimes\mathbb{1}^2)(\Phi))\cdot v_{\bullet(((\bullet\bullet)\bullet)\bullet)}(\mathbb{1}\otimes(\Delta\otimes\mathbb{1})\Delta\otimes\mathbb{1})(\Phi)\cdot\\
&\quad\cdot((\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi)\otimes 1)\cdot(\Phi\otimes 1^2)\\
&=(1^2\otimes\Phi)\cdot(1\otimes(\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi))\cdot(\mathbb{1}\otimes(\mathbb{1}\otimes\Delta)\Delta\otimes\mathbb{1})(\Phi)\cdot v_{(\bullet(\bullet(\bullet\bullet)))\bullet}\cdot\\
&\quad\cdot((\mathbb{1}^2\otimes\Delta)(\Phi)\otimes 1)\cdot((\Delta\otimes\mathbb{1}^2)(\Phi)\otimes 1)\\
&=(1\otimes(\mathbb{1}^2\otimes\Delta)(\Phi))\cdot(\mathbb{1}\otimes\Delta\otimes\Delta)(\Phi)\cdot v_{(\bullet(\bullet\bullet))(\bullet\bullet)}\cdot((\mathbb{1}\otimes\Delta)\Delta\otimes\mathbb{1}^2)(\Phi)\cdot(\Phi\otimes 1^2)\\
&=(1^2\otimes\Phi)\cdot(\mathbb{1}^2\otimes(\Delta\otimes\mathbb{1})\Delta)(\Phi)\cdot v_{(\bullet\bullet)((\bullet\bullet)\bullet)}(\Delta\otimes\Delta\otimes\mathbb{1})(\Phi)\cdot((\Delta\otimes\mathbb{1}^2)(\Phi)\otimes 1).
\end{aligned}
$$

The last thing we need before giving the definition of our variant of the cobar construction is the following notation.

Let $x \in M^n$ and let $v \in V_b^{\otimes n}$ be, for some bracketing $b \in \mathcal{B}_n$, a representative of x. For $1 \le i \le n$, let $d_i(x)$ be defined to be an element of M^{n+1} whose representative in $V_{b_{[i]}}^{\otimes n+1}$ is $(\mathbb{1}^{i-1} \otimes \Delta \otimes \mathbb{1}^{n-i})(v)$; here $b_{[i]} \in \mathcal{B}_{n+1}$ is the bracketing obtained from b by the replacement $\bullet \leftrightarrow (\bullet\bullet)$ at the i-th place. The following statement is an easy exercise.

Lemma 3.6 *The map $d_i : M^n \to M^{n+1}$ is well defined, i.e. it does not depend on the particular choice of the bracketing $b \in \mathcal{B}_n$ and the representative $v \in V_b^{\otimes n}$.*

Now, define $d_{q\Omega} : M^n \to M^{n+1}$ by

$$d_{q\Omega}(x) = d_0(x) - d_1(x) + \cdots + (-1)^n d_n(x) + (-1)^{n+1} d_{n+1}(x),$$

where $d_1, \ldots, d_n$ are defined above, $d_0(x) := 1 \odot x$ and $d_{n+1} = x \odot 1$.

Lemma 3.7 *For $x \in M^n$,*

$$d_{q\Omega}^2(x) = [1\odot(1\odot x) - (1\odot 1)\odot x] + (-1)^n[(1\odot x)\odot 1 - 1\odot(x\odot 1)] + [x\odot(1\odot 1) - (x\odot 1)\odot 1].$$

Proof will be based on the following formulas

$$
\begin{aligned}
&(30) \qquad & d_k d_l &= d_{l+1} d_k, \quad 1 \le k \le l \le n+1,\\
&(31) \qquad & d_0 d_l &= d_{l+1} d_0, \quad 1 \le l \le n,\\
&(32) \qquad & d_k d_{n+1} &= d_{n+2} d_k, \quad 1 \le k \le n.
\end{aligned}
$$

Notice that (30)–(32) plus the equations $d_0^2 = d_1 d_0$, $d_{n+2}d_0 = d_0 d_{n+1}$ and $d_{n+1}^2 = d_{n+2}d_{n+1}$ would give the usual conditions on coboundary operators in a cosimplicial module.

We discuss (30) first. Let $x \in M^n$ and choose a representative $v \in V_b^{\otimes n}$ of x. Then $d_l(x)$ is, by definition, the class of $(\mathbb{1}^{l-1} \otimes \Delta \otimes \mathbb{1}^{n-l})(v)$ in $V_{b_{[l]}}^{\otimes n+1}$ ($b_{[l]} \in \mathcal{B}_{n+1}$ is defined above), while $d_k d_l(x)$ is the class of

$$(\mathbb{1}^{k-1} \otimes \Delta \otimes \mathbb{1}^{n+1-k})(\mathbb{1}^{l-1} \otimes \Delta \otimes \mathbb{1}^{n-l})(v) \quad \text{in} \quad V_{(b_{[l]})_{[k]}}^{\otimes n+2}. \tag{33}$$

Similarly, $d_{l+1}d_k(x)$ is the class of

$$(\mathbb{1}^{l} \otimes \Delta \otimes \mathbb{1}^{n-l})(\mathbb{1}^{k-1} \otimes \Delta \otimes \mathbb{1}^{n-k})(v) \quad \text{in} \quad V_{(b_{[k]})_{[l+1]}}^{\otimes n+2}. \tag{34}$$

For $l > k$ we easily get from the equations above that

$$d_k d_l(x) \text{ is the class of } (\mathbb{1}^{k-1} \otimes \Delta \otimes \mathbb{1}^{l-k-1} \otimes \Delta \otimes \mathbb{1}^{n-l})(v) \text{ in } V_{(b_{[l]})_{[k]}}^{\otimes n+2}$$

and that

$$d_{l+1} d_k(x) \text{ is the class of } (\mathbb{1}^{k-1} \otimes \Delta \otimes \mathbb{1}^{l-k-1} \otimes \Delta \otimes \mathbb{1}^{n-l})(v) \text{ in } V_{(b_{[k]})_{[l+1]}}^{\otimes n+2}.$$

Observing that $(b_{[l]})_{[k]} = (b_{[k]})_{[l+1]}$ we see that in this case really $d_k d_l = d_{l+1} d_k$.

Suppose $k = l$ and prove that $d_k d_k(x) = d_{k+1} d_k(x)$. By (33) we have that

$$d_k d_k(x) \text{ is the class of } (\mathbb{1}^{k-1} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1}^{n-k})(v) \text{ in } V_{(b_{[k]})_{[k]}}^{\otimes n+2}$$

while (34) gives that

$$d_{k+1} d_k(x) \text{ is the class of } (\mathbb{1}^{k-1} \otimes (\mathbb{1} \otimes \Delta)\Delta \otimes \mathbb{1}^{n-k})(v) \text{ in } V_{(b_{[k]})_{[k+1]}}^{\otimes n+2}.$$

Recall that, by (17), $\Phi \cdot (\Delta \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \Delta)\Delta \cdot \Phi$. An easy consequence of the defining relation (19) is that an element $u \in V_{(b_{[k]})_{[k]}}^{\otimes n+2}$ is identified with $(1^{k-1} \otimes \Phi \otimes 1^{n-k}) \cdot u \cdot (1^{k-1} \otimes \Phi \otimes 1^{n-k})^{-1} \in V_{(b_{[k]})_{[k+1]}}^{\otimes n+2}$ from which we get that $d_k d_k(x) = d_{k+1} d_k(x)$. Thus (30) is proven.

Prove (31). Let again $v \in V_b^{\otimes n}$ be a representative for some $x \in M^n$. Then $d_0 d_l(x)$ is the class of $1 \otimes (\mathbb{1}^{l-1} \otimes \Delta \otimes \mathbb{1}^{n-l})(v)$ in $V_{\bullet(b_{[l]})}^{\otimes n+2}$ (the meaning of the notation $\bullet(h)$ for $h \in \mathcal{B}_*$ being clear) and $d_{l+1} d_0(x)$ is the class of $(\mathbb{1}^{l} \otimes \Delta \otimes \mathbb{1}^{n-l})(1 \otimes v)$ in $V_{(\bullet(b))_{[l+1]}}^{\otimes n+2}$. Because clearly $1 \otimes (\mathbb{1}^{l-1} \otimes \Delta \otimes \mathbb{1}^{n-l})(v) = (\mathbb{1}^{l} \otimes \Delta \otimes \mathbb{1}^{n-l})(1 \otimes v)$ and $\bullet(b_{[l]}) = (\bullet(b))_{[l+1]}$, $d_0 d_l = d_{l+1} d_o$ and we have (31). The proof of (32) is similar.

Using (30)–(32), we can easily reduce the equation

$$d_{q\Omega}^2 = \sum_{0 \le j \le n+2} (-1)^j d_j \sum_{0 \le i \le n+1} (-1)^i d_i$$

to

$$d_{q\Omega}^2 = (d_0^2 - d_1 d_0) + (-1)^n (d_{n+2} d_0 - d_0 d_{n+1}) + (d_{n+1}^2 - d_{n+2} d_{n+1}),$$

which is exactly the formula in our lemma. ■

The following lemma shows that the differential $d_{q\Omega}$ is compatible with the $\bullet$-action of (V, μ) on M^*,

Lemma 3.8 *For any $a, b \in V$ and $x \in M^*$,*

$$d_{q\Omega}(a \bullet x) = a \bullet d_{q\Omega}(x) \text{ and } d_{q\Omega}(x \bullet b) = d_{q\Omega}(x) \bullet b.$$

Proof. To prove the compatibility with the left $\bullet$-action, it is enough to show that

(35) $$d_i(a \bullet x) = a \bullet d_i(x) \text{ for all } a \in V,\ x \in M^* \text{ and } 1 \le i \le n.$$

Let $b \in \mathcal{B}_n$ and let $v \in V_b^{\otimes n}$ be a representative for x. Then, by Lemma 3.1 and the definition of d_i, $(\mathbb{1}^{i-1} \otimes \Delta \otimes \mathbb{1}^{n-i})(\Delta^{\{b\}}(a) \cdot v)$ is a representative of $d_i(a \bullet x)$ in $V_{b_{[i]}}^{\otimes n+1}$. But

$$\begin{aligned}(\mathbb{1}^{i-1} \otimes \Delta \otimes \mathbb{1}^{n-i})(\Delta^{\{b\}}(a) \cdot v) &= (\mathbb{1}^{i-1} \otimes \Delta \otimes \mathbb{1}^{n-i})(\Delta^{\{b\}}(a)) \cdot (\mathbb{1}^{i-1} \otimes \Delta \otimes \mathbb{1}^{n-i})(v) \\ &= \Delta^{\{b_{[i]}\}}(a) \cdot (\mathbb{1}^{i-1} \otimes \Delta \otimes \mathbb{1}^{n-i})(v),\end{aligned}$$

which is, again by Lemma 3.1 and the definition of d_i, a representative of $a \bullet d_i(x)$. We thus have proved (35). The argument for the right $\bullet$-action is similar. ■

Lemma 3.7 shows that $(M^*, d_{q\Omega})$ is not, generally speaking, a complex, i.e. that $d_{q\Omega}^2 \neq 0$ (see also the explicit examples below). However, the next statement says that M^* contains a (nontrivial) subspace $M_I^* \subset M^*$ such that $d_{q\Omega}^2$ is zero on M_I^*.

Proposition 3.9 *Define the subspace $M_I^* \subset M^*$ by $M_I^n = \{x \in M^n; a \bullet x - x \bullet a = 0$ for all $a \in V\}$, where $\bullet$ is the action of $B = (V, \mu)$ on M^* (I for "invariant"). Then $d_{q\Omega}(M_I^*) \subset M_I^*$ and $d_{q\Omega}^2 = 0$ on M_I^*, in other words, $(M_I^*, d_{q\Omega})$ is a cochain complex.*

Proof. The inclusion $d_{q\Omega}(M_I^*) \subset M_I^*$ immediately follows from Lemma 3.8. Let us prove that $1 \odot (1 \odot x) = (1 \odot 1) \odot x$ for any $x \in M_I^n$. By Proposition 3.2 we have

(36) $$\Phi \bullet [1 \odot (1 \odot x)] = [(1 \odot 1) \odot x] \bullet \Phi.$$

On the other hand, if $\Phi = \sum \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ then $\Phi \bullet [1 \odot (1 \odot x)] = \sum \Phi_1 \odot (\Phi_2 \odot \Phi_3 \bullet x) = \sum \Phi_1 \odot (\Phi_2 \odot x \bullet \Phi_3) = [1 \odot (1 \odot x)] \bullet \Phi$, because $\Phi_3 \bullet x = x \bullet \Phi_3$ from the invariance. Combining it with (36), we obtain that $1 \odot (1 \odot x) = (1 \odot 1) \odot x$. Similarly, we can show that $1 \odot (x \odot 1) = (1 \odot x) \odot 1$ and $1 \odot (1 \odot x) = (1 \odot 1) \odot x$, for an arbitrary invariant x. In the light of Lemma 3.7, this gives our proposition. ■

Explicit formulas Proposition 3.3 enables us to identify M^n and $V^{\otimes n}$ in a canonical way. This canonical identification offers the possibility of expressing $d_{q\Omega}$ directly in terms of $V^{\otimes *}$ (i.e. to compute $Jd_{q\Omega}J^{-1}$). We give explicit formulas for this presentation of $d_{q\Omega}$ at least in degrees important for the deformation theory.

• For $v \in V$,

$$d_{q\Omega}(v) = 1 \otimes v - \Delta(v) + v \otimes 1,$$

i.e. $d_{q\Omega}$ coincides with the usual cobar differential.

• For $v \in V^{\otimes 2}$,

$$d_{q\Omega}(v) = (1 \otimes v) \cdot \Phi - \Phi \cdot (\Delta \otimes \mathbb{1})(v) + (\mathbb{1} \otimes \Delta)(v) \cdot \Phi - \Phi \cdot (v \otimes 1).$$

• For $v \in V^{\otimes 3}$,

$$\begin{aligned} d_{q\Omega}(v) = {} & (1\otimes v)\cdot(\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi)\cdot(\Phi\otimes 1) - (\mathbb{1}^2\otimes\Delta)(\Phi)\cdot(\Delta\otimes\mathbb{1}^2)(v) \\ & +(1\otimes\Phi)\cdot(\mathbb{1}\otimes\Delta\otimes\mathbb{1})(v)\cdot(\Phi\otimes 1) - (\mathbb{1}^2\otimes\Delta)(v)\cdot(\Delta\otimes\mathbb{1}^2)(\Phi) \\ & +(1\otimes\Phi)\cdot(\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi)\cdot(v\otimes 1). \end{aligned}$$

• For $v \in V^{\otimes 4}$,

$$\begin{aligned} d_{q\Omega}(v) = {} & (1\otimes v)\cdot(\mathbb{1}\otimes(\Delta\otimes\mathbb{1})\Delta\otimes\mathbb{1})(\Phi)\cdot((\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi)\otimes 1)\cdot(\Phi\otimes 1^2) \\ & -(\mathbb{1}^2\otimes(\mathbb{1}\otimes\Delta)\Delta)(\Phi)\cdot(\Delta\otimes\mathbb{1}^3)(v) + (1\otimes(\mathbb{1}^2\otimes\Delta)(\Phi))\cdot(\mathbb{1}\otimes\Delta\otimes\mathbb{1}^2)(v)\cdot(\Phi\otimes 1^2) \\ & -(1^2\otimes\Phi)\cdot(\mathbb{1}^2\otimes\Delta\otimes\mathbb{1})(v)\cdot((\Delta\otimes\mathbb{1}^2)(\Phi)\otimes 1) + (\mathbb{1}^3\otimes\Delta)(v)\cdot((\Delta\otimes\mathbb{1})\Delta\otimes\mathbb{1}^2)(\Phi) \\ & -(1^2\otimes\Phi)\cdot(1\otimes(\mathbb{1}\otimes\Delta\otimes\mathbb{1})(\Phi))\cdot(\mathbb{1}\otimes(\mathbb{1}\otimes\Delta)\Delta\otimes\mathbb{1})(\Phi)\cdot(v\otimes 1). \end{aligned}$$

These formulas can be obtained as a combination of the definition of $d_{q\Omega}$ and explicit formulas for J as given above. We note that in the special case $\Phi = 1$, the relation $\sim$ identically identifies various copies of $V^{\otimes n}$, the map J is the identity and $d_{q\Omega}$ coincides with the usual cobar construction on (V, Δ). The following lemma gives a description of M_I^* in terms of $V^{\otimes *}$; the proof is an easy exercise.

Lemma 3.10 *An element $v \in V^{\otimes n}$ is invariant (in other words, $J^{-1}(v) \in M_I^n$) if and only if*

$$(\mathbb{1}^{n-2}\otimes \Delta)(\mathbb{1}^{n-3}\otimes \Delta)\cdots(\mathbb{1}\otimes \Delta)\Delta(a)\cdot v = v\cdot(\Delta\otimes\mathbb{1}^{n-2})(\Delta\otimes\mathbb{1}^{n-3})\cdots(\Delta\otimes\mathbb{1})\Delta(a),$$

for any $a \in V$, where $\cdot$ denotes, as usual, the multiplication induced on $V^{\otimes n}$ by μ.

We show now how the cohomology of the complex $(M_I^*, d_{q\Omega})$ is related to deformations of a Drinfel'd algebra $A = (V, \mu, \Delta, \Phi)$ with $\mu_t = \mu$ and $\Delta_t = \Delta$.

To this end, suppose that $\overline{\Phi} = \Phi + t\Phi_1 + \cdots + t^n\Phi_n$ is a "partial" deformation, i.e. that $\overline{\Phi}$ satisfies (17) and (18) mod t^{n+1}. Notice also that, by Lemma 3.10, (17) means that Φ_i, when considered as an element of $V^{\otimes 3}$, is invariant, $1 \leq i \leq n$. Look now for some $\Phi_{n+1} \in V^{\otimes 3}$ such that $\tilde{\Phi} := \overline{\Phi} + t^{n+1}\Phi_{n+1}$ satisfies (17) and (18) mod t^{n+2}, which is the same as looking for an *invariant* $\Phi_{n+1} \in V^{\otimes 3}$ such that $\tilde{\Phi}$ above satisfies (18) mod t^{n+2}. Plugging $\tilde{\Phi}$ into (18), we see that this is equivalent to

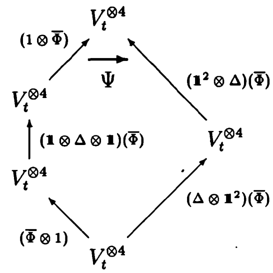
$$d_{q\Omega}(\Phi_{n+1}) = \Psi,$$

where Ψ is defined by the following equation:

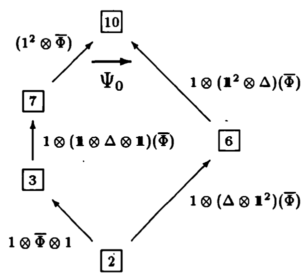
$$t^{n+1}\Psi = (\mathbb{1}^2 \otimes \Delta)(\overline{\Phi}) \cdot (\Delta \otimes \mathbb{1}^2)(\overline{\Phi}) - (1 \otimes \overline{\Phi}) \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\overline{\Phi}) \cdot (\overline{\Phi} \otimes 1) \bmod t^{n+2}.$$

We shall show that $d_{q\Omega}(\Psi) = 0$, without assuming Φ_{n+1} exists. As usual, first interpret Ψ as the deviation of some diagram. If we agree to identify elements v of $V^{\otimes n}$ and maps $V^{\otimes n} \to V^{\otimes n}$ given by left multiplication by v, then Ψ is the deviation of

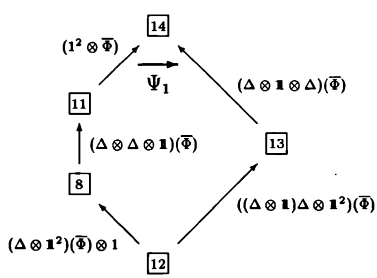
(37)



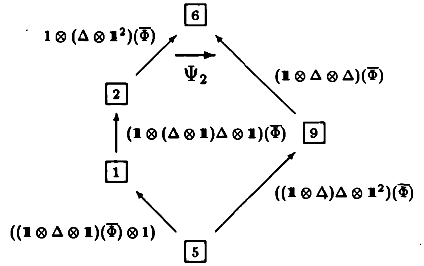
The application of $(1 \otimes *)$ on (37) gives



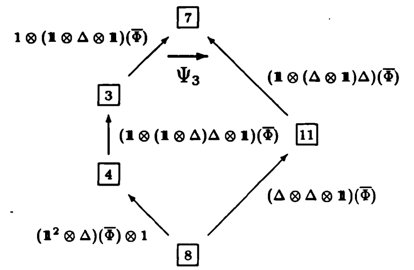
with $\Psi_0 = 1 \otimes \Psi$ (the labels [1] - [14] denote different copies of $V_t^{\otimes 5}$). The application of $(\Delta \otimes \mathbb{1}^3)$ on (37) gives



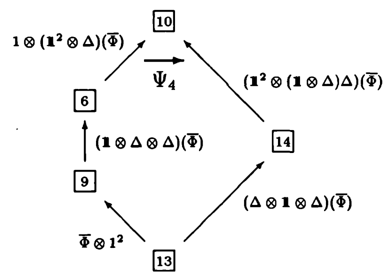
with $\Psi_1 = (\Delta \otimes \mathbb{1}^3)(\Psi)$. Applying $(\mathbb{1} \otimes \Delta \otimes \mathbb{1}^2)$ we get



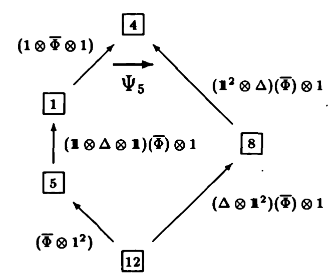
with $\Psi_2 = (\mathbb{1} \otimes \Delta \otimes \mathbb{1}^2)(\Psi)$. The application of $(\mathbb{1}^2 \otimes \Delta \otimes \mathbb{1})$ on (37) gives



with $\Psi_3 = (\mathbb{1}^2 \otimes \Delta \otimes \mathbb{1})(\Psi)$. The application of $(\mathbb{1}^3 \otimes \Delta)$ gives

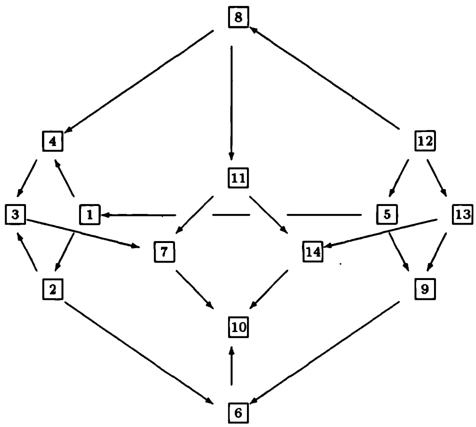


with $\Psi_4 = (\mathbb{1}^3 \otimes \Delta)(\Psi)$ and, finally, tensoring (37) by 1 from the right we get

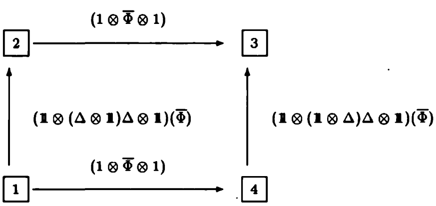


with $\Psi_5 = \Psi \otimes 1$. We can form the following diagram

(38)



which is the "associahedron" introduced in [12]. Notice that we already used this object in Example 3.5. It consists of six pentagonal subdiagrams (whose deviations are discussed above) and three square diagrams. We claim that these square diagrams are commutative. Look, for example, at the diagram



The commutativity of this diagram easily follows from the invariance of $\overline{\Phi}$: if $\overline{\Phi} = \sum \overline{\Phi}_1 \otimes \overline{\Phi}_2 \otimes \overline{\Phi}_3$, then $(1 \otimes \overline{\Phi} \otimes 1) \cdot (\mathbb{1} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1})(\overline{\Phi}) = \sum \overline{\Phi}_1 \otimes \overline{\Phi} \cdot (\Delta \otimes \mathbb{1})\Delta(\overline{\Phi}_2) \otimes \overline{\Phi}_3 = \sum \overline{\Phi}_1 \otimes (\mathbb{1} \otimes \Delta)\Delta(\overline{\Phi}_2) \cdot \overline{\Phi} \otimes \overline{\Phi}_3 = (\mathbb{1} \otimes (\mathbb{1} \otimes \Delta)\Delta \otimes \mathbb{1})(\overline{\Phi}) \cdot (1 \otimes \overline{\Phi} \otimes 1)$. Here we needed the invariance of $\overline{\Phi}$ to have $\overline{\Phi} \cdot (\Delta \otimes \mathbb{1})\Delta(\overline{\Phi}_2) = (\mathbb{1} \otimes \Delta)\Delta(\overline{\Phi}_2) \cdot \overline{\Phi}$. The argument for the remaining square diagrams is the same.

Notice that (38) is again topologically a 2-sphere. Try to apply the principle saying that in this case "the oriented sum of all deviations must be zero". This principle was, in fact, formulated already in [2], where also the rôle of K_5 in the deformation theory of quasi-Hopf algebras was observed for the first time (however, there is no explicit mention of the deformation theory there). Applying this principle in our situation, we would get $\Psi_0 - \Psi_1 + \Psi_2 - \Psi_3 + \Psi_4 - \Psi_5 = 0$, i.e.

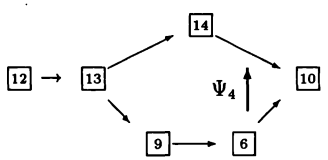
$$1 \otimes \Psi - (\Delta \otimes \mathbb{1}^3)(\Psi) + (\mathbb{1} \otimes \Delta \otimes \mathbb{1}^2)(\Psi) - (\mathbb{1}^2 \otimes \Delta \otimes \mathbb{1})(\Psi) + (\mathbb{1}^3 \otimes \Delta)(\Psi) - \Psi \otimes 1 = 0,$$

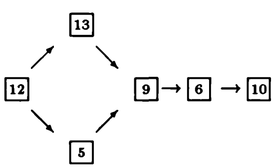
(the usual cobar differential) and this formula is <u>false</u>! The explanation is that in our Proposition 1.2 not only the bare deviations, but deviations composed with corresponding maps, come into play. Notice that in the case discussed in [2] (the case with $\Psi = 1$) this makes no difference as all maps in the corresponding diagrams there are $\mathbb{1}$ mod t.

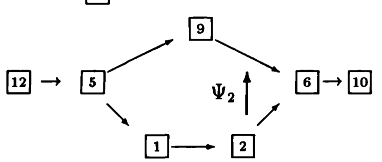
In order to obtain a correct formula, it is necessary to begin once again to think in terms of maps rather than of the deviations alone. First, choose two vertices of (38), a source and a sink. There are no restrictions on our choice (notice that all maps in (38) are invertible) and the resulting formula will depend on our choice simply via multiplication by a nonzero element. However, there is at least a preferred choice: take $\boxed{12}$, which corresponds to $(((\bullet\bullet)\bullet)\bullet)\bullet$, as a source and $\boxed{10}$, corresponding to $\bullet(\bullet(\bullet(\bullet\bullet)))$, as a sink. Consider the following chains of maps from $\boxed{12}$ to $\boxed{10}$:

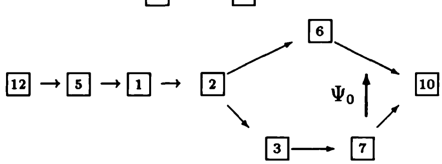
(P_1) $\boxed{12} \to \boxed{13} \to \boxed{14} \to \boxed{10}$
(P_2) $\boxed{12} \to \boxed{13} \to \boxed{9} \to \boxed{6} \to \boxed{10}$
(P_3) $\boxed{12} \to \boxed{5} \to \boxed{9} \to \boxed{6} \to \boxed{10}$
(P_4) $\boxed{12} \to \boxed{5} \to \boxed{1} \to \boxed{2} \to \boxed{6} \to \boxed{10}$
(P_5) $\boxed{12} \to \boxed{5} \to \boxed{1} \to \boxed{2} \to \boxed{3} \to \boxed{7} \to \boxed{10}$
(P_6) $\boxed{12} \to \boxed{5} \to \boxed{1} \to \boxed{4} \to \boxed{3} \to \boxed{7} \to \boxed{10}$
(P_7) $\boxed{12} \to \boxed{8} \to \boxed{4} \to \boxed{3} \to \boxed{7} \to \boxed{10}$
(P_8) $\boxed{12} \to \boxed{8} \to \boxed{11} \to \boxed{7} \to \boxed{10}$
(P_9) $\boxed{12} \to \boxed{8} \to \boxed{11} \to \boxed{14} \to \boxed{10}$
(P_{10}) $\boxed{12} \to \boxed{13} \to \boxed{14} \to \boxed{10}$

where $P_1 = P_{10}$. Their mutual differences are described by the following diagrams ($D_{i,i+1}$ stands for the difference between P_i and P_{i+1}):

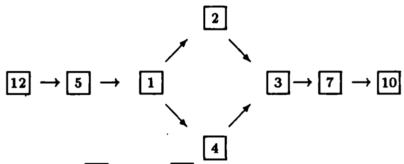
$(D_{1,2})$


$(D_{2,3})$


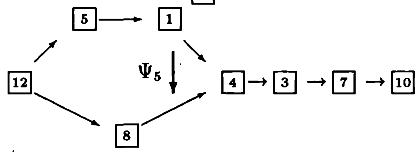
$(D_{3,4})$


$(D_{4,5})$


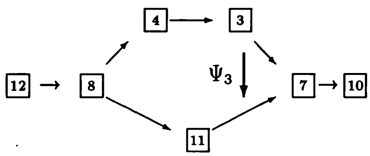
$(D_{5,6})$



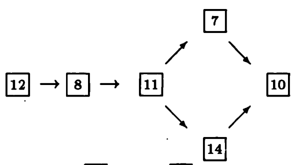
$(D_{6,7})$



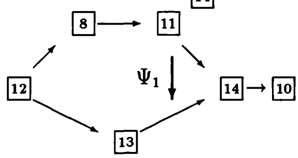
$(D_{7,8})$



$(D_{8,9})$



$(D_{9,10})$



Now we discuss the contributions of the diagrams $D_{1,2}$–$D_{9,10}$ above. The contributions of $D_{2,3}$, $D_{5,6}$ and $D_{8,9}$ are trivial, because the corresponding square diagrams are commutative.

Try to determine, for example, the contribution of $D_{1,2}$. By Proposition 1.2 it is given as the composition of Ψ_4 and the t^0-part of $\boxed{12} \to \boxed{13}$, which is $((\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1}^2)(\Phi)$. Therefore, the contribution of $D_{1,2}$ is

$$C_1 := \Psi_4 \cdot ((\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1}^2)(\Phi) = (\mathbb{1}^3 \otimes \Delta)(\Psi) \cdot ((\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1}^2)(\Phi).$$

Similarly, the contribution of $D_{3,4}$ is

$$C_2 := (1 \otimes (\mathbb{1}^2 \otimes \Delta)(\Phi)) \cdot \Psi_2 \cdot (\Phi \otimes 1^2) = (1 \otimes (\mathbb{1}^2 \otimes \Delta)(\Phi)) \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1}^2)(\Psi) \cdot (\Phi \otimes 1^2),$$

the contribution of $D_{4,5}$ is

$$\begin{aligned} C_3 &= \Psi_0 \cdot (\mathbb{1} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1})(\Phi) \cdot ((\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \otimes 1) \cdot (\Phi \otimes 1^2) \\ &= (1 \otimes \Psi) \cdot (\mathbb{1} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes \mathbb{1})(\Phi) \cdot ((\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \otimes 1) \cdot (\Phi \otimes 1^2), \end{aligned}$$

the contribution of $D_{6,7}$ is (note the minus sign)

$$C_4 : \; = \; -(1^2 \otimes \Phi) \cdot (1 \otimes (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi)) \cdot (\mathbb{1} \otimes (\mathbb{1} \otimes \Delta)\Delta \otimes \mathbb{1})(\Phi) \cdot \Psi_5$$

$$= -(1^2 \otimes \Phi) \cdot (1 \otimes (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi)) \cdot (\mathbb{1} \otimes (\mathbb{1} \otimes \Delta)\Delta \otimes \mathbb{1})(\Phi) \cdot (\Psi \otimes 1),$$

the contribution of $D_{7,8}$ is

$$C_5 = -(1^2 \otimes \Phi) \cdot \Psi_3 \cdot ((\Delta \otimes \mathbb{1}^2)(\Phi) \otimes 1) = -(1^2 \otimes \Phi) \cdot (\mathbb{1}^2 \otimes \Delta \otimes \mathbb{1})(\Psi) \cdot ((\Delta \otimes \mathbb{1}^2)(\Phi) \otimes 1)$$

and, finally, the contribution of $D_{9,10}$ is

$$C_6 := -(\mathbb{1}^2 \otimes (\mathbb{1} \otimes \Delta)\Delta)(\Phi) \cdot \Psi_1 = -(\mathbb{1}^2 \otimes (\mathbb{1} \otimes \Delta)\Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1}^3)(\Psi).$$

Since $P_1 = P_{10}$, $C_1 + C_2 + C_3 + C_4 + C_5 + C_6 = 0$ which is

$$d_{q\Omega}(\Psi) = 0.$$

Before formulating the final result of this chapter, we remark the following two things.

First, the computation above enables one to understand the rôle of multiplicative factors in the formula for $d_{q\Omega}$ – they correspond to the "tails" connecting local deviations with the source and sink. Second, we see that the deviation calculus enables one to derive linear conditions on the obstruction to the integrability without knowing *a priory* the cohomology theory into which everything embeds.

Theorem 3.11 *The primary obstruction to the integrability of a partial deformation $\overline{\Phi}$ is an element $[\Psi] \in H^3(M_I^{*+1}, d_{q\Omega})$.*

REFERENCES

[1] BOARDMAN J.M, VOGT R.M. *"Homotopy Invariant Algebraic Structures on Topological Spaces"*, Springer-Verlag, 1973.

[2] DRINFEL'D V.G. "Kvazichopfovy algebry", *Algebra i Analiz*, **1**,**6**(1989), 114–148.

[3] DRINFEL'D V.G. "O kvazitreugol'nych kvazichopfovych algebrach i odnoj gruppe, tesno svjazannoj s Gal($\overline{\mathbf{Q}}/\mathbf{Q}$)", *Algebra i Analiz*, **2**,**4**(1990), 149-181.

[4] GERSTENHABER M. "On the deformation of rings and algebras", *Ann. of Mathematics*, **79**(1964), 59-104.

[5] GERSTENHABER M., SCHACK S.D. "Algebras, bialgebras, quantum groups and algebraic deformations", preprint.

[6] GERSTENHABER M., SCHACK S.D. "Bialgebra cohomology, deformations, and quantum groups", *Proc. Natl. Acad. Sci. USA*, **87**(1990), 478-481.

[7] MAC LANE S. "Categorical algebra" *Bull. Amer. Math. Soc.*, **71**(1965), 40–106.

[8] MARKL M. "A letter to James Stasheff", unpublished.

[9] MARKL M. "Quasi-Hopf algebras via deformations", in preparation.

[10] SERRE J.-P. *"Lie Algebras and Lie Groups"*, Benjamin, 1965.

[11] SHNIDER S. "Deformation cohomology for bialgebras and quasi-bialgebras", to appear.

[12] STASHEFF J.D. "Homotopy associativity of H-spaces I.,II.", *Trans. Amer. Math. Soc.*, **108**(1963), 275–312.

[13] STERNBERG S. "Notes from the seminar on Drinfel'd algebras", Tel Aviv, Spring 1991.

MATEMATICKÝ ÚSTAV ČSAV
ŽITNÁ 25
115 67 PRAHA 1
CZECHOSLOVAKIA
markl at csearn.bitnet

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
CHAPEL HILL, N.C. 27514
U.S.A.
jds at math.unc.edu