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#### NATURAL TRANSFORMATIONS OF LAGRANGIANS

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#### Introduction.

Let M be a differentiable manifold and let TM be the tangent bundle. A smooth function  $L: TM \longrightarrow \mathbb{R}$  is called a *Lagrangian* on M. We denote by  $C^{\infty}(TM)$  the set of all Lagrangians on M and by  $\Omega^{p}(TM)$  the set of all p-forms on TM.

In this paper we will study natural transformations of Lagrangians into p-forms on the tangent bundle for p=0,1,2. Such a natural transformation A over n-dimensional differentiable manifolds M is a family of maps  $A_M:\mathcal{C}^\infty(TM)\longrightarrow \Omega^p(TM)$  such that for every embedding  $\varphi:M\longrightarrow N$  and for each Lagrangian  $L\in\mathcal{C}^\infty(TN)$  the p-forms  $A_M(L\circ T\varphi)\in\Omega^p(TM)$  and  $A_N(L)\in\Omega^p(TN)$  are  $T\varphi$ -related.

Some natural transformations of Lagrangians into p-forms on the tangent bundle are considered in physics.

Example 1. For an n-dimensional differentiable manifold M let us denote by  $C_M$  the Liouville vector field on  $T\dot{M}$ . Thus for every Lagrangian L on M the energy given by

$$E_M(L) = C_M(L) - L$$

is a natural transformation of Lagrangians into 0-forms on the tangent bundle i. e. a natural transformation of Lagrangians into itself.

**Example 2.** For an n-dimensional differentiable manifold M let us denote by  $J_M$  the canonical tangent structure on TM. Thus for every Lagrangian L on M the Poincaré-Cartan 1-form given by

$$\alpha_M(L) = dL \circ J_M$$

is a natural transformation of Lagrangians into 1-forms on the tangent bundle.

**Example 3.** For every Lagrangian L on M the Poincaré-Cartan 2-form is given by

$$\omega_M(L) = d(\alpha_M(L)) = d(dL \circ J_M).$$

Obviously  $\omega$  is a natural transformation of Lagrangians into 2-forms on the tangent bundle. Poincaré-Cartan 2-forms are very important for theoretical mechanics (see [5]).

It is of interest to know all natural transformations of Lagrangians into p-forms on the tangent bundle for p = 0, 1, 2. In our paper we will study this problem.

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#### Basic definitions.

All manifolds and maps are assumed to be infinitely differentiable.

Let n be a fixed positive integer. A family of maps  $A_M : \mathcal{C}^{\infty}(TM) \longrightarrow \Omega^p(TM)$ , where M is an arbitrary n-dimensional manifold, is called a natural transformation of Lagrangians into p-forms on the tangent bundle if two following conditions hold:

(1) The naturality condition. For every injective immersion  $\varphi: M \longrightarrow N$  of two n-dimensional manifolds M, N and for every Lagrangian  $L \in \mathcal{C}^{\infty}(TN)$  we have

$$\bigwedge {}^p(T(T\varphi^{-1}))^*\circ A_M(L\circ T\varphi)=A_N(L)\circ T\varphi.$$

(2) The regularity condition. For all manifolds M, N such that dim N=n and for every smooth function  $L: M \times TN \ni (t,v) \longrightarrow L_t(v) \in \mathbb{R}$  the map

$$M \times TN \ni (t,v) \longrightarrow A_N(L_t)(v) \in \bigwedge {}^pT^*(TN)$$

is also smooth.

Let A be a natural transformation of Lagrangians into p-forms on the tangent bundle. If U is an open subset of M and if  $\varphi: U \longrightarrow M$  is the inclusion then from the naturality condition we obtain the following implication

$$K|TU = L|TU \Longrightarrow A_M(K)|TU = A_M(L)|TU$$

for all Lagrangians K, L on M. We say that the natural transformation A satisfies the locality condition if for every n-dimensional manifold M, for every open subset V of TM and for all Lagrangians K, L on M the following implication

$$K|V = L|V \Longrightarrow A_M(K)|V = A_N(L)|V$$

holds. The following examples show that there are natural transformations of Lagrangians into itself which don't satisfy the locality condition (see [1]).

**Example 4.** Let  $f:[0,1] \longrightarrow \mathbb{R}$  be a continuous function. We define

$$A_M(L)(v) = \int_0^1 L(f(t)v)dt.$$

Example 5. We put

$$A_M(L)=d(L\circ 0_M)$$

where  $0_M$  is the zero section of TM and  $d(L \circ 0_M)$  is considered as a function on TM.

We say that the natural transformation A is of order r if for every n-dimensional manifold M, for every  $v \in TM$  and for all Lagrangians K, L on M the following implication

$$j_v^r K = j_v^r L \Longrightarrow A_M(K)(v) = A_M(L)(v)$$

holds.

It is clear that if a natural transformation has a order r then it satisfies the locality condition. Using the standard methods and so-called Borel Lemma (see [6]) we can prove that if a natural transformation of Lagragians into p-forms on the tangent bundle satisfies the locality condition then this natural transformation is of order  $\infty$ . The following example shows that there are natural transformations of Lagrangians into itself which satisfy the locality condition and have no finite order (see [1]).

**Example 6.** Suppose that  $f: \mathbf{R} \longrightarrow \mathbf{R}$  is a smooth function such that  $f|(-\infty, 0] = 0$  but  $f \neq 0$ . We define

$$A_M(L)(v) = \sum_{i=1}^{\infty} f(L(v) - \sum_{j=1}^{i} (1 + (C_M^j(L)(v))^2))$$

where  $C_M^j = (C_M \circ \ldots \circ C_M)(L)$  (j times). It is seen at once that this definition makes sense because in a neighbourhood of arbitrary  $v \in TM$  the first sum has a finite number of non-zero therms.

## Natural transformations of Lagrangians into itself.

We have the following characterization of natural transformations of Lagrangians into itself (see [1]).

THEOREM 1. Let  $n \geq 2$  and let r be a positive integer. If A is a natural transformation of order r of Lagrangians into itself then there is one and only one smooth function  $f: \mathbb{R}^{r+1} \longrightarrow \mathbb{R}$  such that  $A_M(L) = f(L, C_M^1(L), \ldots, C_M^r(L))$  for every n-dimensional manifold M and for every Lagrangian L on M.

The following example shows that the assumption that  $n \geq 2$  in Theorem 1 is necesseary.

Example 7. Let  $\varphi$  be a local coordinate system on a 1-dimensional manifold M and let L be a Lagrangian on M. Setting

$$(A_M(L) \circ T\varphi^{-1})(x,v) = \frac{\partial^2 (L \circ T\varphi^{-1})}{\partial x \partial v} (x,v) \frac{\partial (L \circ T\varphi^{-1})}{\partial v} (x,v) v^3$$

$$- \frac{\partial^2 (L \circ T\varphi^{-1})}{\partial v^2} (x,v) \frac{\partial (L \circ T\varphi^{-1})}{\partial x} (x,v) v^3$$

$$- \frac{\partial (L \circ T\varphi^{-1})}{\partial x} (x,v) \frac{\partial (L \circ T\varphi^{-1})}{\partial v} (x,v) v^2$$

we obtain a natural transformation of Lagrangians into itself which has the order two. It is clear that A is not of the form discribed in Theorem 1.

Natural transformations of Lagrangians into 1-forms on the tangent bundle.

Let  $R^r$  denotes the set of all natural transformations of order r of Lagrangians into itself. It is evident that  $R^r$  with the sum and product

$$(A+B)_M(L) = A_M(L) + B_M(L),$$
  
$$(A \cdot B)(L) = A_M(L)B_M(L)$$

is a ring. Let  $M_p^r$  denotes the set of all natural transformations of order r of Lagrangians into p-forms on the tangent bundle. It is evident that  $M_p^r$  is a module over  $R^r$  if we define

$$(A+B)_M(L) = A_M(L) + B_M(L),$$
  
$$(\Gamma \cdot A)_M(L) = \Gamma_M(L)A_M(L)$$

for all  $A, B \in M_p^r$ ,  $\Gamma \in \mathbb{R}^r$ , for every *n*-dimensional manifold M and for every Lagrangian L on M. We can verify (see [4]) that  $M_p^r$  is a free module and we have

THEOREM 2. Let  $n \geq 3$  and let r be a positive integer. The natural transformations given by formulas

$$d(C_M^i(L)) for i = 0, ..., r-1,$$

$$d(C_M^i(L)) \circ J_M$$
 for  $i = 0, \ldots, r-1$ ,

for every n-dimensional manifold M and for every Lagrangian L on M, form a basis of the module  $M_1^r$ .

Natural transformations of Lagrangians into 2-forms on the tangent bundle.

Using the similar methods as in [1], [2] and [4] we can prove

THEOREM 3. Let  $n \geq 4$  and let r be a positive integer. The natural transformations given by formulas

$$\begin{split} d(C_M^i(L)) \wedge d(C_M^j(L)) & \text{for } 0 \leq i < j \leq r-1, \\ d(C_M^i(L)) \wedge (d(C_M^j(L)) \circ J_M) & \text{for } i,j = 0, \dots, r-1, \\ (d(C_M^i(L)) \circ J_M) \wedge (d(C_M^j(L)) \circ J_M) & \text{for } 0 \leq i < j \leq r-1, \\ d(d(C_M^i(L)) \circ J_M) & \text{for } i = 0, \dots, r-2, \end{split}$$

for every n-dimensional manifold M and for every Lagrangian L on M, form a basis of the module  $M_2^r$ .

We can show this theorem also for n=3 but the method of verification is more complicated.

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