Martin Markl
Deformations and the koherence

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DEFORMATIONS AND THE COHERENCE

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Introduction

It is well-known that a good cohomology theory which controls the deformations of schemes is the cohomology of the cotangent complex (see [20] or [11]). The aim of this paper is to construct a similar object for a *category* in order to obtain a cohomology theory which would control the deformations of objects of this category. The range of definition of our construction will be wide enough to cover both classical and new examples, see 0.2 and 1.2.

0.1 Motivations In the majority of classical examples, the experience always provides one with a natural candidate for the cohomology controlling the deformations.

For example, one knows that associative algebras are somehow related with the Hochschild cohomology (or Shukla or Mac Lane cohomology, but these are all the same if we assume we work over a field of characteristic zero), and it is really true that this cohomology captures the deformations of associative algebras (see [8]). In the worst case, the deformation cohomology can be constructed as a suitable combination and modification of classical constructions (again based on the experience), see for example the case of bialgebras discussed in [10]. But the recent development of quantum group theory confronts one with a lot of natural examples (such as Drinfel'd algebras) where this approach fails. We think the failure is substantial as the constructions related with the deformation cohomology are sometimes very complicated, see for example [25] or [28].

So, the first natural question was whether there exists a construction which, when blindly applied to an algebraic object, would provide us with a deformation cohomology for this object. This question was immediately followed with the observation that we should first understand what we mean by 'cohomology which controls the deformation'.

There already have been some attempts to construct such a 'good deformation cohomology'. We should quote at least the classical paper of Gerstenhaber [8] and also the papers [6], [5] of Fox based on the (co)triple approach. The common drawback is the limited range of applications and also very limited possibilities of explicit calculations. Notice that the 'good deformation cohomology' for associative algebras in the sense of Gerstenhaber is isomorphic to the Hochschild cohomology by [8, Theorem 6], while the approach of Fox gives the cotriple cohomology which is again isomorphic, this time by [1], to the same Hochschild cohomology. In both cases the existence of such an isomorphism is a nontrivial fact.

0.2 Here we explain briefly to which kind of objects our construction applies. By an *algebra* we mean a (in most cases) finite dimensional vector space, say V, together with a set of multilinear operations, say $\{m_i\}_{i\in I}$, $m_i : V^{\otimes a_i} \to V^{\otimes b_i}$, which have to satisfy some axioms which are created from $\{m_i\}$ by the composition, tensor product, linear combination and the 'switch' $S : V^{\otimes 2} \to V^{\otimes 2}$, $S(x \otimes y) = y \otimes x$. Notice that we use the word 'algebra' in a very liberal way, for example, a coalgebra is also an algebra in our sense. The variety of such algebras forms then an (algebraic) *theory*, say A; the formal definition is given in 1.2.

Our definition differs from the classical definition of an algebraic theory of Lawvere [19], it is, in a sense, more general. It covers all natural examples of algebras (associative, commutative, Lie, ...), A_m-algebras, coalgebras, bialgebras, and even such exotic objects as Drinfel'd algebras. But this generality has one unpleasant consequence: the forgetful functor $(V, \{m_i\}) \to V$ need not have a left adjoint (example: coalgebras). Notice also that there are natural examples where the set of operations, as well as the set of axioms, is infinite (example: A_∞-algebras).

0.3 The basic idea of the construction is the following. Having an algebra A (in the sense of 0.2), we can consider A as a point of some variety $\mathcal{M}$ of structure constants. A deformation of A can be then interpreted as a deformation of the point A *in* $\mathcal{M}$. Such deformations are then related, by [20], with the cotangent cohomology $T^*(\mathbf{k}[\mathcal{M}]/\mathbf{k}, \mathbf{k})$ of the affine coordinate ring $\mathbf{k}[\mathcal{M}]$ of $\mathcal{M}$. The problem is that, if $\mathcal{M}$ is 'too special', this object need not be isomorphic with the 'good deformation cohomology', for example, if $A = (V, \mu)$, $\mu : V^{\otimes 2} \to V$, is an associative algebra and $\mathcal{M}$ the variety of associative multiplications on V, then $T^*(\mathbf{k}[\mathcal{M}]/\mathbf{k}, \mathbf{k})$ is not, in general, isomorphic to the Hochschild cohomology (we are indebted for this observation to M. Schlessinger).

So, the idea is, very roughly speaking, to replace $\mathbf{k}[\mathcal{M}]$ by something like the affine coordinate ring of the 'universal' variety of structure constants for algebras of the theory $\mathcal{A}$, the rôle of this object being played by a certain strict symmetric monoidal category, see 1.2. This, of course, in turn implies the necessity to define the notion of the *cotangent complex of a category*. The possibility of such a definition was made possible by the recent development of the theory of monoidal categories, see [14], [15] or [30]. Here we point out also the definition of a *module* over a strict monoidal category, given in 1.17. It was only after we realized what the notion of a module is (though the definition is very natural, modules are, as usual, abelian group objects in a suitable over-category), when the full development of the theory was possible.

0.4 As for practical computation, it may be already clear from the hints of 0.3 that the main problem is related with the description of 'relations among axioms'. One way to detect these relations is the 'deviation calculus' described (and also applied to some explicit calculations) in [25]. Another way to visualize these relations is the 'naive' abstract tensor calculus in the sense of [17], this approach will be used in the present paper.

The next problem is, having already found some relations, to prove that they generate all relations. This problem is related with a general coherence problem for a category as it is presented in [18]. For example, we prove in §4 that, as a nontrivial consequence of some arguments used in the proof of the celebrated Mac Lane-Stasheff coherence theorem [21], [29], the 'good deformation theory' for associative algebras is isomorphic with the Hochschild cohomology. See also the discussion in 4.5.

All the computations and considerations of the present paper seems to be a shade of something like 'a homological algebra of the monoidal category', except that, as far as we know, there is nothing like that in the literature; what is usually meant by the cohomology of a category or of a theory in the sense of, for example, [26] or [13], seems to have nothing to do with our calculations.

Plan of the paper:

1. Theories and Modules
2. Resolutions and the Cohomology
3. Deformations and the Cohomology
4. Resolutions and the Coherence (with some open questions)

1. Theories and Modules

In this paragraph we introduce the basic technical tools of our theory. We always work over a field $\mathbf{k}$ of characteristic zero, though we believe that a good deal of the theory makes sense over a field of an arbitrary characteristic. For two maps f and g let fg or $f(g)$ denote the usual (i.e. used by human beings) composition g followed by f. We shall sometimes use also the covariant notation $g \circ f$ for the same composition, so one must take into account the switch $g \circ f = fg = f(g)$.

1.1 For the convenience of the reader we recall here some more or less classical notions. A *strict monoidal category* is a triple $\mathcal{C} = (\mathcal{C}, \otimes, I)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor and $I \in \mathcal{C}$ is an object such that

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \text{ for all } A, B, C \in \mathcal{C} \text{ (associativity), and}$$
$$A \otimes I = A \text{ and } I \otimes A = A \text{ for all } A \in \mathcal{C} \text{ (I is the two-sided identity).}$$

Recall that a strict monoidal category is a special case of a monoidal category (see [22, VII.1] or [21] for the definition), but we will not need this more complex notion here. As usual, the associativity enables us to write multiple products, such as $A \otimes B \otimes C$, without parentheses.

Let $\mathcal{C}$ be a strict monoidal category as above. By a *symmetry* on $\mathcal{C}$ we mean a functorial map $S = S_{A,B} : A \otimes B \to B \otimes A$ given for any two objects $A, B \in \mathcal{C}$, such that the diagrams ($1\!\!1$ denotes the identity)

$$\begin{tikzcd} A \otimes B \arrow[rr, "1\!\!1"] \arrow[dr, "S_{A,B}"'] & & A \otimes B \\ & B \otimes A \arrow[ur, "S_{B,A}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A\otimes B\otimes C \arrow[rr, "S_{A,B\otimes C}"] \arrow[dr, "S_{A,B} \otimes 1\!\!1"'] & & B\otimes C\otimes A \\ & B\otimes A \otimes C \arrow[ur, "1\!\!1 \otimes S_{A,C}"'] & \end{tikzcd}$$

commute. The functoriality of S means that for any two maps $f : A \to A'$ and $g : B \to B'$ the following diagram

$$\begin{tikzcd} A \otimes B \arrow[r, "f \otimes g"] \arrow[d, "S_{A,B}"'] & A' \otimes B' \arrow[d, "S_{A',B'}"] \\ B \otimes A \arrow[r, "g \otimes f"] & B' \otimes A' \end{tikzcd}$$

commutes. Notice that a symmetry on a monoidal category is a special case of a braiding, see [14]. By a *functor* $F : \mathcal{C} \to \mathcal{D}$ between two strict symmetric monoidal categories we always mean a *strict symmetric monoidal* functor, i.e. a functor which respects the strict symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, respectively.

A typical example of a strict symmetric monoidal category is the category **Vect** of **k**-vector spaces. The operation $\otimes$ is the usual tensor product over **k**, the identity object I is **k** itself and the symmetry $S_{U,V} : U \otimes V \to V \otimes U$ takes $u \otimes v$ onto $v \otimes u$.

1.2 By a *theory* we mean a strict symmetric monoidal category, say A, such that

- the objects are indexed by (or identified with) the set N of natural numbers,
- $m \otimes n = m + n$, for any $m, n \in \mathsf{N} = \mathrm{Ob}(\mathsf{A})$ (hence $I = 0$), and
- each hom-set $\mathsf{A}(m,n)$ carries a structure of a k-vector space and all the operations of the category A (the composition $\circ$, $\otimes$ and S) are compatible with this structure.

Theories in the sense of the above definition and their (strict symmetric monoidal) functors form a category which we denote by $\mathfrak{Theories}$.

By an A-*algebra* we mean a (strict symmetric monoidal) functor $A : \mathsf{A} \to \mathbf{Vect}$. It is clear that the value $A(1)$ can be interpreted as the underlying vector space of the algebra A. The category of A-algebras will be denoted by A-**algebras**. Notice that there is a resemblance of our definition and the classical definition of Lawvere [19], compare also the comments in 0.2.

1.3 Let $\mathfrak{Sets}$ denote the category of sets and consider a map $X : \mathsf{N} \otimes \mathsf{N} \to \mathfrak{Sets}$, $X = \{X(m,n)\}_{m,n \geq 0}$, we will call such a map a *core*. Denote by $\mathfrak{Cores}$ the corresponding category. Using the terminology of [22, II.6] we can define $\mathfrak{Cores}$ simply as the category $(\mathsf{N} \times \mathsf{N} \downarrow \mathfrak{Sets})$ of objects under $(\mathsf{N} \times \mathsf{N}) \in \mathfrak{Sets}$. We have the 'forgetful' functor $\Box : \mathfrak{Theories} \to \mathfrak{Cores}$ defined simply by ignoring the **k**-vector space structure on the sets $\mathsf{A}(m,n)$. Using the same kind of arguments as, for example, in [22, II.7] or [30], we can easily show that this functor has a left adjoint $\mathcal{F} : \mathfrak{Cores} \to \mathfrak{Theories}$; it is natural to call the object $\mathcal{F}(X)$ the *free theory* on the core X. Let us give a few of examples to illustrate this notion.

1.4 Example: Sym and $\mathcal{F}(\emptyset)$. Let S_m be, for $m > 0$, the symmetric group on m elements and put, by definition, $S_0 = \{1\}$. Let us define

$$\mathbf{Sym}(m,n) = \begin{cases} \mathbf{k}[S_m], & m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is immediately to see that the usual composition of permutations, linearly extended over the group ring $\mathbf{k}[S_m]$, defines a category with N as the set of objects and $\mathbf{Sym}(m,n)$ as the hom-sets. We denote this category by **Sym**. To define a symmetric monoidal structure on **Sym**, we put, for two permutations $\sigma \in S_m$,

$\sigma : \{1, \dots, m\} \to \{1, \dots, m\}$ and $\nu \in S_n$, $\nu : \{1, \dots, n\} \to \{1, \dots, n\}$,

$$(\sigma \otimes \nu)(i) = \begin{cases} \sigma(i),\ 1 \leq i \leq m, \text{ and} \\ m + \nu(i - m) \text{ for } m < i \leq n + m. \end{cases}$$

Finally, we define the 'switch' $S_{m,n} : m + n \to n + m$ by

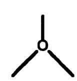
$$S_{m,n}(i) = \begin{cases} n + i \text{ for } 1 \leq i \leq m \text{ and} \\ i - m \text{ for } m < i \leq m + n \end{cases}$$

It is an easy exercise to prove that the operations defined above (or, more precisely, their linear extensions) equip **Sym** with the structure of a theory (denoted again by **Sym**).

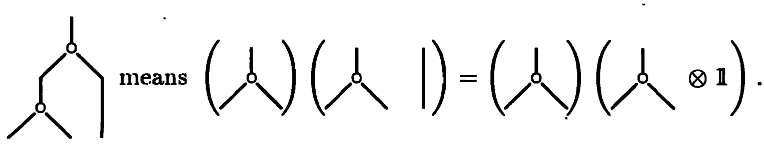
Let $\emptyset$ denote the empty core (= the sets $X(m,n)$ are empty for all $m, n \geq 0$). Then it is more or less clear that **Sym** is the free theory on the empty core $\emptyset$, $\mathbf{Sym} = \mathcal{F}(\emptyset)$, compare the arguments in [14].

For an arbitrary core X, there is a natural map $\emptyset \to X$ which implies the existence of an *embedding* $\mathbf{Sym} = \mathcal{F}(\emptyset) \mapsto \mathcal{F}(X)$. The existence of such an embedding enables us to consider elements of $\mathbf{k}[S_m]$ in a natural way as elements of $\mathcal{F}(X)(m,m)$ for any natural m. There is a very easy explicit description of this embedding: let σ_i be the generator of S_m interchanging i-th and $(i+1)$-th elements. Then σ_i is mapped onto $\mathbb{1}^{i-1} \otimes S_{1,1} \otimes \mathbb{1}^{n-i-1}$.

1.5 In what follows we will use the 'naive' abstract tensor calculus [17, I.8]. For example, the symbol



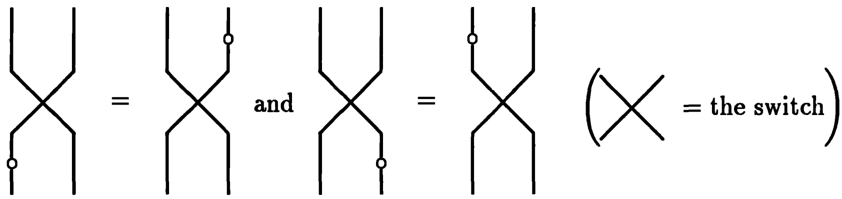
will denote a map having two 'imputs' and one 'output', i.e. an element of $\mathsf{A}(2,1)$ of some theory A. Also the composition will be denoted in a natural way, for instance



1.6 The following notation will be useful in the sequel. Having an index set J and a couple (s_j, t_j) of natural numbers, given for any $j \in J$, then any sequence $\{\xi_{s_j,t_j}\}$ determines a core X with $X(m,n) = \{\xi_{s_j,t_j};\ s_j = m \text{ and } t_j = n\}$. For example, the one-element sequence $\{\xi_{11}\}$ determines a core with $X(1,1) = \{\xi_{11}\}$ and $X(m,n) = \emptyset$ otherwise.

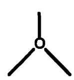
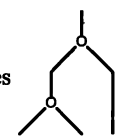
1.7 Example: $\mathcal{F}(\xi_{11})$. We claim that $\mathcal{F}(\xi_{11})(m,n) = \emptyset$ for $m \neq n$ (this is trivial) and that every element of $\mathcal{F}(\xi_{11})(m,m)$ can be written as a **k**-linear combination of elements of the form $(\xi_{11}^{l_1} \otimes \cdots \otimes \xi_{11}^{l_m})\sigma$, where $l_1, \ldots, l_m$ are natural numbers, $\sigma \in S_m$, and ξ_{11}^{l} denotes the composition $\xi_{11} \circ \cdots \circ \xi_{11}$ (l-times).

The proof is an easy exercise; it is based on the functoriality property of the switch S (see 1.1) which implies that the relation



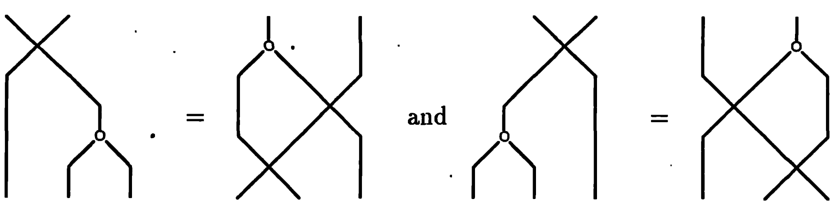
is satisfied, for  $\in \mathsf{A}(1,1)$, in each theory A.

1.8 Example: $\mathcal{F}(\xi_{21})$. Let $\mathcal{B}_n$ denote, for $n \geq 1$, the set of all (full) bracketings of n nonassociative indeterminates. The fact that $\coprod \mathcal{B}_n$ with the evident multiplication is the free monoid on one element [27, I.4] implies that the elements of $\mathcal{B}_n$ can be used to encode n-fold compositions of ξ_{21}. For example, $(\bullet)$ denotes the identity,

$(\bullet,\bullet)$ denotes  $= \xi_{21}$, $((\bullet,\bullet),\bullet)$ denotes , and so on,

see also [2, Chapt. I] or [29] or 4.3 of this paper. This makes possible to identify the elements of $\mathcal{B}_n$ with corresponding maps in $\mathcal{F}(\xi_{21})(n,1)$.

Again as a consequence of the functoriality of S, which implies that



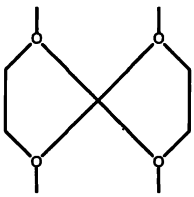
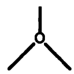
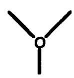
we get that, for any $m \geq n$, every element of $\mathcal{F}(\xi_{21})(m,n)$ decomposes as a **k**-linear combination of elements of the form

$$(\alpha_1 \otimes \cdots \otimes \alpha_n)\sigma,\ \sigma \in S_m,\ \alpha_i \in \mathcal{B}_{l_i},\ l_i \geq 1 \text{ and } \sum l_i = m,$$

for $m < n$ is the set $\mathcal{F}(\xi_{21})(m,n)$, of course, empty.

The dual case of $\mathcal{F}(\xi_{12})$ can be handled analogically just by 'turning all the arguments upside down'. This is a way in which the duality works in our theory, see also the comments in 3.14.

1.9 In all the examples above we showed that an arbitrary map can be written as a k-linear combination of elements either of the form $\phi\sigma$ or $\sigma\phi$ (the former one in the case of $\mathcal{F}(\xi_{12})$), where $\sigma \in S_m$ for some $m \geq 1$ and the map ϕ 'does not contain' the switch. This is, however, not always true. As an example take the element

 (with  $= \xi_{21}$ and  $= \xi_{12}$)

of $\mathcal{F}(\xi_{21}, \xi_{12})$. This element will play an important rôle in the theory of bialgebras, see 1.15.

1.10 The examples of algebras related with free theories are very plain. An $\mathcal{F}(\xi_{11})$-algebra is nothing but a couple (V, d) of a vector space V and an endomorphism d, an $\mathcal{F}(\xi_{21})$-theory is a couple (V, μ) where $\mu : V^{\otimes 2} \to V$ is a bilinear map, an $\mathcal{F}(\xi_{03})$-theory is a couple (V, Φ) where Φ is an element of $V^{\otimes 3}$, etc.

1.11 Let A be a theory and J an index set. Suppose we are given a sequence $r_j \in \mathsf{A}(m_j, n_j)$. Then it is easy to see that there exist a theory $\mathsf{A}/(r_j; j \in J)$ together with a map $\pi : \mathsf{A} \to \mathsf{A}/(r_j; j \in J)$ having the property that, for any theory B and for any map $f : \mathsf{A} \to \mathsf{B}$ such that $f(r_j) = 0$ in $\mathsf{B}(m_j, n_j)$, $j \in J$, there exists a unique $\psi : \mathsf{A}/(r_j; j \in J) \to \mathsf{B}$ such that the diagram

$$\begin{tikzcd} \mathsf{A} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathsf{B} \\ \mathsf{A}/(r_j; j \in J) \arrow[ur, "\psi"'] & \end{tikzcd}$$

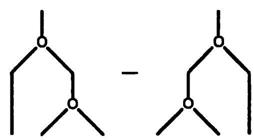
commutes.

1.12 Example: Complexes. Let $\mathsf{A} = \mathcal{F}(\xi_{11})/(r_1)$, where

$r_1 =$ 

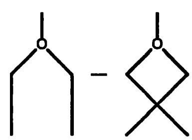
Then an A-algebra is a couple (V, d) of a vector space V and an endomorphism $d : V \to V$ such that $d^2 = 0$.

1.13 Example: Associative algebras. Consider $\mathsf{A} = \mathcal{F}(\xi_{21})/(r_1)$ with

$r_1 =$ 

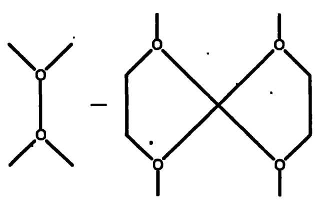
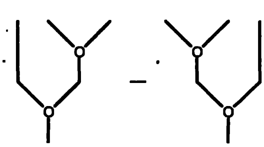
Then an A-algebra is a couple (V, μ) of a vector space and an associative multiplication $\mu : V^{\otimes 2} \to V$.

1.14 Example: Commutative associative algebras. Consider $\mathsf{A} = \mathcal{F}(\xi_{21})/(r_1, r_2)$ with r_1 as in 1.13 and

$r_2 =$ 

Then an A-algebra is an associative commutative algebra.

1.15 Example: Bialgebras. Let $\mathsf{A} = \mathcal{F}(\xi_{21}, \xi_{12})/(r_1, r_2, r_3)$, where r_1 is the same as in 1.13,

$r_2 =$  and $r_3 =$ 

Algebras of the theory A are exactly the bialgebras in the sense of, for example, [25], r_1 encodes the associativity, r_2 encodes the compatibility of the multiplication and the comultiplication and r_3 encodes the coassociativity.

1.16 Example: Dridfel'd algebras. As the last examle of this kind we discuss briefly Drinfel'd algebras. Following [3] and [4], by a *Drinfel'd algebra* (or a *quasi-bialgebra* in the terminology of [9]) we mean an object of the form $A = (V, \mu, \Delta, \Phi)$, where V is a k-linear space, $\mu : V \otimes V \to V$ (the product) and $\Delta : V \to V \otimes V$ (the coproduct) are linear maps and $\Phi \in V \otimes V \otimes V$ is an invertible (in the natural product structure induced on $V \otimes V \otimes V$ by μ) element. Moreover, μ is supposed to be associative, μ and Δ are supposed to be compatible (see the relation r_2 of 1.15) and we also assume that the product μ has an unit $1 \in V$ and that $\Delta(1) = 1 \otimes 1$. The usual coassociativity condition on Δ is in the Drinfel'd case replaced by

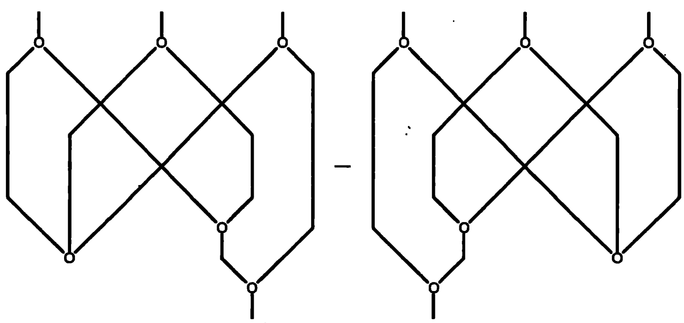
$$(1) \qquad (\mathbb{1} \otimes \Delta)\Delta \cdot \Phi = \Phi \cdot (\Delta \otimes \mathbb{1})\Delta$$

and, moreover, the validity of the following 'pentagon' condition is supposed:

$$(2) \qquad (\mathbb{1}^2 \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1}^2)(\Phi) = (1 \otimes \Phi) \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \cdot (\Phi \otimes 1).$$

In both equations above $\cdot$ denotes the multiplication induced by μ.

If we neglect, for simplicity, the rôle of the unit, the structure of a Drinfel'd algebra is encoded by the theory of the form $\mathcal{F}(\xi_{12}, \xi_{12}, \xi_{03})/(r_1, r_2, r_3, r_4)$, where r_1 and r_2 are as in 1.15. To describe the condition (1) we put

$r_3 =$ 

We leave the relation r_4, encoding (2), as an exercise to the reader. The pictures based on the abstract tensor calculus are in the case of Drinfel'd algebras already too baroque. More appropriate way to handle this case is the deviation calculus of [25].

1.17 Let A be a theory. Modules over A are, as usual, abelian group objects in the over-category $\mathfrak{Theories}/\mathsf{A}$, but we prefer to give the following explicit definition.

By a *module* over a theory A we mean a core $\mathbf{M} = \{\mathbf{M}(m,n)\}_{m,n\geq 0}$ such that each $\mathbf{M}(m,n)$ has, for $m, n \geq 0$, a structure of a vector space and there are bilinear operations

$$\begin{aligned}
\circ = \circ_l &: \mathsf{A}(m,n) \times \mathbf{M}(n,k) \to \mathbf{M}(m,k), && \text{(left composition)}\\
\circ = \circ_r &: \mathbf{M}(m,n) \times \mathsf{A}(n,k) \to \mathbf{M}(m,k), && \text{(right composition)}\\
\otimes = \otimes_l &: \mathsf{A}(m_1,n_1) \times \mathbf{M}(m_2,n_2) \to \mathbf{M}(m_1+m_2, n_1+n_2), && \text{(left tensoring)}\\
\otimes = \otimes_r &: \mathbf{M}(m_1,n_1) \times \mathsf{A}(m_2,n_2) \to \mathbf{M}(m_1+m_2, n_1+n_2), && \text{(right tensoring)}
\end{aligned}$$

for all natural m, n, k, m_1, m_2, n_1 and n_2.

These operations must satisfy the following axioms (notice the shorthand, it gives us almost for free the proper axioms for modules, compare also [23]):

$$f \circ (g \circ h) = (f \circ g) \circ h,\ f \in \Box_1(m,n),\ g \in \Box_2(n,k) \text{ and } h \in \Box_3(k,l),$$

where $(\Box_1, \Box_2, \Box_3) = (\mathsf{A}, \mathsf{A}, \mathbf{M})$, $(\mathsf{A}, \mathbf{M}, \mathsf{A})$ or $(\mathbf{M}, \mathsf{A}, \mathsf{A})$, $m, n, k, l \geq 0$. To elucidate the use of this shorthand, take, for example, $(\Box_1, \Box_2, \Box_3) = (\mathsf{A}, \mathsf{A}, \mathbf{M})$. Then the above axiom reads as

$$f \circ_l (g \circ_l h) = (f \circ g) \circ_l h$$

for $f \in \mathsf{A}(m,n)$, $g \in \mathsf{A}(n,k)$ and $h \in \mathbf{M}(k,l)$. Next, we require

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h,\ f \in \Box_1(m_1,n_1),\ g \in \Box_2(m_2,n_2) \text{ and } h \in \Box_3(m_3,n_3),$$

where $(\Box_1, \Box_2, \Box_3) = (\mathsf{A}, \mathsf{A}, \mathbf{M})$, $(\mathsf{A}, \mathbf{M}, \mathsf{A})$ or $(\mathbf{M}, \mathsf{A}, \mathsf{A})$ and $m_1, n_1, m_2, n_2, m_3, n_3 \geq 0$. The last condition is

$$(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2),$$

for $f_1 \in \Box_1(k_1,k_2)$, $f_2 \in \Box_2(k_2,k_3)$, $g_1 \in \Box_3(l_1,l_2)$ and $g_2 \in \Box_4(l_2,l_3)$; $(\Box_1, \Box_2, \Box_2, \Box_4) = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \mathbf{M})$, $(\mathsf{A}, \mathsf{A}, \mathbf{M}, \mathsf{A})$, $(\mathsf{A}, \mathbf{M}, \mathsf{A}, \mathsf{A})$ or $(\mathbf{M}, \mathsf{A}, \mathsf{A}, \mathsf{A})$, $k_1, k_2, k_3, l_1, l_2, l_3 \geq 0$.

1.18 We denote by A-**biMod** the category of A-modules in the sense of the above definition. It is easy to see that A-**biMod** is an abelian category. The vocabulary from the category of (usual) modules over a (usual) algebra translates 'coordinatewise' in an obvious way. For example, for $\mathbf{M}, \mathbf{N} \in \mathsf{A}$-**biMod**, we say that $\mathbf{M}$ is a *submodule* of $\mathbf{N}$ if $\mathbf{M}(m,n) \subset \mathbf{N}(m,n)$ for all $m, n \geq 0$. Similarly, for a map $f : \mathbf{M} \to \mathbf{N}$ we let $\mathrm{Ker}(f)$ to be the submodule of $\mathbf{M}$ defined by $\mathrm{Ker}(f)(m,n) = \mathrm{Ker}(f|_{\mathbf{M}(m,n)} : \mathbf{M}(m,n) \to \mathbf{N}(m,n))$, $m, n \geq 0$, and so on. Finally, having a module $\mathbf{M}$ and a system $S = \{s_j \in \mathbf{M}(m_j, n_j);\ j \in J\}$, we define the submodule *generated* by S (in $\mathbf{M}$) to be the intersection of all submodules $\mathbf{N}$ of $\mathbf{M}$ such that $s_j \in \mathbf{N}(m_j, n_j)$ for all $j \in J$.

1.19 Let A be a theory and let $\mathbf{M}$ be an A-module. Let $\mathrm{Der}(\mathsf{A}, M)$ be the set of all sequences $\theta = \{\theta_{mn}\}_{m,n \geq 0}$ of linear maps $\theta_{mn} : \mathsf{A}(m,n) \to \mathbf{M}(m,n)$ such that

$$\begin{aligned} \theta(f \circ g) &= f \circ_l \theta(g) + \theta(f) \circ_r g,\ f \in \mathsf{A}(m,n),\ g \in \mathsf{A}(n,k) \text{ and} \\ \theta(f \otimes g) &= f \otimes_l \theta(g) + \theta(f) \otimes_r g,\ f \in \mathsf{A}(m_1,n_1) \text{ and } g \in \mathsf{A}(m_2,n_2), \end{aligned}$$

where m, n, k, m_1, m_2, n_1 and n_2 are natural numbers.

It is obvious that both Der(A, $\mathbf{M}$) (the set of derivations) and $\mathrm{Hom}_{\mathsf{A}}(\mathbf{M},\mathbf{N})$ (the set of A-algebra homomorphisms from $\mathbf{M}$ to $\mathbf{N}$) has a natural structure of a vector space.

1.20 Example: Free modules and the 'basic field'. Similarly as in the case of theories, there exists a 'forgetful' functor $\square$: A-biMod $\to$ $\mathfrak{Cores}$. This functor has again a left adjoint which we denote by $\mathsf{A}\langle - \rangle : \mathfrak{Cores} \to \mathsf{A}$-biMod. We call $\mathsf{A}\langle X \rangle$ the *free* A-*module* on the core X.

Next, let $\mathbf{k}_V$ be, for a given vector space V, the full subcategory of **Vect** whose objects are tensor powers $V^{\otimes n}$, $n \geq 0$ (i.e, $\mathbf{k}_V$ is the subcategory of **Vect**, $\otimes$-generated by V). It is obvious that $\mathbf{k}_V$ inherits from **Vect** the structure of a strict monoidal category. It is equally obvious that the image of an A-algebra $A : \mathsf{A} \to \mathbf{Vect}$ with $A(1) = V$ belongs to $\mathbf{k}_V$ and that A defines on $\mathbf{k}_V$ the structure of an A-module. As we will see later, the category $\mathbf{k}_V$ plays in our theory the rôle of the basic field (this is why we choose the ambiguous notation $\mathbf{k}_V$ for it) and that the algebra A can be considered as an 'augmentation' of the theory A.

2. Resolutions and the Cohomology

The goal of this paragraph is to define an analog of the 'classical' cotangent cohomology functor $T^i(-;-)$, $i = 0, 1, 2$, which was introduced, for example, in [20], see also the discussion in 0.3. Most important for our purpose is the first cohomology, which is intimately related with deformations (see Proposition 3.10), therefore all the exposition aims toward this object.

2.1 Let A be a theory. A *pre-resolution* of A is an object $\mathcal{R}$ of the form

$$\mathsf{A} \xleftarrow{\pi} \mathcal{F}(X) \xleftarrow{\alpha} \mathcal{F}(X)\langle Y \rangle \xleftarrow{\beta} \mathcal{F}(X)\langle Z \rangle, \tag{3}$$

where X, Y and Z are cores, π is an epimorphism (i.e. a map such that π_{mn} : $\mathcal{F}(X)(m,n)$
$\to \mathsf{A}(m,n)$ is epic for all $m, n \geq 0$) of theories, α and β are morphisms of $\mathcal{F}(X)$-modules, $\alpha\beta = 0$ and $\mathrm{Im}(\alpha) = \mathrm{Ker}(\pi)$.

2.2 Let $\mathcal{R}$ be a pre-resolution as in 2.1 and consider the submodule $\mathbf{O} \subset \mathcal{F}(X)\langle Y \rangle$ generated by elements
$\alpha(a) \circ_l b - a \circ_r \alpha(b)$, $a \in \mathcal{F}(X)\langle Y \rangle(m,n)$, $b \in \mathcal{F}(X)\langle Y \rangle(n,k)$, $m, n, k \geq 0$, and
$\alpha(a) \otimes_l b - a \otimes_r \alpha(b)$, $a \in \mathcal{F}(X)\langle Y \rangle(m_1,n_1)$, $b \in \mathcal{F}(X)\langle Y \rangle(m_2,n_2)$, $m_1, m_2, n_1, n_2 \geq 0$,

and call $\mathbf{O}$ the *submodule of obvious relations*. It is immediate to see that $\alpha(\mathbf{O}) = 0$. We say that $\mathcal{R}$ is a *resolution* of A, if $\mathrm{Ker}(\alpha)$ is generated by $\mathrm{Im}(\beta)$ and $\mathbf{O}$. The fact that for any theory there exists a resolution is an easy consequence of the following lemma, whose proof is an exercise.

Lemma 2.3 *For any theory* A *there exist a core* X *and an epimorphism* $\mathcal{F}(X) \to \mathsf{A}$. *Similarly, for any module* $\mathbf{M} \in \mathsf{A}$-$\mathbf{biMod}$ *there exist a core* U *and an epimorphism* $\mathsf{A}\langle U\rangle \to \mathbf{M}$.

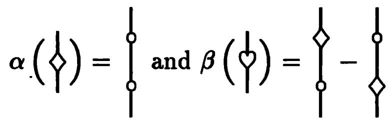
2.4 Example Let us consider $\mathsf{A} = \mathcal{F}(\xi_{11})/(r_1)$, $r_1 = \xi_{11} \circ \xi_{11}$, as in 1.12. Let us denote

$\xi_{11} =$  and let $\eta_{11} =$  and $\zeta_{11} =$  be new independent variables.

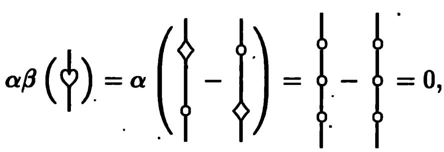
We claim that the object

(4) $$\mathsf{A} \xleftarrow{\pi} \mathcal{F}(\xi_{11}) \xleftarrow{\alpha} \mathcal{F}(\xi_{11})\langle\eta_{11}\rangle \xleftarrow{\beta} \mathcal{F}(\xi_{11})\langle\zeta_{11}\rangle$$

where



is a resolution of A. It follows from the definition of A that $\mathrm{Im}(\alpha) = \mathrm{Ker}(\pi)$. As for $\alpha\beta = 0$, it is clearly enough to verify this condition on the generator ζ_{11}. We have



which proves $\alpha\beta = 0$.

The most difficult task is to prove that the condition of 2.2 is satisfied. This is, by definition, the same as to show that an arbitrary element of $\mathrm{Ker}(\alpha)$ is zero modulo the the submodule of $\mathcal{F}(\xi_{11})\langle\eta_{11}\rangle$ generated by the relations

(5) $$\eta_{11} \circ a \circ \xi_{11}^2 = \xi_{11}^2 \circ a \circ \eta_{11},\ a \in \mathcal{F}(\xi_{11})(1,1)$$

(6) $$(\phi_1 \circ \eta_{11} \circ \phi_2) \otimes b \otimes (\psi_1 \circ \xi_{11}^2 \circ \psi_2) = (\phi_1 \circ \xi_{11}^2 \circ \phi_2) \otimes b \otimes (\psi_1 \circ \eta_{11} \circ \psi_2),$$

$$\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{F}(\xi_{11})(1,1),\ b \in \mathcal{F}(\xi_{11})(l,l),\ l \geq 0, \text{ and}$$

(7) $$\xi_{11} \circ \eta_{11} = \eta_{11} \circ \xi_{11}$$

(the first relation is obviously a consequence of the third one).

Let us introduce first some notation. Let $\mathcal{F}_0(\xi_{11})\langle\eta_{11}\rangle(m,m)$ denote, for $m \geq 1$, the vector subspace of $\mathcal{F}(\xi_{11})\langle\eta_{11}\rangle(m,m)$ generated by all elements of the form $a_1 \otimes \cdots a_m$, $a_i \in \mathcal{F}(\xi_{11})(m,m)$ or $a_i \in \mathcal{F}(\xi_{11})\langle\eta_{11}\rangle(m,m)$, $1 \leq i \leq m$. We can show, similarly as in Example 1.7, that an arbitrary element of $\mathcal{F}(\xi_{11})\langle\eta_{11}\rangle(m,m)$ can be written as a linear combination of elements of the form $u\sigma$ with $u \in \mathcal{F}_0(\xi_{11})\langle\eta_{11}\rangle(m,m)$ and $\sigma \in S_m$. The equation $\alpha(u\sigma) = \alpha(u)\sigma$, following from the fact that α is an $\mathcal{F}(\xi_{11})$-module homomorphism, enables us to infer that an arbitrary element of $\mathrm{Ker}(\alpha)(m,m)$ is of the form $\sum_{j\in J} u^j\sigma^j$ with $u^j \in \mathrm{Ker}(\alpha)(m,m) \cap \mathcal{F}_0(\xi_{11})\langle\eta_{11}\rangle(m,m)$ and $\sigma^j \in S_m$, $j \in J$. This enables us to restrict our attention to the intersection $\mathrm{Ker}(\alpha)(m,m) \cap \mathcal{F}_0(\xi_{11})\langle\eta_{11}\rangle(m,m)$ only.

An arbitrary element u of $\mathcal{F}_0(\xi_{11})\langle\eta_{11}\rangle(m,m)$, $m \geq 1$, can be plainly written as

$$u = \sum_{j\in J} A_j \cdot (\omega_1^j \otimes \cdots \otimes \omega_{l_j-1}^j \otimes \epsilon^j \otimes \omega_{l_j+1}^j \otimes \cdots \otimes \omega_m^j), \tag{8}$$

where $\omega_i^j \in \mathcal{F}(\xi_{11})(1,1)$, $\epsilon^j \in \mathcal{F}(\xi_{11})\langle\eta_{11}\rangle(1,1)$ and $A_j \in \mathbf{k}$, for $j \in J$. Let us introduce also the notation $\{r\} = \xi_{11}^r \in \mathcal{F}(\xi_{11})(1,1)$ for $r \geq 0$ and $\{p,q\} = \xi_{11}^{q-1} \circ \eta_{11} \circ \xi_{11}^{p-q} \in \mathcal{F}(\xi_{11})\langle\eta_{11}\rangle(1,1)$, for $p \geq 1$ and $1 \leq q \leq p$. Notice that $\{p,q\} = \{p,1\}$ modulo the relation (7). Thus we can suppose that the element u from (8) is of the form

$$u = \sum_{j\in J} A_j \cdot (\{r_1^j\} \otimes \cdots \{r_{l_j-1}^j\} \otimes \{p^j, 1\} \otimes \{r_{l_j+1}^j\} \otimes \cdots \otimes \{r_m^j\}$$

for some $r_i^j \geq 0$ and $p^j \geq 1$. We immediately have

$$\alpha(u) = \sum_{j\in J} A_j \cdot (\{r_1^j\} \otimes \cdots \{r_{l_j-1}^j\} \otimes \{p^j + 1\} \otimes \{r_{l_j+1}^j\} \otimes \cdots \otimes \{r_m^j\}.$$

Clearly $\alpha(u) = 0$ if and only if

$$\sum_{j\in J} A_j \cdot (r_1^j, \ldots, r_{l_j-1}^j, p^j + 1, r_{l_j+1}^j, \ldots, r_m^j) = 0 \tag{9}$$

as an element of $\mathbf{k}^m = \mathbf{k} \oplus \cdots \oplus \mathbf{k}$ (m times). Let us decompose u as $\sum u_{\mathbf{n}}$, where the summation is taken over all m-tuples $\mathbf{n} = (n_1, \ldots, n_m)$, $n_i \geq 0$, $1 \leq i \leq m$ and

$$u_{\mathbf{n}} = \sum_{j\in J_{\mathbf{n}}} A_j \cdot (\{r_1^j\} \otimes \cdots \{r_{l_j-1}^j\} \otimes \{p^j, 1\} \otimes \{r_{l_j+1}^j\} \otimes \cdots \otimes \{r_m^j\},$$

where $J_{\mathbf{n}} = \{j \in J; (r_1^j, \cdots, r_{l_j-1}^j, p^j + 1, r_{l_j+1}^j, \cdots, r_m^j) = (n_1, \ldots, n_m)\}$. The equation (9) implies that $u \in \mathrm{Ker}(\alpha)(m,m)$ if and only if $u_{\mathbf{n}} \in \mathrm{Ker}(\alpha)(m,m)$ for all $\mathbf{n}$.

Let us fix some $\mathbf{n} = (n_1, \dots, n_m)$. It follows from the very definition that $u_{\mathbf{n}}$ may be nonzero only if there exists some i_0, $1 \le i_0 \le m$ with $n_{i_0} \ge 2$. We can see now that, modulo the relation (6), $u_{\mathbf{n}}$ can be written as

$$u_{\mathbf{n}} = \sum_{j \in J_{\mathbf{n}}} A_j \cdot (\{n_1\} \otimes \cdots \otimes \{n_{i_0-1}\} \otimes \{n_{i_0}, 1\} \otimes \{n_{i_0+1}\} \otimes \cdots \{n_m\}$$

and for such a $u_{\mathbf{n}}$ clearly $\alpha(u_{\mathbf{n}}) = 0$ if and only if $u_{\mathbf{n}} = 0$. We have proved that (4) is a resolution of $\mathsf{A} = \mathcal{F}\xi_{11})/(\xi_{11} \circ \xi_{11})$.

2.5 Let $\mathcal{R}$ be a resolution (3) of A and let $\mathbf{M} \in \mathsf{A}$-**biMod**. The map $\pi : \mathcal{F}(X) \to \mathsf{A}$ induces on $\mathbf{M}$ the structure of an $\mathcal{F}(X)$-module. We can relate with $\mathcal{R}$ and $\mathbf{M}$ the following complex of vector spaces, $\mathcal{E}^* = \mathcal{E}^*(\mathcal{R}, \mathbf{M})$:

$$(10) 0 \longrightarrow \mathrm{Der}(\mathcal{F}(X), \mathbf{M}) \xrightarrow{\delta_1} \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Y\rangle, \mathbf{M}) \xrightarrow{\delta_2} \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Z\rangle, \mathbf{M}),$$

where δ_1 is defined by $\delta_1(\theta)_{mn} = \theta_{mn}(\alpha_{mn})$ and δ_2 is defined by $\delta_2(f)_{mn} = f_{mn}(\beta_{mn})$, $m, n \ge 0$. It is easy to check that these definitions are correct and that $\delta_2\delta_1 = 0$. We define

$$\mathcal{T}^i(\mathsf{A}; \mathbf{M}) = H^i(\mathcal{E}^*(\mathcal{R}, \mathbf{M}), \delta_*),\ i = 0, 1,$$

and call it the *cotangent cohomology* of the theory A with coefficients in $\mathbf{M}$. The usual acyclicity argument (recall that $\mathcal{R}$ is a *resolution*) together with the freeness of the objects under consideration enables us to prove that:

Proposition 2.6 *The definition of the cotangent cohomology does not depend on the particular choice of a resolution.*

We define also the second cotangent cohomology group, though we do not need it in the paper. Let $\mathcal{R}$ be a resolution as in (3) and let $U = \mathrm{Ker}(\alpha)$. Then the inclusion $U \hookrightarrow \mathcal{F}(X)\langle Y\rangle$ induces the map $\psi : \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Y\rangle, \mathbf{M}) \to \mathrm{Hom}_{\mathcal{F}(X)}(U, \mathbf{M})$. We put $\mathcal{T}^2(\mathsf{A}; \mathbf{M}) = \mathrm{coKer}(\psi)$.

2.7 As a matter of fact, it was possible to define the cotangent cohomology without making use of the third term $\mathcal{F}(X)\langle Z\rangle$ of (3), using only the kernel U (the module of 'relations among the axioms') similarly as it was done in [20] for the 'classical' cotangent cohomology. We could even avoid the use of the resolution and *define* $\mathcal{T}^1(-;-)$ by the similar formula as that of [20, Lemma 3.1.2] for $T^1(-,-)$. The problem is that all attempts to obtain an explicit description of $\mathcal{T}^1(-;-)$ in concrete examples require in fact all the information necessary for the construction of a resolution.

2.8 Exercise: Prove from the definition that $\mathcal{T}^0(\mathsf{A},\mathbf{M}) = \mathrm{Der}(\mathsf{A},\mathbf{M})$.

2.9 It will be useful (see 3.15) to consider the construction above also for a *pre-resolution* $\mathcal{R}$. Notice that the definition of $\mathcal{E}^*(\mathcal{R},\mathbf{M})$ still makes sense and put

$$\mathcal{T}^i_{\mathcal{R}}(\mathsf{A};\mathbf{M}) = H^i(\mathcal{E}^*(\mathcal{R},\mathbf{M}),\delta_*),\ i = 0,1.$$

Proposition 2.10 *For any pre-resolution $\mathcal{R}$ we have $\mathcal{T}^0_{\mathcal{R}}(\mathsf{A};\mathbf{M}) = \mathcal{T}^0(\mathsf{A};\mathbf{M})$ (this is obvious) and $\mathcal{T}^1_{\mathcal{R}}(\mathsf{A};\mathbf{M}) \supset \mathcal{T}^1(\mathsf{A};\mathbf{M})$.*

Proof To prove the inclusion, it is enough to realize that any pre-resolution (3) can be completed into a resolution by adding new generators to the core Z.

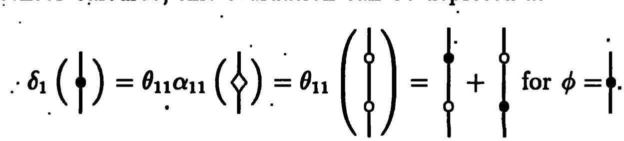
2.11 Example: Continuation of Example 2.4. Let $\mathsf{A} = \mathcal{F}(\xi_{11})/(\xi_{11}\circ\xi_{11})$ be as in Example 2.4. Let us compute, for $\mathbf{M} \in \mathsf{A}$-**biMod**, the cohomology $\mathcal{T}^1(\mathsf{A};\mathbf{M})$ as introduced above. The complex (10) has the form

$$(11)\ 0 \to \mathrm{Der}(\mathcal{F}(\xi_{11}),\mathbf{M}) \xrightarrow{\delta_1} \mathrm{Hom}_{\mathcal{F}(\xi_{11})}(\mathcal{F}(\xi_{11})\langle\eta_{11}\rangle,\mathbf{M}) \xrightarrow{\delta_2} \mathrm{Hom}_{\mathcal{F}(\xi_{11})}(\mathcal{F}(\xi_{11})\langle\zeta_{11}\rangle,\mathbf{M}).$$

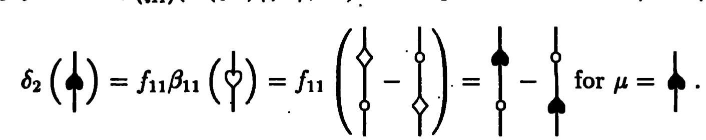
Each $\theta \in \mathrm{Der}(\mathcal{F}(\xi_{11}),\mathbf{M})$ is uniquely determined by its value on the generator ξ_{11}, which gives the identification $\mathrm{Der}(\mathcal{F}(\xi_{11}),\mathbf{M}) \cong \mathbf{M}(1,1)$. Similarly, an element $f \in \mathrm{Hom}_{\mathcal{F}(\xi_{11})}(\mathcal{F}(\xi_{11})\langle\eta_{11}\rangle,\mathbf{M})$ is determined by its value on the generator η_{11}, which gives that $\mathrm{Hom}_{\mathcal{F}(\xi_{11})}(\mathcal{F}(\xi_{11})\langle\eta_{11}\rangle,\mathbf{M}) \cong \mathbf{M}(1,1)$. In the same vain we may get also that $\mathrm{Hom}_{\mathcal{F}(\xi_{11})}(\mathcal{F}(\xi_{11})\langle\zeta_{11}\rangle,\mathbf{M}) \cong \mathbf{M}(1,1)$. Thus (11) can be written as

$$(12)\qquad 0 \longrightarrow \mathbf{M}(1,1) \xrightarrow{\delta_1} \mathbf{M}(1,1) \xrightarrow{\delta_2} \mathbf{M}(1,1).$$

To describe the value of δ_1 on some element $\phi \in \mathbf{M}(1,1)$ it is enough to evaluate $\theta_{11}\alpha_{11}(\eta_{11})$, where $\theta \in \mathrm{Der}(\mathcal{F}(\xi_{11}),\mathbf{M})$ is the derivation whose value at ξ_{11} is ϕ. Using the abstract tensor calculus, this evaluation can be depicted as



Similarly, to compute $\delta_2(\mu)$ for $\mu \in \mathbf{M}(1,1)$ we shall evaluate $f_{11}\beta_{11}(\zeta_{11})$, where we denoted by $f \in \mathrm{Hom}_{\mathcal{F}(\xi_{11})}(\mathcal{F}(\xi_{11})\langle\eta_{11}\rangle,\mathbf{M})$ the map whose value at η_{11} is μ. We have



Expanding the above pictures into formulas we get $\delta_1(\phi) = \phi \circ \xi_{11} + \xi_{11} \circ \phi$ and $\delta_2(\mu) = \mu \circ \xi_{11} - \xi_{11} \circ \mu$.

It is very instructive to verify that $\delta_2\delta_1(\phi) = 0$. We have

$$\begin{aligned} \delta_2\delta_1(\phi) &= \delta_2(\phi \circ \xi_{11} + \xi_{11} \circ \phi) =. \\ &= (\phi \circ \xi_{11}) \circ \xi_{11} + (\xi_{11} \circ \phi) \circ \xi_{11} - \xi_{11} \circ (\phi \circ \xi_{11}) - \xi_{11} \circ (\xi_{11} \circ \phi) = \\ &= \phi \circ (\xi_{11} \circ \xi_{11}) - (\xi_{11} \circ \xi_{11}) \circ \phi \end{aligned}$$

and we shall prove that this is zero. This is, however, easy: it is enough to realize that $\mathbf{M} \in \mathsf{A}$-biMod and that the $\mathcal{F}(\xi_{11})$-module structure on $\mathbf{M}$ is induced via the projection π, which sends $\xi_{11} \circ \xi_{11}$ to zero. We warmly recommend to try to make the above calculations using the abstract tensor calculus, it is much more quicker (though a bit longer in the written form).

Summing up the results, we infer that

$$(13)\mathcal{T}^1(\mathcal{F}(\xi_{11})/(\xi_{11} \circ \xi_{11}); \mathbf{M}) = \frac{\operatorname{Ker}\left\{(-) \circ \xi_{11} - \xi_{11} \circ (-) : \mathbf{M}(1,1) \to \mathbf{M}(1,1)\right\}}{\operatorname{Im}\left\{(-) \circ \xi_{11} + \xi_{11} \circ (-) : \mathbf{M}(1,1) \to \mathbf{M}(1,1)\right\}}.$$

It is also instructive to verify that indeed $\mathcal{T}^0(\mathcal{F}(\xi_{11})/(\xi_{11} \circ \xi_{11}); \mathbf{M}) = \operatorname{Der}(\mathcal{F}(\xi_{11})/(\xi_{11} \circ \xi_{11}), \mathbf{M})$, as it was stated in 2.8.

For another examples, see the following paragraph.

3. Deformations and the Cohomology

We assume that the reader is already familiar with the basic notions of the deformation theory as it is discussed, for example, in [7], [9], [6], [25], ... It is not difficult to verify that the following definitions, given in terms of our calculus, are in concrete examples equivalent with the classical ones.

3.1 There are two main kind of problems of deformation theory: the *equivalence problem* (see 3.3) and the *integrability problem* (see 3.6). As far as the first problem is concerned, it is very simple, at least from the conceptual point of view – it is related with the isomorphism problem in the moduli space of the form M/U, where the algebraic group U acts unipotently on the variety of structure constants M, and this problem has a nice linearization (see [7] or [24]). We still decided to discuss it here briefly, in order to show how our calculus works. The second problem is much more delicate. Let us introduce first some notation.

3.2 Let A be a theory over a field $\mathbf{k}$ and let R be an $\mathbf{k}$-algebra. Then $\mathsf{A} \otimes_{\mathbf{k}} R$ denotes the theory with $(\mathsf{A} \otimes_{\mathbf{k}} R)(m,n) = \mathsf{A}(m,n) \otimes_{\mathbf{k}} R$, $m,n \geq 0$, with $\circ$ and $\otimes$ defined as the R-linear extensions of the corresponding operations in A. We use the notation $\mathsf{A}[[t]]$ for $\mathsf{A} \otimes_{\mathbf{k}} \mathbf{k}[[t]]$ (t is an independent variable) and $\mathsf{A}[[t]]/(t^N)$, $N \geq 1$, for $\mathsf{A} \otimes_{\mathbf{k}} \mathbf{k}[[t]]/(t^N)$. The similar notation will be used also for modules over A.

3.3 Let $A : \mathsf{A} \to \mathbf{k}_V$ be an A-algebra. By a *deformation* of A we mean a theory

$$\tilde{A} = A + tA_1 + t^2 A_2 + \cdots : \mathsf{A}[[t]] \to \mathbf{k}_V[[t]],$$

where the sum above means that $\tilde{A}_{mn} = A_{mn} + t(A_1)_{mn} + \cdots$ for some linear maps $(A_i)_{mn} : \mathsf{A}(m,n) \to \mathbf{k}_V(m,n)$, $m,n \geq 0$. Denote by $\mathrm{Def}(A)$ the set of all deformations of the algebra A.

Let $G = \mathrm{Aut}(V[[t]])$. The group G acts on $\mathrm{Def}(A)$ by $(g\tilde{A})_{mn} = g^{\otimes n}\tilde{A}_{mn}(g^{\otimes m})^{-1}$, $m,n \geq 0$. Finally, let $U = \{g \in \mathrm{G}; g = 1\!\!1 + t\phi_1 + t^2\phi_2 + \cdots, \phi_i \in \mathrm{End}(V)\}$. Then the first problem of 3.1 can be formulated as the problem of the decision whether a deformation $\tilde{A} \in \mathrm{Def}(A)$ is equivalent to the 'trivial' deformation $A = A + t0 + \cdots$ in $\mathrm{Def}(A)/U$. We state without proof (which is actually very easy) the following

Proposition 3.4 *The obstructions to the triviality problem as formulated in 3.3 are elements of the (linear) moduli space*

$$\mathrm{Der}(\mathsf{A},\mathbf{k})/\mathrm{End}(V), \tag{14}$$

where $\phi \in \mathrm{End}(V)$ acts on $\theta \in \mathrm{Der}(\mathsf{A},\mathbf{k})$ by

$$(\phi \circ \theta)_{mn} = \phi^{[n]}\theta_{mn}(\phi^{[m]})^{-1},\ m,n \geq 0,$$

with $\phi^{[l]} = \sum_{1 \leq i \leq l}(1\!\!1^{i-1} \otimes \phi \otimes 1\!\!1^{l-i})$.

3.5 Example Let $\mathsf{A} = \mathcal{F}(\xi_{21},\xi_{12})/(r_1,r_2,r_3)$ be the theory of bialgebras as in 1.15. Let $A = (V,\mu,\Delta)$ be an A-algebra (= a bialgebra), $\mu = A(\xi_{21})$ and $\Delta = A(\xi_{12})$. Any derivation $\theta \in \mathrm{Der}(\mathsf{A},\mathbf{k}_V)$ is uniquely determined by a couple $\psi_1 = \theta_{21}(\xi_{21}) \in \mathrm{Hom}(V^{\otimes 2}, V)$ and $\psi_2 = \theta_{12}(\xi_{12}) \in \mathrm{Hom}(V, V^{\otimes 2})$. Expressing the condition $\theta(r_i) = 0$, $i = 1,2,3$, we get

$$\mu(1\!\!1 \otimes \psi_1) - \psi_1(\mu \otimes 1\!\!1) + \psi_1(1\!\!1 \otimes \mu) - \mu(1\!\!1 \otimes \psi_1) = 0,$$
$$(\mu \otimes \psi_1)(1\!\!1 \otimes S \otimes 1\!\!1(\Delta \otimes \Delta) - \Delta\psi_1 + (\psi_1 \otimes \mu)(1\!\!1 \otimes S \otimes 1\!\!1(\Delta \otimes \Delta) +$$
$$+(\mu \otimes \mu)(1\!\!1 \otimes S \otimes 1\!\!1(\Delta \otimes \psi_2) - \psi_2\Delta + (\mu \otimes \mu)(1\!\!1 \otimes S \otimes 1\!\!1)(\psi_2 \otimes \Delta) = 0, \text{ and}$$
$$(1\!\!1 \otimes \psi_2)\Delta - (\Delta \otimes 1\!\!1)\psi_2 + (1\!\!1 \otimes \Delta)\psi_2 - (\psi_2 \otimes 1\!\!1)\Delta = 0$$

($S(x \otimes y) = y \otimes x$ for $x, y \in V$), which can be written as

$$d_{\mathrm{Hoch}}(\psi_1) = 0,\ d_{\mathrm{coH}}(\psi_1) + d_{\mathrm{Hoch}}(\psi_2) = 0 \text{ and } d_{\mathrm{coH}}(\psi_2) = 0,$$

where d_{Hoch} (d_{coH}) is the Hochschild (coHochschild) differential, see [9] or [25] for details. The action of $\phi \in \mathrm{End}(V)$ on (ψ_1, ψ_2) is described as

$$\psi_1 \mapsto \psi_1 - \mu(\mathbb{1} \otimes \phi) + \phi\mu - \mu(\phi \otimes \mathbb{1}) = \psi_1 - d_{\mathrm{Hoch}}(\phi) \text{ and}$$
$$\psi_2 \mapsto \psi_2 + (\phi \otimes \mathbb{1})\Delta - \Delta\phi + (\mathbb{1} \otimes \phi)\Delta = \psi_2 + d_{\mathrm{coH}}(\psi_2).$$

It is clear from this description that the space (14) of Proposition 3.4 is isomorphic to $\hat{H}^2_b(A; A)$, the bialgebra cohomology introduced by M. Gerstenhaber and S. Schack in [9], see also [25].

3.6 Let $A : \mathsf{A} \to \mathbf{k}_V$ be an A-algebra and consider a 'partial' deformation $\hat{A} = A + tA_1 + \cdots + t^N A_N$, $N \geq 1$, $\hat{A} : \mathsf{A}[[t]]/(t^{N+1}) \to \mathbf{k}_V[[t]]/(t^{N+1})$ (the meaning of this notation being the same as in 3.3). The *integrability problem* for $\hat{A}$ is the problem of the existence of a honest deformation in the sense of 3.3 which extends $\hat{A}$. The primary obstruction to the existence of such an extension is related to the problem of finding some A_{N+1} such that $\bar{A} = \hat{A} + t^{N+1}A_{N+1} : \mathsf{A}[[t]]/(t^{N+2}) \to \mathbf{k}_V[[t]]/(t^{N+2})$ is an $\mathsf{A}[[t]]/(t^{N+2})$-algebra. We show that the obstruction to the existence of such an A_{N+2} is an element $[\mathcal{O}]$ of $\mathcal{T}^1(\mathsf{A}; \mathbf{k}_V)$.

3.7 Let

$$\mathcal{R}: \qquad \mathsf{A} \xleftarrow{\pi} \mathcal{F}(X) \xleftarrow{\alpha} \mathcal{F}(X)\langle Y\rangle \xleftarrow{\beta} \mathcal{F}(X)\langle Z\rangle,$$

be a resolution of A as in (2.1). Define the homomorphism $\hat{\mathcal{A}} : \mathcal{F}(X)[[t]] \to \mathbf{k}_V[[t]]$ of theories by $\hat{\mathcal{A}}_{mn}(x) = \hat{A}_{mn} \circ \pi_{mn}(x)$ for a generator $x \in X(m,n)$, $m,n \geq 0$ ($\hat{\mathcal{A}}$ is a kind of a lift of $\hat{A}$). The crucial observation here is that $\hat{\mathcal{A}}_{mn}(u) \neq \hat{A}_{mn} \circ \pi_{mn}(u)$ for a general $u \in \mathcal{F}(X)(m,n)$, but still $\hat{\mathcal{A}}(\alpha) = 0 \pmod{t^{N+1}}$ – an easy consequence of the fact that $\hat{A}$ is an 'algebra mod t^{N+1}'. Finally, define $\mathcal{O} = \mathcal{O}(\hat{A}) \in \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Y\rangle; \mathbf{k}_V)$ by

$$\mathcal{O}_{mn}(v) = \text{coefficient at } t^{N+1} \text{ in } \hat{\mathcal{A}}_{mn}(\alpha_{mn}(v)),\ v \in \mathcal{F}(X)\langle Y\rangle(m,n),\ m,n \geq 0.$$

Lemma 3.8 *The definition of $\mathcal{O}$ is correct, i.e. $\mathcal{O}$ defined as above is a homomorphism of $\mathcal{F}(X)$-modules.*

Proof We shall show that

$$\begin{aligned} &\mathcal{O}_{mk}(u \circ v) = A_{mn}(u) \circ \mathcal{O}_{nk}(v),\ \mathcal{O}_{mk}(a \circ b) = \mathcal{O}_{mn}(a) \circ A_{nk}(b), \\ (15)\qquad &\mathcal{O}_{m_1+m_2,n_1+n_2}(u' \otimes v') = A_{m_1n_1}(u') \otimes \mathcal{O}_{m_2n_2}(v') \text{ and} \\ &\mathcal{O}_{m_1+m_2,n_1+n_2}(a' \otimes b') = \mathcal{O}_{m_1n_1}(a') \otimes A_{m_2n_2}(b'), \end{aligned}$$

for $u \in \mathcal{F}(X)(m,n)$, $v \in \mathcal{F}(X)\langle Y\rangle(n,k)$, $a \in \mathcal{F}(X)\langle Y\rangle(m,n)$, $b \in \mathcal{F}(X)(n,k)$, $u' \in \mathcal{F}(X)(m_1,n_1)$, $v' \in \mathcal{F}(X)\langle Y\rangle(m_2,n_2)$, $a' \in \mathcal{F}(X)\langle Y\rangle(m_1,n_1)$ and $b' \in \mathcal{F}(X)(m_2,n_2)$, for all natural numbers m, n, m_1, n_1, m_2, and n_2.

Let us prove the first equation of (15). By definition, $\mathcal{O}_{mk}(u \circ v)$ = coefficient at t^{N+1} in $\hat{\mathcal{A}}_{mk}(\alpha_{mk}(u \circ v))$, and $\hat{\mathcal{A}}_{mk}(\alpha_{mk}(u \circ v)) = \hat{\mathcal{A}}_{mk}(u \circ \alpha_{nk}(v)) = \hat{\mathcal{A}}_{mn}(u) \circ \hat{\mathcal{A}}_{nk}(\alpha_{nk}(v))$. Since $\hat{\mathcal{A}}_{nk}(\alpha_{nk}(v)) = 0 \bmod t^{N+1}$, we infer that

$$\begin{aligned}
&\text{coefficient at } t^{N+1} \text{ in } \hat{\mathcal{A}}_{mn}(u) \circ \hat{\mathcal{A}}_{nk}(\alpha_{nk}(v)) = \\
&\quad = \left\{\text{coefficient at } t^0 \text{ in } \hat{\mathcal{A}}_{mn}(u)\right\} \circ \left\{\text{coefficient at } t^{N+1} \text{ in } \hat{\mathcal{A}}_{nk}(\alpha_{nk}(v))\right\} \\
&\quad = A_{mn}(u) \circ \mathcal{O}_{nk}(v),
\end{aligned}$$

which gives the first equation of (15). The argument for the remaining equations is the same. ■

Proposition 3.9 *Let $\mathcal{E}^* = \mathcal{E}^*(\mathcal{R}, \mathbf{k}_V)$ be as in (10). Then $\mathcal{O} \in \mathcal{E}^1(\mathcal{R}, \mathbf{k}_V)$ defined above is a cocycle, i.e. $\delta_2(\mathcal{O}) = 0$.*

Proof By the definition of δ_2, we must show that $\mathcal{O}_{mn} \circ \beta_{mn}(w) = 0$ for all $w \in \mathcal{F}(X)\langle Z\rangle(m,n)$, $m, n \geq 0$. But $\mathcal{O}_{mn} \circ \beta_{mn}(w)$ is, by definition, the coefficient at t^{N+1} in $\hat{\mathcal{A}}_{mn}(\alpha_{mn} \circ \beta_{mn}(w))$, which is zero, because $\alpha \circ \beta = 0$. ■

Proposition 3.10 *The cohomology class $[\mathcal{O}] \in \mathcal{T}^1(\mathsf{A}; \mathbf{k}_V)$ constructed above is the (primary) obstruction to the integrability problem as formulated in 3.6.*

Proof We shall show that $[\mathcal{O}] = 0$ is equivalent to the existence of an A_{N+1} from 3.6. So, suppose that we have such an A_{N+1}, i.e., a system of linear maps $\{(A_{N+1})_{mn}\}_{m,n \geq 0}$, $(A_{N+1})_{mn} : \mathsf{A}(m,n) \to \mathbf{k}_V(m,n)$, such that $\bar{A} = \hat{A} + t^{N+1}A_{N+1}$ is an 'algebra mod t^{N+2}' (in the evident sense). Define the derivation $\mathcal{A}_{N+1} \in \mathrm{Der}(\mathcal{F}(X), \mathbf{k}_V)$ by

$$(\mathcal{A}_{N+1})_{mn}(x) = (A_{N+1})_{mn} \circ \pi_{mn}(x) \text{ for } x \in X(m,n),\ m,n \geq 0.$$

Let also $\bar{\mathcal{A}} : \mathcal{F}(X)[[t]] \to \mathbf{k}_V[[t]]$ be the 'lift' of $\bar{A}$ constructed in exactly the same way as the 'lift' $\hat{\mathcal{A}}$ of $\hat{A}$ from 3.7. We show that $\bar{\mathcal{A}} = \hat{\mathcal{A}} + t^{N+1}\mathcal{A}_{N+1} \bmod t^{N+2}$. Notice first that this equation is certainly fulfilled on the generators of $\mathcal{F}(X)$. Thus it is enough to prove that $\hat{\mathcal{A}} + t^{n+1}\mathcal{A}_{N+1}$ is a 'homomorphism mod t^{N+2}'. We have the following equations mod t^{N+2}:

$$\begin{aligned}\left(\hat{\mathcal{A}} + t^{N+1}\mathcal{A}_{N+1}\right)_{mk}(u \circ w) &= \hat{\mathcal{A}}_{mk}(u \circ w) + t^{N+1}(\mathcal{A}_{N+1})_{mk}(u \circ w)\\ &= \hat{\mathcal{A}}_{mn}(u) \circ \hat{\mathcal{A}}_{nk}(w) + t^{N+1}\left(\mathcal{A}_{mn}(u) \circ (\mathcal{A}_{N+1})_{nk}(w) + (\mathcal{A}_{N+1})_{mn}(u) \circ \mathcal{A}_{nk}(w)\right)\\ &= \left(\hat{\mathcal{A}} + t^{N+1}\mathcal{A}_{N+1}\right)_{mn}(u) \circ \left(\hat{\mathcal{A}} + t^{N+1}\mathcal{A}_{N+1}\right)_{nk}(w)\end{aligned}$$

for all $u \in \mathcal{F}(X)(m,n)$, $v \in \mathcal{F}(X)(n,k)$ and $m,n,k \geq 0$. This shows that $\hat{\mathcal{A}} + t^{N+1}\mathcal{A}_{N+1}$ behaves as a homomorphism mod t^{N+2} with respect to $\circ$. We can show in exactly the same way that it behaves as a homomorphism with respect to $\otimes$, as well. So we know that $\bar{\mathcal{A}} = \hat{\mathcal{A}} + t^{N+1}\mathcal{A}_{N+1}$ mod t^{N+1}. We also have $\bar{\mathcal{A}}_{mn}(\alpha_{mn}(v)) = 0$ mod t^{N+2}, since $\bar{A}$ is an 'algebra mod t^{N+2}'. Summing up the above considerations, we get that

$$\begin{aligned}0 = \text{coefficient at } t^{N+1} &\text{ in } \bar{\mathcal{A}}_{mn}(\alpha_{mn}(v)) =\\ \text{coefficient at } t^{N+1} &\text{ in } \left(\hat{\mathcal{A}}_{mn}(\alpha_{mn}(v)) + (\mathcal{A}_{N+1})_{mn}(\alpha_{mn}(v))\right)\\ &= \mathcal{O}_{mn}(v) + (\delta_1\mathcal{A}_{N+1})_{mn}(v)\end{aligned}$$

for any $v \in \mathcal{F}(X)\langle Y\rangle$, $m,n \geq 0$, which is the same as $\mathcal{O} = -\delta_1(\mathcal{A}_{N+1})$ and this implies that $[\mathcal{O}] = 0$.

On the other hand, suppose $[\mathcal{O}] = 0$, i.e. suppose that $\mathcal{O}_{mn}(v) = \omega_{mn} \circ (\alpha_{mn}(v))$ for all $v \in \mathcal{F}(X)\langle Y\rangle(m,n)$, $m,n \geq 0$, with some $\omega \in \mathrm{Der}(\mathcal{F}(X), \mathbf{k}_V)$. Define $\bar{A} : \mathsf{A}[[t]]/(t^{N+2}) \to \mathbf{k}_V[[t]]/(t^{N+2})$ by $\bar{A}_{mn}(\pi_{mn}(u)) = \hat{\mathcal{A}}_{mn}(u) - t^{N+1}\omega_{mn}(u)$ for all $u \in \mathcal{F}(X)$, $m,n \geq 0$. It is immediate to verify that this formula defines correctly an extension of the partial deformation $\hat{A}$. ■

3.11 Example: Deformations of differential spaces. Let $\mathsf{A} = \mathcal{F}(\xi_{11})/(\xi_{11} \circ \xi_{11})$ be the theory of differential complexes as in 1.12. Let $A : \mathsf{A} \to \mathbf{k}_V$ be an A-algebra, i.e. a couple (V,d) of a vector space V and a differential d. Theorem 3.10 together with the formula (13) gives that deformations of this object are controlled by

$$\mathcal{T}^1(\mathsf{A}; \mathbf{k}_V) = \frac{\mathrm{Ker}\{\delta_2 : \mathrm{Hom}(V,V) \to \mathrm{Hom}(V,V)\}}{\mathrm{Im}\{\delta_1 : \mathrm{Hom}(V,V) \to \mathrm{Hom}(V,V)\}},$$

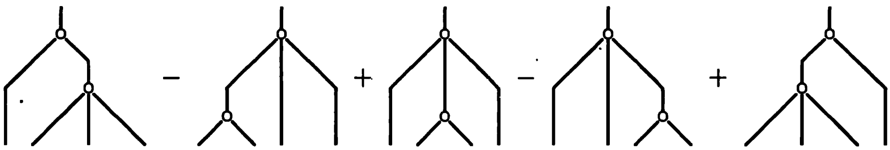
where $\delta_1(\phi) = \phi(d) + d(\phi)$ and $\delta_2(\psi) = \psi(d) - d(\psi)$, $\phi, \psi \in \mathrm{Hom}(V,V)$.

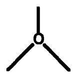
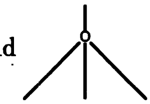
3.12 Example: Deformations of associative algebras.
Let $\mathsf{A} = \mathcal{F}(\xi_{21})/(r_1)$ be the theory of associative algebras as in 1.13, Consider the

object

(16) $\mathcal{R}: \quad \mathsf{A} \xleftarrow{\pi} \mathcal{F}(\xi_{21}) \xleftarrow{\alpha} \mathcal{F}(\xi_{21})\langle\eta_{31}\rangle \xleftarrow{\beta} \mathcal{F}(\xi_{21})\langle\zeta_{41}\rangle,$

where $\alpha(\eta_{31}) = r_1$ and $\beta(\eta_{41}) =$



where  denotes ξ_{21} and  denotes η_{31}.

As we prove in §4, this object is a resolution of the theory of associative algebras.

Let $A \in \mathsf{A}$**-algebras** be an associative algebra, $A = (V, \mu)$, where $\mu V^{\otimes 2} \to V$ is an associative multiplication. The complex $\mathcal{E}^*(\mathcal{R}, \mathbf{k}_V)$ of (10) has obviously the form

$$0 \longrightarrow \mathrm{Hom}(V^{\otimes 2}, V) \xrightarrow{\delta_1} \mathrm{Hom}(V^{\otimes 3}, V) \xrightarrow{\delta_2} \mathrm{Hom}(V^{\otimes 4}, V).$$

We can easily compute (compare the arguments in 2.11) that

$$\begin{aligned} \delta_1(m) &= \mu(\mathbb{1} \otimes m) - m(\mu \otimes \mathbb{1}) + m(\mathbb{1} \otimes \mu) - \mu(m \otimes \mathbb{1}) \text{ and} \\ \delta_2(f) &= \mu(\mathbb{1} \otimes f) - f(\mu \otimes \mathbb{1}^2) + f(\mathbb{1} \otimes \mu \otimes \mathbb{1}) - f(\mathbb{1}^2 \otimes \mu) + \mu(f \otimes \mathbb{1}), \end{aligned}$$

i.e. that δ_1 and δ_2 are the usual Hochschild differentials and

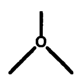
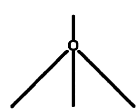
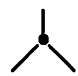
$$\mathcal{T}^1(\mathsf{A}; \mathbf{k}_V) = \mathrm{Hoch}^3(A; A),$$

the third Hochschild cohomology group of the algebra A with coefficients in itself.

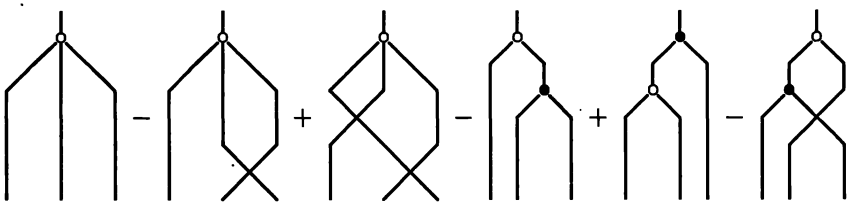
3.13 Example: Deformations of commutative associative algebras.
Let $\mathsf{A} = \mathcal{F}(X)/(r_1, r_2)$ be the theory of commutative associative algebras as in 1.14. Consider the object

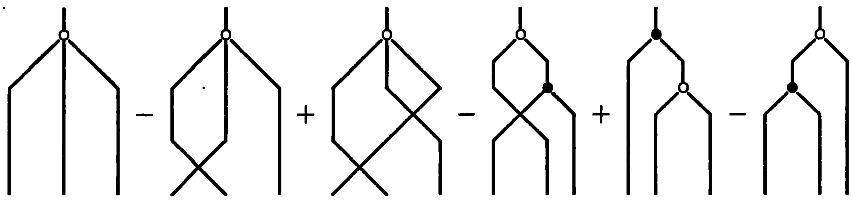
(17) $\mathcal{R}: \quad \mathsf{A} \xleftarrow{\pi} \mathcal{F}(\xi_{21}) \xleftarrow{\alpha} \mathcal{F}(\xi_{21})\langle\eta_{31}, \eta_{21}\rangle \xleftarrow{\beta} \mathcal{F}(\xi_{21})\langle\zeta_{41}, \zeta_{31}, \zeta'_{31}, \zeta_{21}\rangle,$

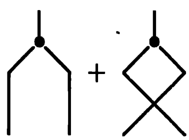
where $\alpha(\eta_{31}) = r_1$, $\alpha(\eta_{21}) = r_2$ and $\beta(\eta_{41})$ is as in 3.12. To define β on the remaining generators, let us introduce the notation

 $= \xi_{21}$,  $= \eta_{31}$ and  $= \eta_{21}$.

Then, let

$\beta(\zeta_{31}) =$ 

$\beta(\zeta'_{31}) =$ 

$\beta(\zeta_{21}) =$ 

We recommend to the reader to verify that $\alpha\beta = 0$. It is possible to show, by a slight generalization of the arguments of §4, that (17) is indeed a resolution of A. It is immediate to see that

$$
\begin{aligned}
\mathcal{E}^1(\mathcal{R}, \mathbf{k}_V) &= \mathrm{Hom}(V^{\otimes 2}, V), \\
\mathcal{E}^2(\mathcal{R}, \mathbf{k}_V) &= \mathrm{Hom}(V^{\otimes 3}, V) \oplus \mathrm{Hom}(V^{\otimes 2}, V), \text{ and that} \\
\mathcal{E}^3(\mathcal{R}, \mathbf{k}_V) &= \mathrm{Hom}(V^{\otimes 4}, V) \oplus \mathrm{Hom}(V^{\otimes 3}, V) \oplus \mathrm{Hom}(V^{\otimes 3}, V) \oplus \mathrm{Hom}(V^{\otimes 2}, V).
\end{aligned}
$$

To describe the differential, consider the diagram

$$
\begin{tikzcd}
\mathrm{Hom}(V^{\otimes 2}, V) \arrow[d, "d_{\mathrm{Hoch}}"'] \arrow[drr, "M_1"] & & & & & & \\
\mathrm{Hom}(V^{\otimes 3}, V) \arrow[d, "d_{\mathrm{Hoch}}"'] \arrow[drr, "Sh_1"'] \arrow[drrrr, "Sh_2"] & \bigoplus & \mathrm{Hom}(V^{\otimes 2}, V) \arrow[d, "J_1"] \arrow[drr, "J_2"] \arrow[drrrr, "M_2"] & & & & \\
\mathrm{Hom}(V^{\otimes 4}, V) & \bigoplus & \mathrm{Hom}(V^{\otimes 3}, V) & \bigoplus & \mathrm{Hom}(V^{\otimes 3}, V) & \bigoplus & \mathrm{Hom}(V^{\otimes 2}, V)
\end{tikzcd}
$$

where d_{Hoch} is the Hochschild differential as in 3.12, $M_1(m) = m(\mathbb{1} - S)$ (S = the switch), $M_2(n) = n(\mathbb{1} + S)$, $J_1(n) = -\mu(\mathbb{1} \otimes n) + n(\mu \otimes \mathbb{1}) - \mu(n \otimes \mathbb{1})(\mathbb{1} \otimes S)$, $J_2(n) = -\mu(\mathbb{1} \otimes n)(S \otimes \mathbb{1}) + n(\mathbb{1} \otimes \mu) - \mu(n \otimes \mathbb{1})$, $Sh_1(f) = f(\mathbb{1}^3 - \mathbb{1} \otimes S + (S \otimes \mathbb{1})(\mathbb{1} \otimes S))$ and $Sh_2(f) = f(\mathbb{1}^3 - S \otimes \mathbb{1} + (\mathbb{1} \otimes S)(S \otimes \mathbb{1}))$, for $m, n \in \mathrm{Hom}(V^{\otimes 2}, V)$ and $f \in \mathrm{Hom}(V^{\otimes 3}, V)$. The differentials are described as

$$\begin{aligned} \delta_1(m) &= d_{\mathrm{Hoch}}(m) \oplus M_1(m) \text{ and} \\ \delta_2(f \oplus n) &= d_{\mathrm{Hoch}}(f) \oplus (Sh_1(f) + J_1(n)) \oplus (Sh_2(f) + J_2(n)) \oplus M_2(n). \end{aligned}$$

Let $\omega = f \oplus n$ be a cocycle in $\mathcal{E}^1(\mathcal{R}, \mathbf{k}_V)$, i.e.

$$d_{\mathrm{Hoch}}(f) = 0,\ Sh_1(f) + J_1(n) = 0,\ Sh_2(f) + J_2(n) = 0 \text{ and } M_2(n) = 0. \tag{18}$$

From $M_2(n) = 0$ we easily get that $n = M_1(m)$ for $m = \frac{1}{2}n$. We can now take $\omega - \delta_1(m)$ instead of ω without changing the cohomology class.

Using the trick above, we can suppose that every cocycle in $\mathcal{E}^1(\mathcal{R}, \mathbf{k}_V)$ is of the form $f \oplus 0$, $f \in \mathrm{Hom}(V^{\otimes 3}, V)$. This implies, by (18), that $Sh_1(f) = Sh_2(f) = 0$, which means exactly that f is zero on the decomposables of the shuffle product in $V^{\otimes *}$. We see that

$$\mathcal{T}^1(\mathsf{A}; \mathbf{k}_V) = \mathrm{Harr}^3(A; A),$$

the Harrison cohomology of A with coefficients in A, see [12].

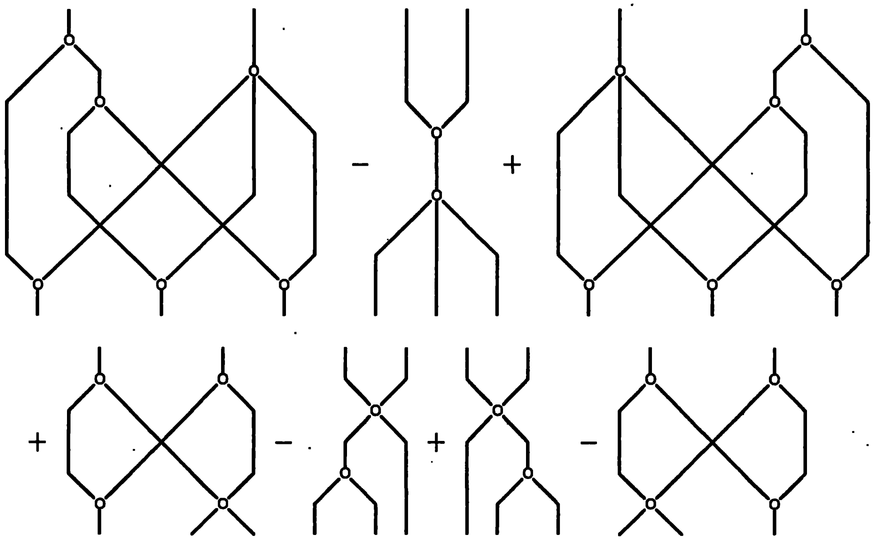
3.14 Remark As we have already mentioned, the dualization in our theory is extremely easy – it is nothing but just turning all diagrams upside down. Using this innocuous trick, we can easily infer, dualizing the arguments of 3.12, that the cotangent cohomology of the theory of coassociative coalgebras is exactly the coHochschild cohomology (the notion of coHochschild is a kind of folk lore, it has occurred for example in [9] or [25]).The statements of this kind are not much surprising. Much more important application of this kind of duality are offered by self-dual theories, such as the theory of bialgebras (Example 3.15). Here the duality may lead to substantial simplifications of arguments.

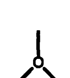
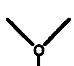
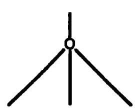
The second remark is related to the construction of the obstruction $[\mathcal{O}]$ (Theorem 3.10). All the arguments used in the construction remain valid if we start not with a *resolution*, but with a *pre-resolution* $\mathcal{R}$ only. The element $[\mathcal{O}_{\mathcal{R}}]$ thus constructed will then belong to $\mathcal{T}^1_{\mathcal{R}}(\mathsf{A}; \mathbf{k}_V)$. It is an easy consequence of the functoriality of the construction that $[\mathcal{O}_{\mathcal{R}}] = [\mathcal{O}]$ as elements of $\mathcal{T}^1(\mathsf{A}; \mathbf{k}_V) \subset \mathcal{T}^1_{\mathcal{R}}(\mathsf{A}; \mathbf{k}_V)$. The importance of this remark is related to the fact that we are not always able to prove that a pre-resolution is a resolution (see Example 3.15), but we still have an obstruction cohomology, though it may not be the best possible.

3.15 Example: Deformations of bialgebras. Let $\mathsf{A} = \mathcal{F}(\xi_{21}, \xi_{12})/(r_1, r_2, r_3)$ be the theory of bialgebras as in Example 1.15. Consider the object

$$(19)\ \mathcal{R}: \qquad \mathsf{A} \xleftarrow{\pi} \mathcal{F}(\xi_{21}) \xleftarrow{\alpha} \mathcal{F}(\xi_{21})\langle \eta_{31}, \eta_{22}, \eta_{13}\rangle \xleftarrow{\beta} \mathcal{F}(\xi_{21})\langle \zeta_{41}, \zeta_{32}, \zeta_{23}, \zeta_{14}\rangle,$$

where $\alpha(\eta_{31}) = r_1$, $\alpha(\eta_{22}) = r_2$ and $\alpha(\eta_{13}) = r_3$. The map β is defined by: $\beta(\zeta_{41}) =$ the same as in 3.12, $\beta(\zeta_{14})$ is dual to $\beta(\zeta_{41})$ in the sense of 3.14, $\beta(\zeta_{32})$ is defined by



where  $= \xi_{21}$,  $= \xi_{12}$,  $= \eta_{31}$ and  $= \eta_{22}$,

and $\beta(\eta_{23})$ is defined as the dual to $\beta(\eta_{32})$ in the sense of 3.14.

It is quite easy to verify that (19) is a pre-resolution, but we are not able to prove that it is a resolution, albeit we are almost sure that this is true. Having and A-algebra $A : \mathsf{A} \to \mathbf{k}_V$, we can easily compute that

$$\mathcal{T}^1_{\mathcal{R}}(\mathsf{A}; \mathbf{k}_V) = \hat{H}^3_b(A; A),$$

the bialgebra cohomology of A with coefficients in A, introduced in [10].

4. Resolutions and the Coherence

The aim of this paragraph is to prove that (16) is indeed a resolution of the theory of associative algebras. We formulate also some open problems related with the similar questions.

4.1 Let $\mathcal{F}_0(\xi_{21})(m,n)$ denotes, for $m,n \geq 0$, the subspace of $\mathcal{F}(\xi_{21})(m,n)$ consisting of elements which 'do not contain' the switch S. This definition is correct – the set $\mathcal{F}(\xi_{21})(m,n)$ can be presented as the set of (meaningful) words in ξ_{21} and S factored by the relations which should be satisfied in a theory, $\mathcal{F}_0(\xi_{21})(m,n)$ then corresponds to those words which do not contain S. The meaning of the notation $\mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(m,n)$ and $\mathcal{F}_0(\xi_{21})\langle\zeta_{41}\rangle(m,n)$ is similar. Denote also by $\mathbf{O}_0(n,1)$ the subspace of $\mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1)$ generated by elements of the form $\alpha(a)\circ b - a\circ\alpha(b)$, $a \in \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,l)$, $b \in \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(l,1)$, $1 \leq l \leq n$. Notice that $\mathbf{O}_0(n,1) = \mathbf{O}(n,1)\cap\mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1)$. Another way to define $\mathbf{O}_0(n,1)$ is to put

$$\mathrm{pre}\mathbf{O}_0(n,1) = \bigoplus_{l=1}^{n} \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,l) \oplus \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(l,1)$$

and introduce the map $\gamma : \mathrm{pre}\mathbf{O}_0(n,1) \to \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1)$ by $\gamma(a\oplus b) = \alpha(a)\circ b - b\circ\alpha(b)$. Then $\mathbf{O}_0(n,1) = \mathrm{Im}(\gamma)$.

4.2 Let $\mathcal{B}_{n,i}$ denotes, for $n \geq 2$ and $0 \leq i \leq n-2$, the set of all (meaningful) insertions of i pairs of brackets between n nonassociative variables. For $b \in \mathcal{B}_{n,i}$, let $b_{[i]}$ (resp. $b_{[[i]]}$, resp. $b_{[[[i]]]}$) denote the bracketing obtained from b by the replacement $(\bullet\bullet) \mapsto \bullet$ (resp. $(\bullet\bullet\bullet) \mapsto \bullet$, resp. $(\bullet\bullet\bullet\bullet) \mapsto \bullet$) at the i-th position.

4.3 As we have already observed in 1.8, the set $\mathcal{B}_n = \mathcal{B}_{n,n-2}$ can be used to index the elements of $\mathcal{F}_0(\xi_{21})(n,1)$. To describe this correspondence in a more formal way, observe that all elements of $\mathcal{B}_n$ can be obtained by a successive application of the operation $b \mapsto b_{[i]}$ defined above on $\bullet\bullet \in \mathcal{B}_2$. The correspondence $\mathcal{B}_n \ni b \mapsto \{b\} \in \mathcal{F}_0(\xi_{21})(n,1)$ can be then described inductively as:

- $\{\bullet\bullet\} = \xi_{21} \in \mathcal{F}(\xi_{21})(2,1)$ and,
- if $b \in \mathcal{B}_m$, then $\{b_{[i]}\} = \{b\}(\mathbb{1}^{i-1}\otimes\xi_{21}\otimes\mathbb{1}^{m-i})$ $(m \geq 2,\ 1 \leq i \leq m)$.

Following excactly the same lines, the set $\mathcal{B}_{n,n-3}$ can be used to index the elements of $\mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1)$, $n \geq 3$. To describe this correspondence, observe that an arbitrary element of $\mathcal{B}_{n,n-3}$ can be obtained

– either by successive applications of the operation $b \mapsto b_{[i]}$ on $\bullet\bullet\bullet \in \mathcal{B}_{3,0}$, or

– by successive applications of the replacements $b \mapsto b_{[i]}$ or $b \mapsto b_{[[j]]}$ on $\bullet\bullet \in \mathcal{B}_2$, the second replacement being used exactly once.

The correspondence $\mathcal{B}_{n,n-3} \ni b \mapsto \{b\} \in \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1)$ is then inductively described by the rules: $\{\bullet\bullet\} = \xi_{21} \in \mathcal{F}_0(\xi_{21})(2,1)$ and $\{\bullet\bullet\bullet\} = \eta_{31} \in \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(3,1)$, $\{b_{[i]}\} = \{b\}(\mathbb{1}^{i-1} \otimes \xi_{21} \otimes \mathbb{1}^{m-i})$ and $\{b_{[[j]]}\} = (\mathbb{1}^{j-1} \otimes \eta_{31} \otimes \mathbb{1}^{m-j})$, for $b \in \mathcal{B}_{m,*}$, $1 \le i,j \le m$ and $* = m-2$ or $m-3$.

Finally, consider $\mathcal{B}_{n,n-4}$, $n \ge 4$. Again, an arbitrary element of this set can be created

i) either from $\bullet\bullet$ by successive applications of the operations $b \mapsto b_{[i]}$ and $b \mapsto b_{[[j]]}$, the second one applied exactly twice, or

ii) from $\bullet\bullet$ by successive applications of the operations $b \mapsto b_{[i]}$ and $b \mapsto b_{[[[l]]]}$, the second operation applied exactly once, or

iii) from $\bullet\bullet\bullet$ by successive applications of the operations $b \mapsto b_{[i]}$ and $b \mapsto b_{[[l]]}$, the second one applied exactly once, or

iv) from $\bullet\bullet\bullet\bullet$ by successive applications of the operation $b \mapsto b_{[i]}$.

Clearly $\mathcal{B}_{n,n-4}$ can be written as the disjoint union of two subsets $\mathcal{B}'_{n,n-4}$ and $\mathcal{B}''_{n,n-4}$, the first one consisting of elements of $\mathcal{B}_{n,n-4}$ constructed by the procedure of ii) or iv) above, the second one consisting of elements constructed as in i) or iii) above.

Now any $b \in \mathcal{B}'_{n,n-4}$ can be identified with some $\{b\} \in \mathcal{F}_0(\xi_{21})\langle\zeta_{41}\rangle(n,1)$. The identification is described by the inductive rules: $\{\bullet\bullet\} = \xi_{21} \in \mathcal{F}_0(\xi_{21})(2,1)$, $\{\bullet\bullet\bullet\bullet\} = \zeta_{41} \in \mathcal{F}_0(\xi_{21})\langle\eta_{41}\rangle(4,1)$, $\{b_{[i]}\} = \{b\}(\mathbb{1}^{i-1} \otimes \xi_{21} \otimes \mathbb{1}^{m-i})$ and $\{b_{[[[l]]]}\} = \{b\}(\mathbb{1}^{l-1} \otimes \eta_{41} \otimes \mathbb{1}^{m-l})$ for $b \in \mathcal{B}_{m,*}$, $1 \le i,l \le m$ and $* = m-2$ or $m-4$.

Next observation is that for any $b \in \mathcal{B}''_{n,n-4}$ there are some l,k and i, $l+k = n+1$ and $1 \le i \le l$, and some $b_1 \in \mathcal{B}_{l,l-3}$ and $b_2 \in \mathcal{B}_{k,k-3}$ such that b is obtained from b_1 by the replacement $b_2 \mapsto \bullet$ at the i-th position. The numbers k,l,i and the elements b_1 and b_2 are not unique, but we may suppose they are chosen, once for all, for any $b \in \mathcal{B}''_{n,n-4}$. Let us define $\{b\}_1$ as $\{b_1\} \in \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(l,1)$ and $\{b\}_2$ as $\mathbb{1}^{i-1} \otimes \{b_2\} \otimes \mathbb{1}^{l-i} \in \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,l)$. The correspondence $b \mapsto \{b\}_2 \oplus \{b\}_1$ defines then a map from $\mathcal{B}''_{n,n-4}$ to $\mathrm{pre}\mathbf{O}_0(n,1)$.

Using the same arguments as in Example 2.4, we can easily infer that it is enough to prove the following proposition (as a matter of fact, this is a consequence of a general reduction principle which holds for all theories whose axioms 'do not contain' the switch; we intend to discuss this phenomena in a forthcoming paper).

Proposition 4.4 *Consider the sequence*

$$\mathcal{F}_0(\xi_{21})(n,1) \xleftarrow{\alpha_0} \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1) \xleftarrow{\beta_0} \mathcal{F}_0(\xi_{21})\langle\zeta_{41}\rangle(n,1),$$

where α_0 *and* β_0 *are the obvious restrictions,* $n \geq 1$*. Then* $\mathrm{Ker}(\alpha_0)(n,1)$ *is generated by* $\mathrm{Im}(\beta_0)(n,1)$ *and* $\mathbf{O}_0(n,1)$.

Proof Let us begin the proof with a brief discussion of the properties of the polyhedron K_n (so-called *associahedron*) introduced in [29] (see also [2] or [25], a nice exposition of the properties of K_n can be found also in [16, Chapter II]). K_n is, for $n \geq 2$, an $(n-2)$-dimensional polyhedron whose i-cells are, for $0 \leq i \leq n-2$, indexed by the elements of $\mathcal{B}_{n,n-i-2}$. If s is a cell corresponding to $b \in \mathcal{B}_{n,n-i-2}$, then the faces of s are indexed by those $b' \in \mathcal{B}_{n,n-i-1}$ which are obtained from b by (meaningful) inserting one more pair of brackets. It can be shown that the relation $b_1(b_2b_3) > (b_1b_2)b_3$, $b_j \in \mathcal{B}_{n_j}$, $j = 1,2,3$, induces a partial order on the vertices of K_n.

Let $C_i(K_n;\mathbf{k})$ denote, for $i \geq 0$, the vector space of i-dimensional cellular chains of K_n with values in $\mathbf{k}$. If we identify the i-cells of K_n with the corresponding elements of $\mathcal{B}_{n,n-2-i}$, then the elements of $C_i(K_n;\mathbf{k})$ are of the form $\sum A_j \cdot b^j$ (finite sum), $A_j \in \mathbf{k}$, $b^j \in \mathcal{B}_{n,n-2-i}$. Using the correspondences constructed in 4.3, we can define the map $\Omega_0 : C_0(K_n;\mathbf{k}) \to \mathcal{F}_0(\xi_{21})(n,1)$ by $\Omega_0(\sum A_j \cdot b^j) = \sum a_j \cdot \{b^j\}$; the map $\Omega_1 : C_1(K_n;\mathbf{k}) \to \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1)$ is defined by the similar way.

We have the decomposition $C_2(K_n;\mathbf{k}) = C_2'(K_n;\mathbf{k}) \oplus C_2''(K_n;\mathbf{k})$, where $C_2'(K_n;\mathbf{k})$ is defined to be the part of $C_2(K_n;\mathbf{k})$ corresponding to 2-cells of K_m indexed by $\mathcal{B}'_{n,n-4}$ and, similarly, $C_2''(K_n;\mathbf{k})$ is the part corresponding to $\mathcal{B}''_{n,n-4}$. This decomposition has a nice geometrical interpretation. There are two types of 2-cells: the cells of the first type are of the pentagonal shape while the cells of the second type have four vertices. Then $C_2'(K_n;\mathbf{k})$ corresponds to the cells of the first type and $C_2''(K_n;\mathbf{k})$ corresponds to the cells of the second type. The correspondence $\mathcal{B}'_{n,n-4} \ni b \mapsto \{b\} \in \mathcal{F}_0(\xi_{21})\langle\zeta_{41}\rangle(n,1)$ defines the map $\Omega_2' : C_2'(K_n;\mathbf{k}) \to \mathcal{F}_0(\xi_{21})\langle\zeta_{41}\rangle(n,1)$ and the correspondence $\mathcal{B}''_{n,n-4} \ni b \mapsto \{b\}_2 \oplus \{b\}_1$ defines the map $\Omega_2'' : C_2''(K_n;\mathbf{k}) \to \mathrm{pre}\mathbf{O}_0(n,1)$. Let us consider the diagram

$$\begin{tikzcd}
\mathcal{F}_0(\xi_{21})(n,1) & \mathcal{F}_0(\xi_{21})\langle\eta_{31}\rangle(n,1) \arrow[l, "\alpha_0"'] & \mathcal{F}_0(\xi_{21})\langle\zeta_{41}\rangle(n,1) \oplus \mathrm{pre}\mathbf{O}_0(n,1) \arrow[l, "\beta_0+\gamma"'] \\
C_0(K_n;\mathbf{k}) \arrow[u, "\Omega_0"] & C_1(K_n;\mathbf{k}) \arrow[l, "\partial_1"] \arrow[u, "\Omega_1"] & C_2(K_n;\mathbf{k}) \arrow[l, "\partial_2"] \arrow[u, "\Omega_2' \oplus \Omega_2''"]
\end{tikzcd} \tag{20}$$

The crucial observation is that the diagram commutes. It is an easy consequence of the definitions of the maps involved and the incidence relations of the polyhedron K_n, and we leave the verification to the reader. It is also clear that the maps Ω_0 and Ω_1 are isomorphisms. The claim of our proposition is equivalent with the exacteness of the upper row of (20). But this follows immediately from the above remarks and from the exacteness of the bottom row of (20) which follows from the asphericity of K_n (see [29] or [16]). ■

4.5 Open questions and remarks As we have already seen, the most difficult task related with the construction of the cotangent cohomology of a theory was to find the 'relations among relations'. Here we shall say that it is not so difficult to find *some* of them, it would be principally possible to write a computer program to do that. Also the 'abstract tensor calculus' or the 'deviation calculus' of [25] may be used for the visualization of these relations. The problem is to prove that we have found all of them (or, more precisely, a generating system). There are certain indications that this problem is closely related with a coherence problem in some category, but it is not exactly the same. For example, in the proof of Proposition 4.4 we did not use the coherence theorem in the formulation of for example [21], but we use the asphericity of the associahedra, the property which implies the coherence. So the *first natural question* is whether the construction of a resolution of a theory is related with a coherence property of some category. It may also well happen that there exists an easier and more direct way to prove the acyclicity.

As we have already explained in the introduction, for the majority of classical examples there exists a candidate for the cotangent cohomology – as the bialgebra cohomology [9] for bialgebras, see 3.15. The *second question* is whether these cohomology theories coincide with the cotangent cohomology introduced here, i.e. whether they are the best possible in the sense of the inclusion of 3.14.

The *third question* is related with the existence of a natural Lie algebra structure. Again, as it was shown by many authors, the cohomology theory related with the classical objects are naturally defined in all degrees and have a natural structure of a graded Lie algebra, although we think that this need not be always true (a candidate is the deformation cohomology for Drinfel'd algebras). The question is whether our construction can be, at least in some cases, extended to higher degrees to obtain this structure on our cotangent cohomology. Here is one remarkable hint: the 'classical' cotangent cohomology of [20] can be extended in such a way provided the characteristic of the ground field is zero.

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