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Hopf Structures on the Borel subalgebra of $sl(2)$

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Abstract

Moduli space of Hopf structures on the Borel subalgebra of $sl(2)$ is a line with the origin (the point $\beta = 0$, where β is a parameter on the line) excluded, and with two points at $\beta = 1$, the classical cocommutative algebra and a Hopf algebra $\mathcal{J}$. The algebra $\mathcal{J}$ is triangular. We describe its properties and give its differential operator realization on the Jordanian quantum plane.

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1 Moduli of Hopf structures

The simplest nonabelian Lie algebra is the algebra with two generators τ_3, π_- satisfying

$$[\tau_3, \pi_-] = -2\pi_- \ . \tag{1}$$

It has the standard cocommutative coproduct, $\Delta_o\tau_3 = \tau_3 \otimes 1 + 1 \otimes \tau_3$, $\Delta_o\pi_- = \pi_- \otimes 1 + 1 \otimes \pi_-$. The theory of Hopf algebras generalizes the theory of Lie algebras like the theory of nonabelian Lie algebras generalizes the theory of abelian Lie algebras. For Hopf algebras the coproduct becomes noncocommutative.

In the simplest situation of the algebra (1) one can ask what is the most general coproduct for this algebra. It is easier to consider this question from the dual point of view. Let $\mathcal{A}$ be the coalgebra of linear functionals on the algebra $\mathcal{U} = \mathcal{U}(\tau_3, \pi_-)$ of polynomials in τ_3, π_-. The comultiplication law on $\mathcal{A}$ is given by $(\Delta\phi, u \otimes v) = (\phi, uv)$, $\phi \in \mathcal{A}$, $u, v \in \mathcal{U}(\tau_3, \pi_-)$. Consider two linear functionals t and p, given by

$$(t, \tau_3^k \pi_-^l) = 2^k \delta_0^l \ , \ (p, \tau_3^k \pi_-^l) = \delta_0^k \delta_1^l \ . \tag{2}$$

Computing values of $\Delta t, \Delta p$ on monomials we obtain using $\pi_-^j \tau_3^k = (\tau_3 + 2j)^k \pi_-^j$,

$$(\Delta t, \tau_3^i \pi_-^j \otimes \tau_3^k \pi_-^l) = 2^{i+k} \delta_0^j \delta_0^l \ ,$$

$$(\Delta p, \tau_3^i \pi_-^j \otimes \tau_3^k \pi_-^l) = \delta_0^i (2^k \delta_0^l \delta_1^j + \delta_1^l \delta_0^k \delta_0^j) \ . \tag{3}$$

Therefore

$$\Delta t = t \otimes t \ , \tag{4}$$

$$\Delta p = p \otimes t + 1 \otimes p \ . \tag{5}$$

Thus the coproduct closes on p and t in a simple way. This is the reason for our choice of p and t.

Possible coproducts for the algebra (1) become multiplication laws for the coalgebra $\mathcal{A}$. To find the coproduct on the whole of $\mathcal{A}$ one can directly compute values for every linear functional like in (3). Alternatively one can find some commutation rules for p, t, making $\mathcal{A}$ an algebra, in such a way that $\mathcal{A}$ is generated (in a proper topological sense) by t, p, and Δ is a homomorphism. The precise formulation of our question is: what is the most general ordering prescription preserved by the coproduct (4, 5)? Assume we order functions of p, t in such a way that p stands before t. The ordered monomials $p^i t^j$ we assume to be linearly independent. Then writing

$$tp = \sum_i p^i a_i(t) \tag{6}$$

with the most general r.h.s. and applying Δ we obtain

$$(t \otimes t)(p \otimes t + 1 \otimes p) = \sum (p \otimes t + 1 \otimes p)^i \ a_i(t \otimes t). \tag{7}$$

The ordered expression for the l.h.s. is

$$\sum_i p^i a_i(t) \otimes t^2 + t \otimes \sum_i p^i a_i(t) . \tag{8}$$

Collecting terms not containing p in the first factor, we find

$$a_0(t) \otimes t^2 + t \otimes \sum p^i a_i(t) = \sum (1 \otimes p^i) a_i(t \odot t) . \tag{9}$$

The terms of the form $f(t) \otimes p^i g(t)$ are linearly independent for different i. Writing $a_i(t) = \sum_j a_{ij} t^j$ we obtain for $i > 0$

$$\sum_j a_{ij}(t \otimes t^j - t^j \otimes t^j) = 0 . \tag{10}$$

Therefore $a_{ij} = 0,\ j \neq 1,\ i > 0$.

For $i = 0$ we have $a_0(t) \otimes t^2 + t \otimes a_0(t) = a_0(t \odot t)$, which after decomposing a_0 in power series, $a_0(t) = \sum_j a_{0j} t^j$, implies that $a_{00} = 0,\ a_{02} = -a_{01},\ a_{0i} = 0, i > 2$.

Thus, eqn. (6) takes the form

$$tp = \sum p^i a_{i1} t - a_{01} t^2 . \tag{11}$$

Eqn. (7) reduces to

$$\sum_{i>0} p^i a_{i1} t \otimes t^2 = \sum_{i>0} \{(p \otimes t + 1 \otimes p)^i - 1 \otimes p^i\}\ a_{i1} \cdot (t \otimes t) . \tag{12}$$

Compare terms containing $p^k,\ k \geq 2$, in the first factor. Eqn. (11) shows that $\mathcal{A}t$ is the both-sided ideal. Therefore, terms containing p^k in the first factor in the r.h.s. of (12) have at least t^{k+1} in the second factor. For $k \geq 2$ such terms are absent in the l.h.s. Therefore we find that $a_{k1} = 0,\ k \geq 2$.

Thus, denoting $2\alpha = a_{01},\ \beta = a_{11}$, we conclude that

$$tp = \beta pt + 2\alpha(t - t^2) \tag{13}$$

is the most general ordering prescription compatible with (4, 5). The factor 2 stands for convenience.

We have now to identify equivalent products (13). The coproduct (4, 5) allows automorphisms

$$t \mapsto t \quad , \quad p \mapsto p' = \gamma p + \mu(t - 1) . \tag{14}$$

Then

$$tp' = \beta p' t + (2\gamma\alpha + \mu(\beta - 1))(t - t^2) . \tag{15}$$

Therefore, if $\beta \neq 1$ we can set α to 0. The product $tp = \beta pt$ corresponds to the coproduct $\Delta\tau_3 = \tau_3 \otimes 1 + 1 \otimes \tau_3,\ \Delta\pi_- = \pi_- \otimes 1 + \beta^{\tau_3/2} \otimes \pi_-$ for the algebra (1). This coproduct appears in the standard q-deformation of the Borel subalgebra of $sl(2)$ [1].

For $\beta = 1$ one can set α to 1 by the transformations (14). But since the classical limit corresponds to $\alpha = 0$, it is convenient to leave α free. Below it is shown that this bialgebra structure actually extends to the Hopf structure.

For $\beta = 0$ one can set α to 0 by the transformations (14). Then $tp = 0$. The counit ϵ should obey $(\epsilon \otimes \mathrm{id})\Delta t = t$, or $\epsilon(t)t = t$. Since the ordered monomials are linearly independent it follows that $\epsilon(t) = 1$. Then $t\epsilon(t) = t$ is satisfied as well. One of the properties of the antipode S is $\epsilon(t) = m(S \otimes \mathrm{id})\Delta t$, where m is the multiplication. Thus we should have $1 = S(t)t$. Multiplying this by p from the right one finds $p = 0$. Therefore for $\beta = 0$ the antipode does not exist, so this bialgebra does not admit a Hopf structure.

To conclude, Hopf structures (admitting ordering in which the ordered monomials are linearly independent) form a family described by the parameter $\beta \neq 0, 1$, and two points for $\beta = 1$, one is the classical algebra, $\alpha = 0$, another corresponds to $\alpha = 1$. In the topology induced by the transformations (14), neighbourhoods of the point $\alpha = 1$ are open sets of the line of β, but neighbourhoods of $\alpha = 0$ include an open set of the line of β and always the point $\alpha = 1$. Thus the point $\alpha = 1$ can be considered as a nonstandard "classical limit" of the q-deformation. Note that the classification of the Hopf structures coincides with the classification of quantum groups in two dimensions admitting left and right quantum spaces [2].

2 Algebra $\mathcal{J}$

We now study the bialgebra structure with $\beta = 1$, $\alpha \neq 0$.

Lemma 1.1 The product $[t, p] = 2\alpha(t - t^2)$ corresponds to the coproduct

$$\begin{aligned} \Delta\tau_3 &= \tau_3 \otimes \Lambda + 1 \otimes \tau_3 , \\ \Delta\pi_- &= \pi_- \otimes 1 + \Lambda^{-1} \otimes \pi_- . \end{aligned} \tag{16}$$

for the algebra (1), where

$$\Lambda^{-1} = 1 - 2\alpha\pi_- . \tag{17}$$

Proof. One checks that Δ given by (16) is a homomorphism, and that it is coassociative. We will show that the coproduct Δ induces the needed multiplication rules for t, p.

The product on the coalgebra $\mathcal{A}$ is defined by $(\phi\psi, u) = (\phi \otimes \psi, \Delta u)$, $\phi, \psi \in \mathcal{A}$, $u \in \mathcal{U}$.

Denote by $\mathcal{K}$ a two-sided ideal $\mathcal{U}\pi_- \otimes \mathcal{U} + \mathcal{U} \otimes \mathcal{U}\pi_-^2$ of $\mathcal{U} \otimes \mathcal{U}$. We have

$$\begin{aligned} &\Delta(\tau_3^i\pi_-^j) = (\tau_3 \otimes \Lambda + 1 \otimes \tau_3)^i(\pi_- \otimes 1 + 1 \otimes \pi_-)^j \\ &\equiv \delta_0^j(\tau_3 \otimes \Lambda + 1 \otimes \tau_3)^i + \delta_1^j(\tau_3 \otimes \Lambda + 1 \otimes \tau_3)^i(1 \otimes \pi_-)(\mathrm{mod}\ \mathcal{K}) \\ &\equiv \delta_0^j(\tau_3 \otimes 1 + 1 \otimes \tau_3 + 2\alpha\tau_3 \otimes \pi_-)^i \\ &+\delta_1^j(\tau_3 \otimes 1 + 1 \otimes \tau_3)^i(1 \otimes \pi_-)(\mathrm{mod}\ \mathcal{K}) . \end{aligned} \tag{18}$$

If $[h,x] = -2x$ then $(h+2\alpha x)^i \equiv h^i + \alpha((h+2)^i - h^i)x \pmod{x^2}$. Using this with $h = \tau_3 \otimes 1 + 1 \otimes \tau_3$, $x = \tau_3 \otimes \pi_-$, we find

$$\begin{aligned}
&(tp, \tau_3^i \pi_-^j) = \delta_0^j (t \otimes p, (\tau_3 \otimes 1 + 1 \otimes \tau_3)^i + \alpha((\tau_3 \otimes 1 + 1 \otimes \tau_3 + 2)^i \\
&-(\tau_3 \otimes 1 + 1 \otimes \tau_3)^i) \cdot (\tau_3 \otimes \pi_-)) \\
&+\delta_1^j (t \otimes p, (\tau_3 \otimes 1 + 1 \otimes \tau_3)^i \cdot (1 \otimes \pi_-)) \\
&= 2\alpha \delta_0^j (2^{2i} - 2^i) + \delta_1^j 2^i ,
\end{aligned} \tag{19}$$

since p takes a non-zero value only on the linear monomial π_-.

It is simpler to compute

$$\begin{aligned}
(pt, \tau_3^i \pi_-^j) &= \delta_1^j 2^i , \\
(t^2 - t, \tau_3^i \pi_-^j) &= \delta_0^j (2^{2i} - 2^i) ,
\end{aligned} \tag{20}$$

and we obtain $tp - pt = 2\alpha(t^2 - t)$, as required. □

The bialgebra (1), (16) has an antipode

$$S(\tau_3) = -\tau_3 \Lambda^{-1} \quad , \quad S(\pi_-) = -\pi_- \Lambda , \tag{21}$$

and a counit

$$\epsilon(\tau_3) = \epsilon(\pi_-) = 0 . \tag{22}$$

Thus, the algebra (1) with the coproduct (16) is a Hopf algebra.

Definition. We denote by $\mathcal{J}_0$ the Hopf algebra generated by elements $\Lambda, \Lambda^{-1}, \tau_3$ with the product

$$\Lambda\Lambda^{-1} = 1 \quad , \quad [\Lambda, \tau_3] = 2\Lambda(\Lambda - 1) , \tag{23}$$

the coproduct

$$\begin{aligned}
\Delta\tau_3 &= \tau_3 \otimes \Lambda + 1 \otimes \tau_3 , \\
\Delta\Lambda &= \Lambda \otimes \Lambda ,
\end{aligned} \tag{24}$$

the antipode

$$S(\tau_3) = -\tau_3 \Lambda^{-1} \quad , \quad S(\Lambda) = \Lambda^{-1} , \tag{25}$$

and the counit

$$\epsilon(\tau_3) = 0 \quad , \quad \epsilon(\Lambda) = 1 . \square \tag{26}$$

It is convenient to take the logarithm of Λ. There is a minimal completion for which the logarithm is defined. Define the vector space $\mathcal{J}$ of finite sums $\sum_i f_i(\tau_3) g_i(\sigma)$, where f_i are polynomials and for each g_i there exists such $n \in \mathbb{Z}$ that $g_i(\sigma)$ differs from $\exp(n\sigma)$ by at most a polynomial in σ.

Definition. The commutation relation

$$[\tau_3, \sigma] = 2(1 - e^\sigma) \tag{27}$$

together with the coalgebra structure

$$\Delta\sigma = \sigma\otimes 1 + 1\otimes\sigma \quad , \quad \Delta\tau_3 = \tau_3\otimes e^{\sigma} + 1\otimes\tau_3 \; . \tag{28}$$

and the antipode and counit

$$S(\sigma) = -\sigma \quad , \quad S(\tau_3) = -\tau_3 e^{-\sigma} \quad , \epsilon(\sigma) = \epsilon(\tau_3) = 0 \; . \tag{29}$$

make $\mathcal{J}$ a Hopf algebra. □

The Hopf algebra $\mathcal{J}_0$ is a dense Hopf subalgebra in $\mathcal{J}$, the embedding is given by $\Lambda = e^{\sigma}$.

We now describe some properties of $\mathcal{J}$. Below $\mathcal{J}^{opp}$ denotes the Hopf algebra obtained from $\mathcal{J}$ by taking the opposite comultiplication (and the same multiplication), and $\mathcal{J}_{opp}$ - the Hopf algebra obtained by taking the opposite multiplication and the same comultiplication.

Lemma 1.2 (a) $\mathcal{J}$ is isomorphic to $\mathcal{J}^{opp}$.
(b) $\tau_3 \mapsto -\tau_3$, $\sigma \mapsto \sigma$ is an isomorphism between $\mathcal{J}$ and $\mathcal{J}_{opp}$.

Proof. (a) Let $\tilde{\tau}_3 = \tau_3 e^{-\sigma}$, $\tilde{\sigma} = -\sigma$. Then

$$[\tilde{\tau}_3, \tilde{\sigma}] = [\tau_3 e^{-\sigma}, -\sigma] = 2(1 - e^{\tilde{\sigma}}) \tag{30}$$

and

$$\begin{aligned} \Delta\tilde{\tau}_3 &= (\tau_3\otimes e^{\sigma} + 1\otimes\tau_3)(e^{-\sigma}\otimes e^{-\sigma}) = \tilde{\tau}_3\otimes 1 + e^{\tilde{\sigma}}\otimes\tilde{\tau}_3 \; , \\ \Delta\tilde{\sigma} &= \tilde{\sigma}\otimes 1 + 1\otimes\tilde{\sigma} \; , \end{aligned} \tag{31}$$

which proves (a). (b) is obvious. □

The composition of the mappings (a) and (b) is the antipode S.

Any other isomorphism between $\mathcal{J}$ and $\mathcal{J}^{opp}$ or $\mathcal{J}_{opp}$ is the composition of the ones given in Lemma with some automorphism of $\mathcal{J}$. Here is the description of the group of automorphisms of $\mathcal{J}$.

Lemma 1.3 Any automorphism of the Hopf algebra $\mathcal{J}$ has the form

$$\sigma \mapsto \sigma \quad , \quad \tau_3 \mapsto \tau_3 + c(1 - e^{\sigma}) \tag{32}$$

with some constant c.

Proof. Choose a basis $\tau_3^i\sigma^j$ of $\mathcal{J}$.
1. Let ϕ be an automorphism of $\mathcal{J}$. Then $\Delta\circ\phi = (\phi\otimes\phi)\circ\Delta$, therefore, $\sigma' = \phi(\sigma)$ must obey $\Delta\sigma' = \sigma'\otimes 1 + 1\otimes\sigma'$. For $\sigma' = \sum_{ij} a_{ij}\tau_3^i\sigma^j \in \mathcal{J}$ this gives

$$\sum a_{ij}(\tau_3\otimes e^{\sigma} + 1\otimes\tau_3)^i(\sigma\otimes 1 + 1\otimes\sigma)^j = \sum a_{ij}\tau_3^i\sigma^j\otimes 1 + 1\otimes\sum a_{ij}\tau_3^i\sigma^j \; . \tag{33}$$

Compare terms not containing τ_3 in the first factor:

$$\sum_{ij} a_{ij}(1\otimes\tau_3^i)(\sigma\otimes 1+1\otimes\sigma)^j = \sum_l a_{0l}\sigma^l\otimes 1 + 1\otimes\sum_{mn} a_{mn}\tau_3^m\sigma^n\ . \tag{34}$$

Denoting $a_i(z)=\sum_j a_{ij}z^j$, $x=\sigma\otimes 1$, $y=1\otimes\sigma$ and comparing terms with τ_3^k we find $a_0(x+y)=a_0(x)+a_0(y)$ and $a_k(x+y)=a_k(y)$, $k>0$. Therefore, $a_0(x)=\gamma_0 x$ and $a_k(x)=\gamma_k$, $k>0$, for some constants γ_i. Hence, $\sigma'=\gamma_0\sigma+\sum_{i>0}\gamma_i\tau_3^i$, and eqn. (33) reduces to

$$\sum_{i>0}\gamma_i(\tau_3\otimes e^\sigma+1\otimes\tau_3)^i=\sum_{i>0}\gamma_i\tau_3^i\otimes 1+1\otimes\sum_{i>0}\gamma_i\tau_3^i\ . \tag{35}$$

Modulo the ideal $\mathcal{J}\otimes\mathcal{J}\sigma$ this implies $\gamma(t+s)=\gamma(t)+\gamma(s)$, where $\gamma(r)=\sum_{i>0}\gamma_i r^i$, $t=\tau_3\otimes 1$ and $s=1\otimes\tau_3$. Therefore $\gamma(r)=\gamma_1 r$. Substituting $\sigma'=\gamma_0\sigma+\gamma_1\tau_3$ in (35) one gets immediately $\gamma_1=0$. Thus, $\sigma'=\gamma_0\sigma$.
2. For $\tau_3'=\sum b_{ij}\tau_3^i\sigma^j$ we obtain

$$\sum b_{ij}(\tau_3\otimes e^\sigma+1\otimes\tau_3)^i(\sigma\otimes 1+1\otimes\sigma)^j=\sum b_{ij}\tau_3^i\sigma^j\otimes e^{\gamma_0\sigma}+1\otimes\sum b_{ij}\tau_3^i\sigma^j\ . \tag{36}$$

Terms not containing τ_3 in the first factor are

$$\sum_{ij} b_{ij}(1\otimes\tau_3^i)(\sigma\otimes 1+1\otimes\sigma)^j=\sum_l b_{0l}\sigma^l\otimes e^{\gamma_0\sigma}+1\otimes\sum_{mn} b_{mn}\tau_3^m\sigma^n\ . \tag{37}$$

It follows then that $b_0(x+y)=b_0(x)e^{\gamma_0 y}+b_0(y)$, $b_k(x+y)=b_k(y)$, $k>0$, where $b_i(z)=\sum_j b_{ij}z^j$. Thus, $b_0(x)=\beta_0(1-e^{\gamma_0 x})$, $b_k(x)=\beta_k$ with some constants β_i. Considering (36) modulo $\mathcal{J}\otimes\mathcal{J}\sigma$ one finds that $\beta_k=0$, $k>1$. Thus, $\tau_3'=\beta_0(1-e^{\gamma_0\sigma})+\beta_1\tau_3$. We have $\beta_1\neq 0$, otherwise the image of ϕ belongs to the proper subalgebra generated by σ only. Now eqn. (36) reduces to

$$\beta_1(\tau_3\otimes e^\sigma+1\otimes\tau_3)=\beta_1\tau_3\otimes e^{\gamma_0\sigma}+1\otimes\beta_1\tau_3\ , \tag{38}$$

which implies $\gamma_0=1$. Thus, $\sigma'=\sigma$, $\tau_3'=\beta_0(1-e^\sigma)+\beta_1\tau_3$.
3. Substituting expressions for σ' and τ_3' in (27) one finds immediately $\beta_1=1$. □

We now return to the dual algebra $\mathcal{A}$. One finds that the linear functionals t,p take the following values on the basis $\tau_3^i\sigma^j$ of $\mathcal{J}$:

$$(t,\tau_3^i\sigma^j)=2^i\delta_0^j\quad,\quad(p,\tau_3^i\sigma^j)=2\alpha\delta_0^i\delta_1^j\ . \tag{39}$$

Introduce also a functional s, $(s,\tau_3^i\sigma^j)=2\delta_1^i\delta_0^j$. By induction one finds then that $(s^k,\tau_3^i\sigma^j)=k!2^k\delta_k^i\delta_0^j$. Therefore, $(e^s,\tau_3^i\sigma^j)=2^i\delta_0^j$, which implies that $t=e^s$. The functionals s^kp^l are characterized by the property that they take nonzero values only on finite number of monomials $\tau_3^i\sigma^j$. So, in this topology they form the dual coalgebra. However, to make it an algebra one has to add the exponent of s. In other words, we define the dual Hopf algebra $\mathcal{J}'$ to be the vector space of finite sums $\sum f_i(s)g_i(p)$, where g_i are polynomials, and for each f_i there exists $n\in\mathbb{Z}$ such that $f_i(s)$ differs from exp(ns) by at most a polynomial in s. Above we have found the algebra structure on $\mathcal{J}'$, which for the generators s, p takes the form

$$[s,p] = 2\alpha(e^s - 1) \ , \tag{40}$$

and its coalgebra structure,

$$\Delta s = s \otimes 1 + 1 \otimes s \quad , \quad \Delta p = p \otimes e^s + 1 \otimes p \ . \tag{41}$$

The antipode and counit translate to $\mathcal{J}'$ as

$$S(s) = -s \ , \ S(p) = -pe^{-s} \ , \ \epsilon(s) = \epsilon(p) = 0 \ . \tag{42}$$

Comparing (27,28,29) with (40,41,42) we conclude:

Lemma 1.4 The map

$$\chi : \tau_3 \mapsto \frac{1}{\alpha} p \quad , \quad \sigma \mapsto s \tag{43}$$

is an isomorphism between the Hopf algebra $\mathcal{J}$ and its dual $\mathcal{J}'$. □

Thus, all eight Hopf algebras which can be obtained from $\mathcal{J}$ by taking opposite multiplication or opposite comultiplication or dual are isomorphic. Together, Lemmas 1.2, 1.3 and 1.4 give the description of the spaces of isomorphisms between them.

In the next section we find the universal $\mathcal{R}$-matrix for $\mathcal{J}$ and show that $\mathcal{J}$ is triangular.

3 Universal $\mathcal{R}$-matrix

We will often use the original generators τ_3, π_- of $\mathcal{J}$.

The universal $\mathcal{R}$-matrix satisfies

$$\mathcal{R}\Delta\tau_3 = \Delta'\tau_3 \, \mathcal{R} \ , \tag{44}$$

$$\mathcal{R}\Delta\pi_- = \Delta'\pi_- \, \mathcal{R} \ , \tag{45}$$

where Δ' is the opposite comultiplication.

Considering α as a variable, one can introduce a grading on the Hopf algebra $\mathcal{J}$ by deg $\tau_3 = 0$, deg $\pi_- = 1$ and deg $\alpha = -1$. (Exactly the fact that α has a nonzero dimension, allows to set it to 1, in contrast to the standard q-deformation, where q is dimensionless.) We assume that $\mathcal{R}$ has the grading zero. This means that it depends on π_- only in the combination $\alpha\pi_-$.

Also, since the classical limit is $\alpha = 0$, we assume that $\mathcal{R}|_{\alpha=0} = 1$.

Lemma 2.1 Let

$$F = \exp(\frac{1}{2}\sigma \otimes \tau_3) \ . \tag{46}$$

Then

$$\mathcal{R} = F_{12}F_{21}^{-1} . \tag{47}$$

Proof. For any dimensionless function $f(\alpha\pi_-)$ we have

$$[\tau_3, f] = -2\alpha\frac{\partial}{\partial\alpha}f . \tag{48}$$

Rewriting (44) in the form

$$[\tau_3 \otimes 1 + 1 \otimes \tau_3, \mathcal{R}] = \mathcal{R}(\tau_3 \otimes (\Lambda - 1)) - ((\Lambda - 1) \otimes \tau_3)\mathcal{R} , \tag{49}$$

using (48) and $\Lambda - 1 = 2\alpha\Lambda\pi_-$, we find, after cancelling by (-2α),

$$\dot{\mathcal{R}} = (\Lambda\pi_- \otimes \tau_3)\mathcal{R} - \mathcal{R}(\tau_3 \otimes \Lambda\pi_-) , \tag{50}$$

where dot means the derivative in α.

We have

$$\dot{F}_{12} = (\Lambda\pi_- \otimes \tau_3)F_{12} \quad , \quad (F_{21}^{-1})^{\cdot} = -F_{21}^{-1}(\tau_3 \otimes \Lambda\pi_-) . \tag{51}$$

Therefore, $\mathcal{R} = F_{12}WF_{21}^{-1}$ with $\dot{W} = 0$. Comparison of values at $\alpha = 0$ gives $W = 1$.

It remains to check (45). We have

$$(\pi_- \otimes 1)\mathcal{R} = e^{\frac{1}{2}\sigma\otimes\tau_3}e^{-\frac{1}{2}(\tau_3+2)\otimes\sigma}(\pi_- \otimes 1) = \mathcal{R}(\pi_- \otimes \Lambda^{-1}) . \tag{52}$$

Here we used that $\pi_- g(\tau_3) = g(\tau_3 + 2)\pi_-$ for any function g. Now,

$$(\Lambda^{-1} \otimes \pi_-)\mathcal{R} = (\Lambda^{-1} \otimes 1)e^{\frac{1}{2}\sigma\otimes(\tau_3+2)}e^{-\frac{1}{2}\tau_3\otimes\sigma}(1 \otimes \pi_-) = \mathcal{R}(1 \otimes \pi_-) . \tag{53}$$

Since $\Delta\pi_- = \pi_- \otimes 1 + \Lambda^{-1} \otimes \pi_- = \pi_- \otimes 1 + 1 \otimes \pi_- - 2\alpha\pi_- \otimes \pi_- = \pi_- \otimes \Lambda^{-1} + 1 \otimes \pi_-$, eqs. (52) and (53) imply (45).□

Lemma 2.2 $(\mathcal{J}, \mathcal{R})$ is triangular. In other words,

$$\mathcal{R}_{12}\mathcal{R}_{21} = 1 , \tag{54}$$

and

$$(\Delta \otimes \mathrm{id})\mathcal{R}_{12} = \mathcal{R}_{13}\mathcal{R}_{23} , \tag{55}$$

$$(\mathrm{id} \otimes \Delta)\mathcal{R}_{12} = \mathcal{R}_{13}\mathcal{R}_{12} . \tag{56}$$

Proof. 1. Eqn. (54) follows from (47).
2. Obviously,

$$(\Delta \otimes \mathrm{id})F_{12} = F_{13}F_{23} . \tag{57}$$

For any c commuting with π_- and τ_3 one has $[\tau_3, \Lambda^c] = -4\alpha c \Lambda^{c+1}\pi_-$ which implies

$$\Lambda^c \tau_3 \Lambda^{-c} = \tau_3 + 4\alpha c \Lambda \pi_- \ . \tag{58}$$

Using this, one finds

$$\begin{aligned} &e^{\frac{1}{2}\tau_3\otimes\sigma\otimes 1}(1\otimes\tau_3\otimes\sigma)e^{-\frac{1}{2}\tau_3\otimes\sigma\otimes 1} = 1\otimes\tau_3\otimes\sigma + \tau_3\otimes(\Lambda-1)\otimes\sigma \\ &= (\Delta\tau_3 - \tau_3\otimes 1)\otimes\sigma \ . \end{aligned} \tag{59}$$

$\Delta\tau_3$ commutes with $\tau_3\otimes 1$. Dividing by 2, taking exponent of eq. (59) and using $\exp(xyx^{-1}) = x\exp(y)x^{-1}$, we obtain $F_{21}F_{32}F_{21}^{-1} = F_{31}^{-1}\cdot(\Delta\otimes \mathrm{id})F_{21}$, which implies

$$(\Delta\otimes\mathrm{id})F_{21}^{-1} = F_{21}F_{32}^{-1}F_{21}^{-1}F_{31}^{-1} \ . \tag{60}$$

Therefore

$$\begin{aligned} (\Delta\otimes\mathrm{id})\mathcal{R}_{12} &= (\Delta\otimes\mathrm{id})(F_{12}F_{21}^{-1}) = F_{13}F_{23}F_{21}F_{32}^{-1}F_{21}^{-1}F_{31}^{-1} \\ &= F_{13}F_{21}\mathcal{R}_{23}F_{21}^{-1}F_{31}^{-1} \ , \end{aligned} \tag{61}$$

since F_{23} commutes with F_{21}.

On the other hand, $(\mathrm{id}\otimes\Delta)F_{21}^{-1} = F_{21}^{-1}F_{31}^{-1}$, $(\mathrm{id}\otimes\Delta')F_{21}^{-1} = F_{21}^{-1}F_{31}^{-1}$. By Lemma 2.1 we have $(\mathrm{id}\otimes\Delta)F_{21}^{-1} = \mathcal{R}_{23}^{-1}\cdot(\mathrm{id}\otimes\Delta')F_{21}^{-1}\cdot\mathcal{R}_{23}$, therefore

$$\mathcal{R}_{23}F_{21}^{-1}F_{31}^{-1} = F_{21}^{-1}F_{31}^{-1}\mathcal{R}_{23} \ . \tag{62}$$

Substituting this in (61) we obtain

$$\begin{aligned} &(\Delta\otimes\mathrm{id})\mathcal{R}_{12} = F_{13}F_{21}\mathcal{R}_{23}F_{21}^{-1}F_{31}^{-1} \\ &= F_{13}F_{31}^{-1}\mathcal{R}_{23} = \mathcal{R}_{13}\mathcal{R}_{23} \ , \end{aligned} \tag{63}$$

which proves (55).

3. When $\mathcal{R}_{12} = \mathcal{R}_{21}^{-1}$, eqn. (56) follows from (55) (if $(\Delta\otimes\mathrm{id})v_{12} = w_{123}$ for some v, w, then $(\mathrm{id}\otimes\Delta)u_{21} = w_{231}$) and we need not check it separately.□

Notes. 1. The element F_{21} twists the classical coproduct,

$$\Delta u = F_{21}\cdot\Delta_0 u\cdot F_{21}^{-1} \tag{64}$$

for any u (to check it for $u = \tau_3$ one uses (58)). The element F satisfies the condition

$$F_{32}\cdot(\mathrm{id}\otimes\Delta_0)F_{21} = F_{21}\cdot(\Delta_0\otimes\mathrm{id})F_{21} \tag{65}$$

ensuring the coassociativity of the twisted coproduct [3]. To prove (65) we notice that

$$e^{\frac{1}{2}\tau_3\otimes\sigma}\cdot(\Delta_0\Lambda^{-1})\cdot e^{-\frac{1}{2}\tau_3\otimes\sigma} = \Lambda^{-1}\otimes\Lambda^{-1} \ . \tag{66}$$

Therefore $e^{\frac{1}{2}1\otimes\tau_3\otimes\sigma} \cdot (\mathrm{id} \otimes \Delta_0)(1 \otimes \Lambda)^c \cdot e^{-\frac{1}{2}1\otimes\tau_3\otimes\sigma} = 1 \otimes \Lambda^c \otimes \Lambda^c$ for any c commuting with $1 \otimes \mathcal{U} \otimes \mathcal{U}$. For $c = \frac{1}{2}\tau_3 \otimes 1 \otimes 1$ this gives

$$F_{32} \cdot (\mathrm{id} \otimes \Delta_0)F_{21} \cdot F_{32}^{-1} = F_{21}F_{31} , \tag{67}$$

which is equivalent to (65) since $(\Delta_0 \otimes \mathrm{id})F_{21} = F_{31}F_{32}$.

After we know the twisting factor F, we can build the Hopf structure on the whole of $sl(2)$ by adding the generator π_+ satisfying $[\pi_+, \pi_-] = \tau_3$, $[\tau_3, \pi_+] = 2\pi_+$, $\Delta\pi_+ = F_{21}\Delta_0\pi_+F_{21}^{-1}$. Using the identity $[\pi_+, \Lambda^c] = 2\alpha c\Lambda^{c+1}\tau_3 - 4c(c+1)\alpha^2\Lambda^{c+2}\pi_-$ which implies $\Lambda^{-c}\pi_+\Lambda^c = \pi_+ + 2\alpha c\Lambda\tau_3 - 4c(c+1)\alpha^2\Lambda^2\pi_-$ one finds explicitly the coproduct on π_+:

$$\Delta\pi_+ = \pi_+ \otimes \Lambda + 1 \otimes \pi_+ - \alpha\tau_3 \otimes \Lambda\tau_3 - \alpha^2(\tau_3^2 - 2\tau_3) \otimes \Lambda^2\pi_- . \tag{68}$$

It looks simpler on a generator $\phi = \pi_+ + \frac{1}{2}\alpha\tau_3^2$,

$$\Delta\phi = \phi \otimes \Lambda + 1 \otimes \phi. \tag{69}$$

We denote this Hopf algebra with generators π_+, π_-, τ_3 by $U^J sl(2)$. Hopf algebras related to $\mathcal{J}$, $U^J sl(2)$ and its dual were appearing in a number of contexts [4], [5], [6], [2].

2. Let e_s be a basis of some Hopf algebra, m_{st}^k, μ_k^{st} are the structure constants of the multiplication and comultiplication respectively, S_s^t is the matrix of the antipode, ϵ^s and ϵ_s are the unit and counit respectively. Let e^s be the dual basis of the dual Hopf algebra, $(e^s, e_t) = \delta_t^s$. For the self-dual Hopf algebra let χ be the isomorphism between the algebra and its dual, in components $\chi(e^s) = g^{st}e_t$. If g^{st} is symmetric, $g^{st} = g^{ts}$, it is natural to call the Hopf algebra symmetric self-dual. The matrix g^{st} relates the structure constants m and μ, it is compatible with the antipode, and maps the unit to the counit:

$$m_{st}^k = g_{sa}g_{tb}g^{kc}\mu_c^{ab} \ , \ (S^{-1})_j^i = g^{it}S_t^k g_{kj} \ , \ \epsilon^s = g^{st}\epsilon_t \ , \tag{70}$$

where g_{st} is the inverse to g^{st}, $g_{st}g^{tk} = \delta_s^k$. The universal $\mathcal{R}$-matrix for the standard q-groups can be obtained by the double construction [1]. In the self-dual situation one can find a "self-double" interpretation of the $\mathcal{R}$-matrix. The element $e^s \otimes e_s$ satisfies the Yang-Baxter equation. This element reverses the coproduct if

$$\mu_a^{bc}m_{ic}^s g^{ij}m_{jb}^t = \mu_a^{uv}m_{um}^s g^{mn}m_{vn}^t . \tag{71}$$

For the algebra $\mathcal{J}$ we take a basis $e_{kl} = \tau_3^k\sigma^l$. One computes then that the dual basis is

$$e^{kl} = \frac{1}{k!l!(2\alpha)^l 2^k}p^l s^k \ , \quad (e^{ij}, e_{kl}) = \delta_k^i\delta_l^j . \tag{72}$$

Thus, $\mathcal{J}$ is symmetric self-dual. One sees immediately that its $\mathcal{R}$-matrix has the form $e^s \otimes e_s$.

4 Differential Realization

Any representation of $\mathcal{J}$ in a space V gives rise to a numerical R-matrix, acting in $V \otimes V$, and a $\hat{R}$-matrix $\hat{R} = PR$, where P is the matrix of permutation, $P(v \otimes w) = w \otimes v$. The triangularity (54) implies $\hat{R}\hat{R} = PRPR = R_{21}R_{12} = 1\!\!1$.

The $\hat{R}$-matrix corresponding to the simplest two-dimensional representation of the relation (1) first time appeared probably in [4]. It defines a quantum group called Jordanian [6], dual to $U^J sl(2)$. The Jordanian quantum group coacts on the noncommutative quantum plane given by the relation

$$yx = xy + \alpha y^2. \tag{73}$$

This relation is defined by one projector entering the $\hat{R}$-matrix, and therefore one can build the derivatives by the formula [7]

$$\partial_i x^j = \delta_i^j + \hat{R}_{il}^{jk} x^l \partial_k \ . \tag{74}$$

This reads explicitly

$$\begin{array}{rclrcl} \partial_x x &=& 1 + x\partial_x + \alpha y \partial_x \ , & \partial_y x &=& x\partial_y - \alpha x \partial_x - \alpha y \partial_y + \alpha^2 y \partial_x \ , \\ \partial_x y &=& y\partial_x \ , & \partial_y y &=& 1 + y\partial_y + \alpha y \partial_x \ . \end{array} \tag{75}$$

Derivatives satisfy

$$\partial_y \partial_x = \partial_x \partial_y + \alpha \partial_x \partial_x. \tag{76}$$

We are going now to build a realization of $U^J sl(2)$ in terms of differential operators on the Jordanian quantum plane, similar to the realization of $U_q sl(2)$ and $U_q sl(2, \mathbf{C})$ [8]. We shall see that the coproduct for $U^J sl(2)$ is naturally implied by the Leibnitz rule for differential operators acting on the space (73).

The operator $1 + 2\alpha y \partial_x$ will turn out to represent the element Λ appearing in (17). Thus we denote it by the same letter, $\Lambda = 1 + 2\alpha y \partial_x$. It obeys

$$\begin{array}{rclrcl} \Lambda x &=& (x + 2\alpha y)\Lambda \ , & \Lambda \partial_x &=& \partial_x \Lambda \ , \\ \Lambda y &=& y \Lambda \ , & \partial_y \Lambda &=& \Lambda(\partial_y + 2\alpha \partial_x) \ . \end{array} \tag{77}$$

The operator Λ is a Jordanian analogue of the multiplicative factors used in [9].

The commutation relations between the operators $x^i \partial_j$ are

$$\begin{array}{rclrcl} [x\partial_x, x\partial_y] &=& x\Lambda\partial_y \ , & [x\partial_y, y\partial_x] &=& x\Lambda\partial_x - y\Lambda\partial_y \ , \\ {[x\partial_x, y\partial_y]} &=& 0 \ , & [x\partial_y, y\partial_y] &=& x\Lambda\partial_y \ , \\ {[x\partial_x, y\partial_x]} &=& -y\Lambda\partial_x \ , & [y\partial_x, y\partial_y] &=& -y\Lambda\partial_x \ . \end{array} \tag{78}$$

Thus three operators

$$\tau_- = y\partial_x \ , \ \tau_3 = x\partial_x - y\partial_y \ , \ \tau_+ = (x - 2\alpha y)\partial_y \tag{79}$$

form a closed subalgebra, We have

$$[\tau_3,\tau_-]=-2\Lambda\tau_- \ , \ [\tau_3,\tau_+]=2\Lambda\tau_+ \ , \ [\tau_+,\tau_-]=\Lambda\tau_3-4\alpha\Lambda\tau_- \tag{80}$$

and $\Lambda = 1+2\alpha\tau_-$. We find a comutiplication for this algebra using the Leibnitz rule: if an operator T obeys $T(fg)=\sum_i a_i(f)b_i(g)$ with uniquely defined a_i and b_i then we set $\Delta T=\sum_i a_i\otimes b_i$. This requires a knowledge of the action of the operators τ_+, τ_- and τ_3 on a whole basis of functions. It turns out to be convenient to work in the basis $F_{mn}=x^{(m}y^n$ where we denoted for any v

$$v^{(m}=v(v+\alpha y)...(v+\alpha(m-1)y) \ , \ v^{(0}=1 \ . \tag{81}$$

The following identities are useful in computations in the basis F_{mn}:

$$\begin{aligned}(x+\alpha my)(x+\alpha ny) &= (x+\alpha(n+1)y)(x+\alpha(m-1)y) \ ,\\ (xy)^n &= x^{(n}y^n \ ,\\ (x\Lambda^{1/2})^n &= x^{(n}\Lambda^{n/2}.\end{aligned} \tag{82}$$

By induction one finds

$$\begin{aligned}\tau_3x^{(n} &= nx^{(n}\Lambda+x^{(n}\tau_3 \ ,\\ \tau_3y^n &= -ny^n\Lambda+y^n\tau_3 \ ,\\ \tau_-x^{(n} &= ny(x+\alpha y)^{(n-1}+(x+2\alpha y)^{(n}\tau_- \ ,\\ \tau_-y^n &= y^n\tau_- \ ,\\ \tau_+x^{(n} &= (x-2\alpha y)^{(n}(\tau_+-n\alpha\tau_3-\alpha^2n(n-2)\tau_-) \ ,\\ \tau_+y^n &= n(x-2\alpha y)y^{n-1}+y^n(\tau_++\alpha n\tau_3-\alpha^2n(n+2)\tau_-) \ .\end{aligned} \tag{83}$$

This easily extends to the basis F_{mn} and we obtain the following formulas for comultiplication:

$$\begin{aligned}\Delta\tau_3 &= \tau_3\otimes\Lambda+1\otimes\tau_3 \ ,\\ \Delta\tau_- &= \tau_-\otimes 1+\Lambda\otimes\tau_- \ ,\\ \Delta\tau_+ &= \tau_+\otimes 1+\Lambda^{-1}\otimes\tau_+-\alpha\Lambda^{-1}\tau_3\otimes\tau_3-\alpha^2\Lambda^{-1}(\tau_3^2-2\alpha\tau_3)\otimes\tau_- \ ,\end{aligned} \tag{84}$$

and $\Delta\Lambda=\Lambda\otimes\Lambda$. Here only the expression for $\Delta\tau_+$ requires an explanation. Introducing a generator $\phi=\Lambda\tau_++\frac{1}{2}\alpha\tau_3^2$ one finds, using (83),

$$\begin{aligned}\phi x^{(n} &= \frac{1}{2}\alpha n^2x^{(n}\Lambda+x^{(n}\phi \ ,\\ \phi y^n &= (nxy^{n-1}+\frac{1}{2}\alpha n^2y^n)\Lambda+y^n\phi \ .\end{aligned} \tag{85}$$

which implies

$$\Delta\phi=\phi\otimes\Lambda+1\otimes\phi \ . \tag{86}$$

This is equivalent to the last line in (84).

To compare with $U^J sl(2)$ we choose another set of generators

$$\pi_- = \tau_- \Lambda^{-1} \ , \ \pi_+ = \Lambda \tau_+ \ . \tag{87}$$

Now $\Lambda^{-1} = 1 - 2\alpha\pi_-$. For the generators π_+, π_- and τ_3 the commutation relations are those of the classical $sl(2)$,

$$[\tau_3, \pi_+] = 2\pi_+ \ , \ [\tau_3, \pi_-] = -2\pi_- \ , \ [\pi_+, \pi_-] = \tau_3 \ , \tag{88}$$

while the coalgebra structure is

$$\Delta\tau_3 = \tau_3 \otimes \Lambda + 1 \otimes \tau_3 \ , \tag{89}$$
$$\Delta\pi_- = \pi_- \otimes 1 + \Lambda^{-1} \otimes \pi_- \ , \tag{90}$$
$$\Delta\pi_+ = \pi_+ \otimes \Lambda + 1 \otimes \pi_+ - \alpha\tau_3 \otimes \Lambda\tau_3 - \alpha^2(\tau_3^2 - 2\tau_3) \otimes \Lambda^2 \pi_- \ , \tag{91}$$

which coincides with the above formulas for $U^J sl(2)$.

Notes. 1. In [9] the isomorphism between certain completions of the rings of differential operators on the q-spaces and the ring of usual differential operators was established. Here it holds as well. Let

$$\xi^1 = x\Lambda^{1/2} \ , \quad \xi^2 = y\Lambda^{-1/2} \ , \tag{92}$$
$$\partial_1 = \Lambda^{-1/2}\partial_x \ , \quad \partial_2 = \Lambda^{1/2}\partial_y \ . \tag{93}$$

Then

$$\xi^1\xi^2 = \xi^2\xi^1 \ , \quad \partial_1\partial_2 = \partial_2\partial_1 \ , \quad \partial_i\xi^j = \delta_i^j + \xi^j\partial_i \ , \tag{94}$$

which gives the needed isomorphism.

2. As shown in Sect. 1, the Hopf algebra $U^J sl(2)$ is a limit of the standard q-deformation. In the dual picture this limiting procedure looks as follows.

Let $\tilde{v} = (\tilde{x}, \tilde{y})^t$ be the quantum vector for $\mathrm{SL}_q(2)$, $\tilde{x}\tilde{y} = q\tilde{y}\tilde{x}$. Introduce a matrix $M = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, where $\mu = \alpha/(1-q)$. The matrix M is singular at $q = 1$. However the components x, y of the vector $v = M\tilde{v}$ satisfy the nonsingular relation $xy = qyx + \alpha y^2$, which in the limit $q \to 1$ defines the Jordanian plane. Similarly if A is the $\mathrm{SL}_q(2)$ quantum matrix then $\lim_{q\to 1} MAM^{-1}$ is finite and satisfies the relations for the Jordanian quantum matrix. Twisting with M and taking the limit one obtains also the Jordanian $\hat{R}$-matrix from the $\hat{R}$-matrix for SL(2).

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