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SPINOR FIELDS ON RIEMANNIAN MANIFOLDS

OLGA POKORNÁ[1]

1. Introduction

Let (M, g) be a connected Riemannian manifold of dimension n with a spin structure $(\tilde{P}, \eta)$, let S be a spinor bundle on M and $\Gamma(S)$ the space of all smooth sections od S.

A spin field $\psi \in \Gamma(S)$ is called Killing spinor with a Killing number $\lambda \in \mathbf{C}$ if the differential equation

$$\nabla^S_X \psi = \lambda X.\psi \tag{1}$$

is satisfied for all vector fields X on M.

A spinor field $\psi \in \Gamma(S)$ is called a twistor spinor if for all vector fields X on M

$$\mathcal{D}\psi = \nabla^S_X \psi + \frac{1}{n} X.D\psi = 0 \tag{2}$$

and such a field ψ is called E-spinor (or so-called Lichnerowicz spinor) if for all vectors fields X on the manifold M

$$E\psi = \nabla^S_X(D\psi) + \frac{R}{4(n-1)} X.\psi = 0, \tag{3}$$

where D denotes the Dirac operator.

The equation (3) was introduced by A. Lichnerowicz in 1988 in connection with a study of spinor fields. At the same time he proved the following important theorems (see [4]).

Theorem 1.1. (Lichnerowicz)

If (M, g) is a connected Riemannian spin manifold of dimension $n \geq 3$ with a nontrivial E-spinor, then the scalar curvature R is constant on M.

Theorem 1.2 (Lichnerowicz)

If (M, g) is a compact Riemannian spin manifold with a nontrivial E-spinor, then

$$Ker(\mathcal{D}) = Ker(E) = \mathbf{K},$$

where we denoted by $\mathbf{K}$ the space of all Killing spinors on M (see [1], e.g.).

[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

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I have succeeded in finding E-spinors on $S^2 \times R^1$ and $H^2 \times R^1$. It is a natural way to construct E-spinors which are not Killing ones. Noncompact Riemannian manifolds $S^2 \times R^1$ and $H^2 \times R^1$ are not the Einstein spaces and that is why Killing spinors do not exist there.

Proposition 1.3

Every solution of the equation (3) on $H^2 \times R^1$ is of the form

$$\psi(x,t) = \{A_0 \cos(\frac{1}{2}t) + A_1 \sin(\frac{1}{2}t)\}\psi^+(x) + \{A_0 \sin(\frac{1}{2}t) - A_1 \cos(\frac{1}{2})\}\psi^-(x) +$$

$$+\{B_0 \cos(\frac{1}{2}t) + B_1 \sin(\frac{1}{2}t)\}\varphi^+(x) + \{-B_0 \sin(\frac{1}{2}t) + B_1 \cos(\frac{1}{2}t)\}\varphi^-(x),$$

where A_0, A_1, B_0, B_1 are arbitrary constants and $\psi = \psi^+ + \psi^-$ resp. $\varphi = \varphi^+ + \varphi^-$ are Killing spinors on H^2 corresponding to $\lambda = \frac{i}{2}$ (resp. $\lambda = -\frac{i}{2}$). Proof.(see [5])

Proposition 1.4

Every solution of the equation (3) on $S^2 \times R^1$ is of the form

$$\psi(x,t) = \{A_0 \cosh(\frac{1}{2}t) + A_1 \sinh(\frac{1}{2}t)\}\psi^+(x) - i\{A_0 \sinh(\frac{1}{2}t) + A_1 cosh(\frac{1}{2}t)\}\psi^-(x) +$$

$$+\{B_0 \cosh(\frac{1}{2}t) + B_1 \sinh(\frac{1}{2}t)\}\varphi^+(x) + +i\{B_0 \sinh(\frac{1}{2}t) + B_1 \cos(\frac{1}{2}t)\varphi^-(x),$$

where A_0, A_1, B_0, B_1 are arbitrary constants and $\psi = \psi^+ + \psi^-$ resp. $\varphi = \varphi^+ + \varphi^-$ are Killing spinors on S^2 corresponding to $\lambda = \frac{1}{2}$ (resp. $\lambda = \frac{1}{2}$). Proof.(see [5])

2. The other relations between $Ker(E)$ and $Ker(\mathcal{D})$.

Theorem 2.1

Let (M, g) be a connected Riemannian manifold of dimension n with a spin structure. If $\psi \in Ker(E) \neq \{0\}$, then

$$R^S(X,Y).D\psi + \frac{R}{4(n-1)}(Y.\nabla^S_X\psi - X.\nabla^S_Y\psi) = 0, \tag{4}$$

where $R^S(X,Y)$ is the curvature tensor of the connection ∇^S on S and X, Y are vector fields on M.

Proof: If $\psi \in Ker(E) \neq \{0\}$, then the Theorem 1.1 implies that the scalar curvature R is constant on M (see [4]).

By differentiation of the equation (3) with respect to Y, we get

$$\nabla^S_Y\nabla^S_X(D\psi) + \frac{R}{4(n-1)}(\nabla^S_Y X).\psi + \frac{R}{4(n-1)}X.\nabla^S_Y\psi = 0. \tag{5}$$

Exchanging X and Y, we get

$$\nabla^S_X\nabla^S_Y(D\psi) + \frac{R}{4(n-1)}(\nabla^S_X Y).\psi + \frac{R}{4(n-1)}Y.\nabla^S_X\psi = 0. \tag{6}$$

The equation (3) is also valid for vector fields $[X, Y]$ on M:

$$\nabla^S_{[X,Y]}(D\psi) + \frac{R}{4(n-1)}[X,Y].\psi = 0. \tag{7}$$

By subtracting the equations (5) and (7) from (6), we get

$$\nabla^S_X\nabla^S_Y(D\psi) - \nabla^S_Y\nabla^S_X(D\psi) - \nabla^S_{[X,Y]}(D\psi) +$$

$$+\frac{R}{4(n-1)}(\nabla^S_X Y - \nabla^S_Y X - [X,Y])\psi +$$

$$+\frac{R}{4(n-1)}(Y.\nabla^S_X\psi - X.\nabla^S_Y\psi) = 0. \qquad \blacksquare$$

For a given $\psi \in \Gamma(S)$, let us define functions

$$C\psi = Re(D\psi, \psi)$$

$$Q\psi = |\psi|^2|D\psi|^2 - C^2\psi - \sum_{i=1}^{n}(Re(D\psi, e_i\psi))^2$$

Then we have

Theorem 2.2

Let (M, g) be a connected Riemannian spin manifold of dimension $n \geq 3$ such that $Ker(E) \neq 0$ and the scalar curvature is nonzero. Then the quadratic forms C and Q are constant on $Ker(E)$.

Proof: Theorem 1.1 implies that the scalar curvature R is constant. Moreover R is nonzero. Then Corollary of Theorem 1 (see[2]) implies that,

$$\dim_C Ker(E) = \dim_C Ker(\mathcal{D}) \leq 2^{[n/2]+1}.$$

On this vector space, there exist quadratic forms C and Q.

For all $X \in T_xM$, $x \in M$ we get

$$X(C\psi) = Re((\nabla^S_X D\psi, \psi) + (D\psi, \nabla^S_X\psi)). \tag{8}$$

Proposition 2 (see [2]) implies, that

$$\nabla^S_X\psi = \frac{2(n-1)}{R(n-2)}\left(\frac{R}{2(n-1)}X - Ric(X)\right).D\psi.$$

We obtain

$$X(C\psi) = \mathrm{Re}\left(-\tfrac{R}{4(n-1)}X.\psi,\psi\right) + \mathrm{Re}\left(D\psi, \tfrac{2(n-1)}{R(n-2)}\left(\tfrac{R}{2(n-1)}X - \mathrm{Ric}(X)\right).D\psi\right) =$$
$$= -\tfrac{R}{4(n-1)}\mathrm{Re}(X.\psi,\psi) + \tfrac{2(n-1)}{R(n-2)}.\mathrm{Re}\left(D\psi, \left(\tfrac{R}{2(n-1)}X - \mathrm{Ric}(X)\right).D\psi\right).$$

Clifford multiplication has the following property with respect to the Hermetian scalar product (,)

$$\mathrm{Re}(X\psi,\psi) = 0 \quad \text{for all} \quad X \in T_xM, x \in M$$

hence $C\psi$ =konst.

Moreover, if $\psi \in Ker(E)$, then Theorem 1 (see [2]) implies that

$$\varphi = D\psi \in Ker(\mathcal{D})$$

hence $Q\varphi =$ konst (see [3]).

But

$$Q\varphi = |\varphi|^2|D\varphi|^2 - C^2\varphi - \sum_{i=1}^{n}(\mathrm{Re}(D\varphi, e_i.\varphi))^2 =$$
$$= |D\psi|^2|D^2\psi|^2 - (\mathrm{Re}(D^2\psi, D\psi))^2 - \sum_{i=1}^{n}(\mathrm{Re}(D^2\psi, e_i.D\psi))^2 =$$
$$= \frac{n^2R^2}{16(n-1)^2}(|D\psi|^2|\psi|^2 - (\mathrm{Re}(\psi, D\psi))^2 - \sum_{i=1}^{n}(\mathrm{Re}(\psi, e_i.D\psi))^2 =$$
$$= \left(\frac{nR}{4(n-1)}\right)^2 Q\psi.$$

Moreover we have

$$\mathrm{Re}(\psi, e_i.D\psi) = -\mathrm{Re}(D\psi, e_i.\psi).$$

Hence $Q\psi$ is constant. ∎

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