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NATURAL OPERATORS LIFTING FUNCTIONS TO COTANGENT BUNDLES OF LINEAR HIGHER ORDER TANGENT BUNDLES

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Abstract. *All natural operators $C^\infty(M) \to C^\infty(T^*T^{(r)}M)$ for n-dimensional manifolds are determined, provided $n \geq 3$.*

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0. From now on we fix two natural numbers r and n. Given a manifold M we denote the space of all r-jets of maps $M \to \mathbf{R}$ with target 0 by $T^{r*}M = J^r(M, \mathbf{R})_0$. This is a vector bundle over M with the source projection. The dual vector bundle $(T^{r*}M)^*$ of $T^{r*}M$ is denoted by $T^{(r)}M$ and called the linear r-tangent bundle of M, c.f. [2]. Every embedding $\varphi : M \to N$ of two n dimensional manifolds (n-manifolds) induces vector bundle homomorphisms $T^{r*}\varphi : T^{r*}M \to T^{r*}N$ over φ defined by composition of jets and $T^{(r)}\varphi : T^{(r)}M \to T^{(r)}N$ dual to $T^{r*}\varphi^{-1}$.

In this paper we study the problem how a map $L : M \to \mathbf{R}$ on a manifold M can induce canonically a map $A_M(L) : T^*T^{(r)}M \to \mathbf{R}$. This problem is reflected in the concept of natural operators $T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ for n-manifolds, cf. [2].

Definition 0.1. A *natural operator $A : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ for n-manifolds*

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is a family of functions

$$A_M : C^\infty(M) \to C^\infty(T^*T^{(r)}M)$$

for any n-manifold M satisfying the following conditions:

(1) For any embedding $\varphi : M \to N$ of two n-manifolds and any map $L : N \to \mathbf{R}$ we have $A_M(L \circ \varphi) = A_N(L) \circ T^*T^{(r)}\varphi$.

(2) If $L_t : M \to \mathbf{R}$, $t \in \mathbf{R}$, is a smoothly parametrized family of maps (i.e. the resulting map $L : \mathbf{R} \times M \to \mathbf{R}$ is smooth), then so is $A_M(L_t)$.

Example 0.1. For every vector bundle $E \to M$, $x \in M$ and $y \in E_x$ we have a natural linear isomorphism between the fibre E_x of E over x and the vertical space $V_yE := T_yE_x$ of E at y given by $v \to \frac{d}{dt}|_{t=0}(y+tv)$. For any vector space W we have $< , >: W^* \times W \to \mathbf{R}$, $< a, v >= a(v)$. Denote

$$S(r) = \{(s_1, s_2) \in (\mathbf{N} \cup \{0\})^2 : 1 \le s_1 + s_2 \le r\} .$$

Let $(s_1, s_2) \in S(r)$ and let $L : M \to \mathbf{R}$, where M is an n-manifold.
Define $\lambda_M^{<s_1,s_2>}(L) : T^*T^{(r)}M \to \mathbf{R}$ by

$$\lambda_M^{<s_1,s_2>}(L)(a) :=< (A^{<s_1,s_2>}(L) \circ \pi)(a), q(a) > ,$$

where $q : T^*T^{(r)}M \to T^{(r)}M$ is the cotangent bundle projection,
$A^{<s_1,s_2>}(L) : (T^{(r)}M)^* \to (T^{(r)}M)^*$ is a fibre bundle map over id_M given by

$$A^{<s_1,s_2>}(L)(j^r_x\gamma) := j^r_x(\gamma^{s_1}(L - L(x))^{s_2}),\ \gamma : M \to \mathbf{R},\ \gamma(x) = 0,\ x \in M ,$$

and $\pi : T^*T^{(r)}M \to (T^{(r)}M)^*$ is a fibre bundle map over id_M given by

$$\pi(a) := a|V_{q(a)}T^{(r)}M \tilde{=} T^{(r)}_xM,\ a \in (T^*T^{(r)})_xM,\ x \in M .$$

Clearly, given a pair $(s_1, s_2) \in S(r)$ the family $\lambda^{<s_1,s_2>} = \{\lambda_M^{<s_1,s_2>}\}$ of functions

$$\lambda_M^{<s_1,s_2>} : C^\infty(M) \to C^\infty(T^*T^{(r)}M) ,\ L \to \lambda_M^{<s_1,s_2>}(L)$$

for any n-manifold M, is a natural operator $T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ for n-manifolds.

Given $L : M \to \mathbf{R}$ we have the vertical lifting $L^V : T^*T^{(r)}M \to \mathbf{R}$ of L defined to be the composition of L with the canonical projection $T^*T^{(r)}M \to M$. The correspondence "$L \to L^V$" gives a natural operator $T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ for n-manifolds

If $H : \mathbf{R}^{S(r)} \times \mathbf{R} \to \mathbf{R}$ is a map, then the family $A^{(H)}$ of functions $A_M^{(H)} : C^\infty(M) \to C^\infty(T^*T^{(r)}M)$,

$$A_M^{(H)}(L) := H \circ ((\lambda_M^{<s_1,s_2>}(L))_{(s_1,s_2)\in S(r)}, L^V)$$

for any n-manifold M, is also a natural operator $T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ for n-manifolds.

We are going to prove

Theorem 0.1. *Let $A : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ be a natural operator for n-manifolds. If $n \geq 3$, then there exists the uniquely determined smooth map $H : \mathbf{R}^{S(r)} \times \mathbf{R} \to \mathbf{R}$ such that $A = A^{(H)}$.*

We see that any constant natural operator $T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ is a natural function on $T^*T^{(r)}$ in the sense of [1] or [3]. On the other hand any natural function g on $T^*T^{(r)}$ for n-manifolds determines a natural operator $A : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$, $A_M(L) = g_M$. Thus we have reobtained the following result of [3].

Corollary 0.1. *All natural functions on $T^*T^{(r)}$ for n-manifolds ($n \geq 3$) are of the form $\{H \circ (\lambda_M^{<1,0>}, ..., \lambda_M^{<r,0>})\}$, where $H \in C^\infty(\mathbf{R}^r)$ is a function of r variables.*

1. The proof of Theorem 0.1 will be given in Item 2. In this item we prove some lemmas.

Let q, π be as in Example 0.1. The usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, ..., x^n$ and the canonical vector fields induced by $x^1, ..., x^n$ on $\mathbf{R}^n$ by $\partial_1, ..., \partial_n$. For any vector field X on M the complete lift of X to $T^{(r)}M$ is denoted by $T^{(r)}X$.

It is clear that $T^{(r)}((x^1)^r\partial_1)$ is vertical over 0. We recall that $j_0^r(x^1) \in T_0^{r*}\mathbf{R}^n \doteq (V_y T^{(r)}\mathbf{R}^n)^*$ for any $y \in T_0^{(r)}\mathbf{R}^n$. We have.

Lemma 1.1. *The set*

$$\{y \in T_0^{(r)}\mathbf{R}^n :< T^{(r)}((x^1)^r\partial_1)(y), j_0^r(x^1) >\neq 0\}$$

is dense in $T_0^{(r)}\mathbf{R}^n$.

Proof. Let φ_t be the flow of $(x^1)^r\partial_1$ near 0. Then we have

$$\begin{aligned} < T^{(r)}((x^1)^r\partial_1)(y), j_0^r(x^1) > &=< \frac{d}{dt}|_{t=0} T_0^{(r)}\varphi_t(y), j_0^r(x^1) > \\ &= \frac{d}{dt} < T^{(r)}\varphi_t(y), j_0^r(x^1) > |_{t=0} \\ &= \frac{d}{dt} < y, j_0^r(x^1 \circ \varphi_t) > |_{t=0} \\ &=< y, j_0^r(\frac{\partial}{\partial t}(x^1 \circ \varphi_t)_{t=0}) > \\ &=< y, j_0^r((x^1)^r) > \end{aligned}$$

for any $y \in T_0^{(r)}\mathbf{R}^n$. Hence our lemma is obvious. □

Now we prove the following lemma.

Lemma 1.2. *Let $A, B : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ be two natural operators for n-manifolds. Assume that $n \geq 2$ and that*

$$A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$$

for all $\alpha \in \mathbf{R}$ and all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ with

$$\pi(a) = j_0^r(x^1) \ . \tag{1.1}$$

($(T^*T^{(r)})_0\mathbf{R}^n$ is the fibre over 0 of the bundle $T^*T^{(r)}\mathbf{R}^n \to \mathbf{R}^n$.) Then $A = B$.

Proof. Consider $L : \mathbf{R}^n \to \mathbf{R}$. Using the invariancy of A and B it suffices to show that $A_{\mathbf{R}^n}(L) = B_{\mathbf{R}^n}(L)$ over $0 \in \mathbf{R}^n$.

Let $a \in (T^*T^{(r)})_0\mathbf{R}^n$. We can write $\pi(a) = j_0^r(\gamma)$ for some $\gamma : \mathbf{R}^n \to \mathbf{R}$ with $\gamma(0) = 0$. Consider two cases.

(1) Suppose that the rank of the differential $d_0(\gamma, L)$ of (γ, L) at 0 is maximal. Then by the rank theorem there is an embedding $\varphi : \mathbf{R}^n \to \mathbf{R}^n$, $\varphi(0) = 0$, such that

$$(\gamma, L) \circ \varphi = (x^1, x^n + L(0))$$

on some neighbourhood of 0. Then $\pi(T^*T^{(r)}\varphi^{-1}(a)) = j_0^r(x^1)$ and $L \circ \varphi = x^n + L(0)$ on some neighbourhood of 0. Now, using the invariancy of A and B with respect to φ and the assumption of the lemma we deduce that $A_{\mathbf{R}^n}(L)(a) = B_{\mathbf{R}^n}(L)(a)$.

(2) Otherwise, there exists a sequence t_m $(m = 1, 2, ...)$ of real numbers tending to 0 such that $a_m = a + j_0^r(t_m x^1) \in (T^*T^{(r)})_0\mathbf{R}^n$ and $L_m = L + t_m x^n$ satisfy the assumption of case (1) with a, L replaced by a_m, L_m for any $m = 1, 2, ...$ Then (by case (1)) $A_{\mathbf{R}^n}(L_m)(a_m) = B_{\mathbf{R}^n}(L_m)(a_m)$ for any m. If $m \to \infty$, then $A_{\mathbf{R}^n}(L)(a) = B_{\mathbf{R}^n}(L)(a)$ because of the regularity condition. □

Using Lemma 1.2 we prove.

Lemma 1.3. *Let $A, B : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ be two natural operators for n-manifolds. Assume that $n \geq 2$ and that*

$$A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$$

for all $\alpha \in \mathbf{R}$ and all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ satisfying the conditions (1.1) and

$$< a, T^{(r)}\partial_i(q(a)) >= 0 \tag{1.2}$$

for $i \in \{3, ..., n\}$. Then $A = B$.

Proof. Consider $a \in (T^*T^{(r)})_0\mathbf{R}^n$ with $\pi(a) = j_0^r(x^1)$. Using Lemma 1.2 it is sufficient to show that $A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$ for any $\alpha \in \mathbf{R}$. We can assume that $n \geq 3$.(For, if $n \leq 2$, then $\{3, ..., n\} = \emptyset$.)

Using the density argument one can assume that $< a, T^{(r)}\partial_2(q(a)) >\neq 0$. Define $\Theta \in T_0^*\mathbf{R}^n$ by

$$< \Theta, Z(0) >=< a, T^{(r)}Z(q(a)) >$$

for all constant vector fields Z on $\mathbf{R}^n$. Then

$$\Theta = \beta_1 d_0x^1 + \beta_2 d_0x^2 + ... + \beta_n d_0x^n$$

for some $\beta_1, ..., \beta_n \in \mathbf{R}$. By the above assumption $\beta_2 \neq 0$. Let $\psi = (x^1, \beta_2 x^2 + ... + \beta_n x^n, x^3, ..., x^n)$. Then $\psi : \mathbf{R}^n \to \mathbf{R}^n$ is an linear isomorphism, $x^1 \circ \psi = x^1$, $x^n \circ \psi = x^n$ and

$$T_0^*\psi(\Theta) = \beta_1 d_0x^1 + d_0x^2 \ .$$

Let $\overline{a} = T^*T^{(r)}\psi(a)$. Since $T^{r*}\psi(j_0^r(x^1)) = j_0^r(x^1)$, $\overline{a}$ satisfies the condition (1.1) with a replaced by $\overline{a}$. Moreover,

$$\begin{aligned} < \overline{a}, T^{(r)}\partial_i(q(\overline{a})) > &=< a, T^{(r)}((\psi^{-1})_*\partial_i)(q(a)) > \\ &=< \Theta, ((\psi^{-1})_*\partial_i)(0) > \\ &=< T^*\psi(\Theta), \partial_i(0) >= 0 \end{aligned}$$

for $i = 3, ..., n$. Then by the assumption of the lemma $A_{\mathbf{R}^n}(x^n + \alpha)(\overline{a}) = B_{\mathbf{R}^n}(x^n + \alpha)(\overline{a})$ for any $\alpha \in \mathbf{R}$. Thus by the invariancy of A and B with respect to ψ we obtain $A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$ for any $\alpha \in \mathbf{R}$. □

Lemmas 1.1 and 1.3 imply the following assertion.

Lemma 1.4. *Let $A, B : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ be two natural operators for n-manifolds. Assume that $n \geq 3$ and that*

$$A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$$

for all $\alpha \in \mathbf{R}$ and all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ satisfying the conditions (1.1) and (1.2) for $i \in \{2, ..., n\}$. Then $A = B$.

Proof. Consider $a \in (T^*T^{(r)})_0\mathbf{R}^n$ with (1.1) and (1.2) for $i \in \{3, ..., n\}$. Let $\alpha \in \mathbf{R}$. By Lemma 1.3 it suffices to show that $A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$.

Using the density argument and Lemma 1.1 we can additionally assume that

$$< T^{(r)}((x^1)^r\partial_1)(q(a)), j_0^r(x^1) >= \frac{1}{\beta_1}$$

for some $\beta_1 \in \mathbf{R}$.

Let $< a, T^{(r)}\partial_2(q(a)) >= \beta_2$. Since $j_0^{r-1}(\partial_2 - \beta_1\beta_2(x^1)^r\partial_1) = j_0^{r-1}(\partial_2)$, there exists an embedding $\varphi : \mathbf{R}^n \to \mathbf{R}^n$, $\varphi(0) = 0$, such that: $j_0^r(\varphi) = j_0^r(id)$, $x^n \circ \varphi = x^n$,

$$germ_0(T\varphi \circ (\partial_2 - \beta_1\beta_2(x^1)^r\partial_1)) = germ_0(\partial_2 \circ \varphi), \text{ and}$$

$$germ_0(T\varphi \circ \partial_i) = germ_0(\partial_i \circ \varphi)$$

for $i = 3, ..., n$, cf. [2].

Let $\overline{a} = T^*T^{(r)}\varphi(a)$. Since φ preserves $j_0^r(x^1)$ and ∂_i for $i = 3, ..., n$, then $\overline{a}$ satisfies the conditions (1.1) and (1.2) for $i = 3, ..., n$. Moreover,

$$\begin{aligned} < \overline{a}, T^{(r)}\partial_2(q(\overline{a})) > &=< a, TT^{(r)}\varphi^{-1}(T^{(r)}\partial_2(q(\overline{a}))) > \\ &=< a, T^{(r)}\partial_2(q(a)) - \beta_1\beta_2 T^{(r)}((x^1)^r\partial_1)(q(a)) > \\ &= \beta_2 - \beta_1\beta_2\frac{1}{\beta_1} = 0 \end{aligned}$$

Then by the assumption of the lemma $A_{\mathbf{R}^n}(x^n + \alpha)(\overline{a}) = B_{\mathbf{R}^n}(x^n + \alpha)(\overline{a})$. Now, by the invariancy of A and B with respect to φ we obtain that $A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$.□

Similarly, one can prove.

Lemma 1.5. *Let $A, B : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ be two natural operators for n-manifolds. Assume that $n \geq 3$ and that*

$$A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$$

for all $\alpha \in \mathbf{R}$ and all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ satisfying the conditions (1.1) and (1.2) for $i \in \{1, ..., n\}$. Then $A = B$.

Proof. The proof of Lemma 1.5 is a replica of the proof of Lemma 1.4. (In the text of the proof of Lemma 1.4 we replace ∂_2 by ∂_1, Lemma 1.3 by Lemma 1.4 and $i = 3, ..., n$ by $i = 2, ..., n$.) $\square$

Now, we prove the main lemma.

Lemma 1.6. *Let $A, B : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ be two natural operators for n-manifolds. Assume that $n \geq 3$ and that*

$$A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$$

for all $\alpha \in \mathbf{R}$ and all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ satisfying the conditions (1.1) and (1.2) for $i \in \{1, ..., n\}$ and

$$(1.3) \qquad < q(a), j_0^r(x^\beta) >= 0$$

for all $\beta = (\beta_1, ..., \beta_n) \in (\mathrm{N} \cup \{0\})^n$ with $1 \leq |\beta| \leq r$ and $\beta_2 + ... + \beta_{n-1} \geq 1$. Then $A = B$.

Proof. Consider $a \in (T^*T^{(r)})_0\mathbf{R}^n$ satisfying the conditions (1.1) and (1.2) for $i = 1, ..., n$. Let $\alpha \in \mathbf{R}$. By Lemma 1.5 it is sufficient to show that $A_{\mathbf{R}^n}(x^n + \alpha)(a) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$.

Let $c_t := (x^1, tx^2, ..., tx^{n-1}, x^n) : \mathbf{R}^n \to \mathbf{R}^n$, $t \neq 0$. It is easy to see that $a^o := lim_{t\to 0}(T^*T^{(r)}c_t(a))$ satisfies (1.1), (1.2) for $i = 1, ..., n$, and (1.3) for all $\beta = (\beta_1, ..., \beta_n) \in (\mathrm{N} \cup \{0\})^n$ with $1 \leq |\beta| \leq r$ and $\beta_2 + ... + \beta_{n-1} \geq 1$. Then using the invariancy of A and B with respect to c_t we deduce that $A_{\mathbf{R}^n}(x^n + \alpha)(a) = A_{\mathbf{R}^n}(x^n + \alpha)(a^o) = B_{\mathbf{R}^n}(x^n + \alpha)(a^o) = B_{\mathbf{R}^n}(x^n + \alpha)(a)$. $\square$

2. We are now in position to prove the theorem. Let $A : T^{(0,0)} \to T^{(0,0)}(T^*T^{(r)})$ be a natural operator for n-manifolds. Define

$$H : \mathbf{R}^{S(r)} \times \mathbf{R} \to \mathbf{R}\ ,\ H(\xi, \alpha) = A_{\mathbf{R}^n}(x^n + \alpha)(a_\xi),$$

where $\xi = (\xi_{(s_1,s_2)}) \in \mathbf{R}^{S(r)}$ and $a_\xi \in (T^*T^{(r)})_0\mathbf{R}^n$ is the unique form satisfying the conditions:

(1.1); (1.2) for $i = 1, ..., n$;

(1.3) for all $\beta \in (\mathrm{N} \cup \{0\})^n$ with $1 \leq |\beta| \leq r$ and $\beta_2 + ... + \beta_{n-1} \geq 1$; and

$$(2.4) \qquad < q(a_\xi), j_0^r((x^1)^{s_1}(x^n)^{s_2}) >= \xi_{(s_1,s_2)}$$

for all $(s_1, s_2) \in S(r)$.

It is clear that H is smooth. We see that

$$\begin{aligned} A_{\mathbf{R}^n}(x^n + \alpha)(a_\xi) &= H \circ ((\lambda^{<s_1,s_2>}_{\mathbf{R}^n}(x^n + \alpha)(a_\xi))_{(s_1,s_2)\in S(r)}, (x^n + \alpha)^V(a_\xi)) \\ &= A^{(H)}_{\mathbf{R}^n}(x^n + \alpha)(a_\xi) \end{aligned}$$

for all $\xi \in \mathbf{R}^{S(r)}$ and all $\alpha \in \mathbf{R}$. Hence by Lemma 1.6 we obtain $A = A^{(H)}$. (For, any a satisfying the conditions of Lemma 1.6 is of the form a_ξ for some ξ as above.)□

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