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INTRODUCTION TO ALGEBRA OVER "BRAVE NEW RINGS"

R.M. VOGT

"Brave New Rings" is a catchphrase introduced by Waldhausen for algebraic structures on topological spaces which are ring structures up to coherent homotopies. "Coherent" means that the homotopies fit together nicely (see below). Such "rings" were introduced in the mid seventies in an effort to analyze the structure of classifying spaces arising from geometry [32].

Waldhausen was the first person to extend classical algebraic constructions to the world of brave new rings. The most famous one is his algebraic K-theory of topological spaces [54], [55] which, in fact, is an extension of Quillen's algebraic K-functor to certain brave new rings. Other algebraic constructions are Bökstedt's topological Hochschild homology [6] or the topological cyclic homology of Goodwillie [14] or Bökstedt-Hsiang-Madsen [7].

Although this extended algebra is still at its initial stage it has already proved to have remarkable applications in geometry and classical algebra.

Let me present a dictionary

Dictionary

classical algebra	brave new algebra	spectra version
monoid	A_∞-space	-
group	loop space	-
abelian monoid	E_∞-space	-
abelian group	infinite loop space	S-module spectrum
ring	A_∞-ring space	monoid in $\mathcal{M}od_{\mathbb{S}}$
commutative ring	E_∞-ring space	commutative monoid in $\mathcal{M}od_{\mathbb{S}}$
ground ring $\mathbb{Z}$	E_∞-ring space $Q(S^0)$	sphere spectrum $\mathbb{S}$
tensor product $\otimes_{\mathbb{Z}}$	?	smash product $\wedge_{\mathbb{S}}$

*The paper is in final form and no version of it will be submitted elsewhere.

To give the reader a feeling for the problems and concepts involved we investigate the simplest cases, namely A_∞-spaces and loop spaces, in some detail, giving at least indications of proofs. We then treat the algebraic structure of E_∞-spaces and infinite loop spaces. These structures were introduced in the sixties [50], [3]. We then pass to homotopy ring spaces. It will soon become clear that algebraic constructions over homotopy ring spaces, though in principle often possible, are quite involved - the structure is very complex. The way out is a black box in terms of spectra, which we will introduce in Chapter 3. The final chapter summarizes some applications which either involve "brave new algebra" directly or which use its methods and ideas.

The paper consists of notes of a series of lectures delivered at the 18th Winter School on Geometry and Physics in Srni, Czech Republic. I would like to thank the organizers for a delightful week in the Bohemian Forest.

Chapter I: A_∞-spaces and loop spaces

We will work in the categories $\mathcal{T}op$ and $\mathcal{T}op^*$ of compactly generated spaces and their based versions, respectively. Limits, colimits and function spaces are taken in these categories. They have the useful properties that quotients commutate with finite products and that the exponential law for function spaces holds in general.

1 The structure of a loop space

1.1 Definition: The *loop space* ΩX of the based space $(X, *)$ is the space of all maps

$$\Omega X = \mathcal{T}op\,((I, \partial I), (X, *)) \cong \mathcal{T}op^*(S^1, X)$$

with the constant map as base point. Here I is the unit interval and S^1 the unit sphere.

Loop spaces have been studied extensively in algebraic topology but also have applications in other fields of mathematics such as differential geometry (Marston Morse's work on geodesics on Riemann n-spheres is nothing but the calculation of the homology of ΩS^n [36]) or, more recently, mathematical physics.

1.2 Elementary properties

(1) If $[X, Y]^*$ denotes the based homotopy classes of based maps, then $[X, \Omega Y]^*$ is a group.

(2) ΩX is a topological group up to homotopy (this is equivalent to (1)).

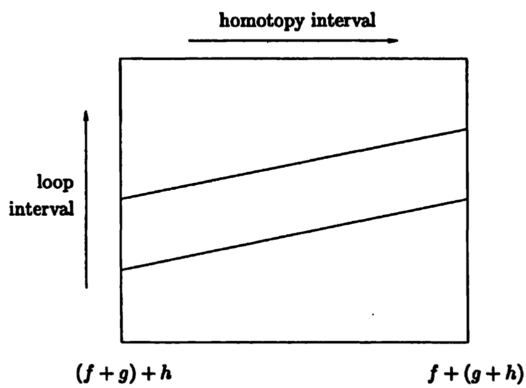
(3) $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ where $\pi_n(Y) = [S^n, Y]^*$ denotes the n-th homotopy group of Y. In particular, the fundamental group $\pi_1(\Omega X)$ is abelian.

(4) Singular homology $H_*(\Omega X)$ with arbitrary coefficients is a graded algebra.

In this section we are interested in the algebraic structure of ΩX and hence expand a little on property (2). We define loop addition by

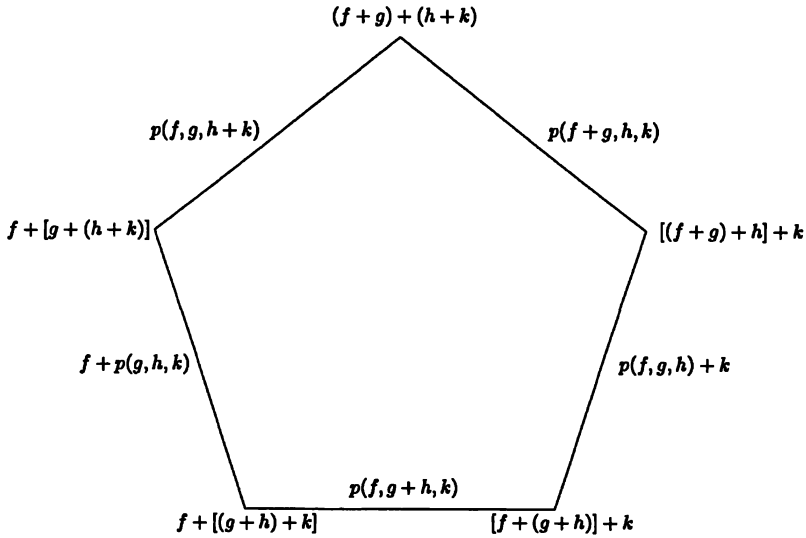
$$f+g : I \to X, \quad t \mapsto \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

i.e. we first run through the loop f and then through the loop g with double speed. Loop addition is only homotopy associative as the following picture shows



We have an associating path $p(f, g, h)$ from $(f+g)+h$ to $f+(g+h)$. Similar pictures show that the constant loop is a two-sided homotopy unit and that the loop $-f$, obtained by running through f in the opposite direction, is homotopy inverse to f. Hence ΩX is a group up to homotopy.

But there is more to the structure of ΩX: if we take four inputs, the associating homotopies assemble to a loop, i.e. we obtain a map $S^1 \times (\Omega X)^4 \to \Omega X$.

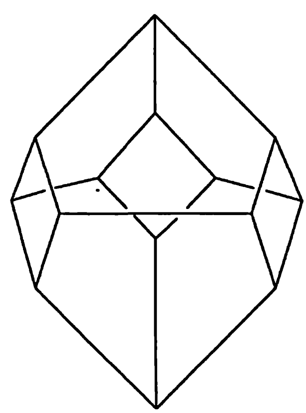


This map can be extended over the 2-ball to a map

$$\mathbb{B}^2 \times (\Omega X)^4 \to \Omega X.$$

If we take five inputs, six pentagons and three squares obtained from products of associating homotopies assemble to a 2-sphere to give a map

$$S^2 \times (\Omega X)^5 \to \Omega X$$



which extends over the 3-ball etc. In other words, the n-dimensional homotopies which arise fit together by an $(n+1)$-dimensional homotopy; they are *(homotopy-) coherent.*

1.3 Definition: A homotopy associative multiplication with homotopy unit in which all homotopies are coherent up to dimension $n-2$ is called an A_n-*structure* ("A_n" for associative with coherence conditions up to n inputs).

Our considerations imply (a more detailed proof will follow):

1.4 Theorem: A loop space admits an A_∞-structure; it is an A_∞-space. Moreover, it has homotopy inverses.

A_n-structures with strict units were introduced and studied by J. Stasheff [50]. He implicitly proved a converse of Theorem 1.4.

1.5 Theorem: A connected A_∞-space is of the weak homotopy type of a loop space.

To give a flavor of our methods we will sketch proofs of these results. Stasheff's associahedra are too complicated to deal with and the additional coherence relations for the homotopy unit make things even worse. Therefore we need a new book-keeping device for the coherence conditions.

2 Operads

2.1 Definition: A *non-Σ-operad* is a category $\mathcal{B}$ with object $0, 1, 2, \dots$, topologized morphism sets $\mathcal{B}(m, n)$ such that composition is continuous, and an associative continuous bifunctor

$$\oplus : \mathcal{B} \times \mathcal{B} \to \mathcal{B}, \quad m \oplus n = m + n$$

such that

$$(*) \quad \coprod_{r_1+\dots+r_k=n} \mathcal{B}(r_1, 1) \times \dots \times \mathcal{B}(r_k, 1) \to \mathcal{B}(n, k)$$

$$(f_1, \dots, f_k) \longmapsto f_1 \oplus f_2 \oplus \dots \oplus f_k$$

is a homeomorphism. For technical reasons we assume that $\{id_1\} \subset \mathcal{B}(1, 1)$ is a closed cofibration. A *functor of operads* is a continuous functor preserving objects and $\oplus$.

2.2 Remark: In [3] non-Σ-operads without condition $(*)$ were called *categories of operators without permutations*, and *in standard form* if $(*)$ is satisfied. Because of $(*)$, $\mathcal{B}$ is uniquely determined by the morphism spaces $\mathcal{B}(n, 1)$, $n \geq 0$, and composition. For these restricted data May in [31] introduced the word (non-Σ) *operad.* We prefer to stick to the older version of a category of operators in standard form but adopt the catchphrase "operad".

2.3 Definition: Let $\mathcal{B}$ be a non-Σ operad. A *$\mathcal{B}$-space* is a continuous functor X : $\mathcal{B} \to \mathcal{T}op$ sending $\oplus$ to $\times$. In particular, $X(n) = X(1)^n$. In abuse of notation we often identify X with the space $X(1)$.

2.4 Example: Let $\mathcal{A}$ be the terminal non-Σ operad: $\mathcal{A}(n, 1)$ consists of a single point μ_n for all n. This forces composition

$$\mu_n \circ (\mu_{r_1} \oplus \dots \oplus \mu_{r_n}) = \mu_{r_1+\dots+r_n}.$$

An $\mathcal{A}$-space is monoid and vice versa:

$$\begin{array}{ll} \mu_n : X^n \longrightarrow X, & (x_1 \dots, x_n) \mapsto x_1 \cdot \dots \cdot x_n,\ n \geq 1 \\ \mu_0 : X^0 = \{*\} \to X, & * \longmapsto \text{unit } e \end{array}$$

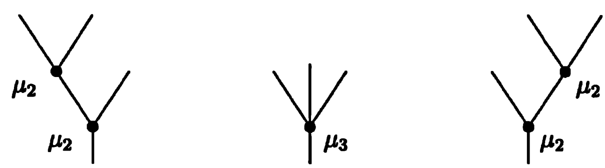
Associativity and unit axiom follow from the commutativity of (note $\mu_1 = id$)

$$\begin{tikzcd} X^3 \arrow[r, "\mu_2\times\mu_1"] \arrow[d, "\mu_1\times\mu_2"'] \arrow[dr, "\mu_3"] & X^2 \arrow[d, "\mu_2"] \\ X^2 \arrow[r, "\mu_2"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mu_0\times\mu_1"] \arrow[dr, "\mu_1"'] & X^2 \arrow[d, "\mu_2"] & X \arrow[l, "\mu_1\times\mu_0"'] \arrow[dl, "\mu_1"] \\ & X & \end{tikzcd}$$

2.5 Definition: A non-Σ operad $\mathcal{B}$ is called A_∞*-operad* if the unique functor of operads $\mathcal{B} \to \mathcal{A}$ is an h-equivalence (i.e. homotopy equivalence) of underlying morphism spaces. In this case we call a $\mathcal{B}$-space an A_∞*-space.*

Our next aim is to construct a homotopy universal A_∞-operad. Given a non-Σ operad $\mathcal{B}$ we think of $\alpha \in \mathcal{B}(n, 1)$ as a black box with n inputs and one output. Composition is represented by wiring boxes together. E.g.

2.6



$(x \cdot y) \cdot z$ $\qquad$ $x \cdot y \cdot z$ $\qquad$ $x \cdot (y \cdot z)$

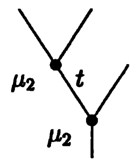
If $\alpha \in \mathcal{B}(0, 1)$, the associated box has no input but one output. E.g.

In $\mathcal{A}$ the three trees of (2.6) represent the same operation. For a general $\mathcal{B}$ they may be different.

We obtain a new non-Σ operad $T\mathcal{B}$, where $T\mathcal{B}(n, 1)$ is the space of all such formal trees with n inputs. A morphism in $T\mathcal{B}(n, k)$ is an ordered k tuple of trees, composition wires trees together, the identity is the trivial tree

in $T\mathcal{B}(1, 1)$ having no box. Clearly, $T\mathcal{B}$ is the free non-Σ operad over $\mathcal{B}$.

If we want to describe a homotopy associative multiplication we have to join the left and right tree of (2.6) by an interval. To do this, we give each edge joining two boxes (call these *internal edges*) a length $t \in I$ and allow edges of length 0 to be shrunk. E.g.

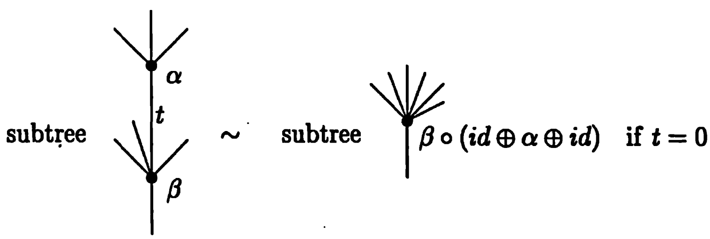


represents $\mu_2 \circ (\mu_2 \oplus id)$ for $t = 1$ and μ_3 for $t = 0$. So both the left tree and the right tree of (2.6) is joined to the middle one by a unit interval.

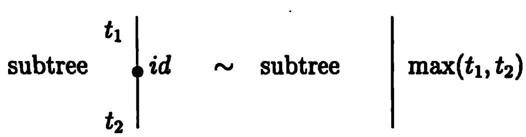
2.7 Construction: Let $\widetilde{T\mathcal{B}}$ be the non-Σ operad obtained from $T\mathcal{B}$ by attaching a length to each internal edge. Composition creates new internal edges which obtain the lengths 1.

Let $W\mathcal{B}$ be the non-Σ quotient operad obtained from $\widetilde{T\mathcal{B}}$ by imposing the relations

(1)

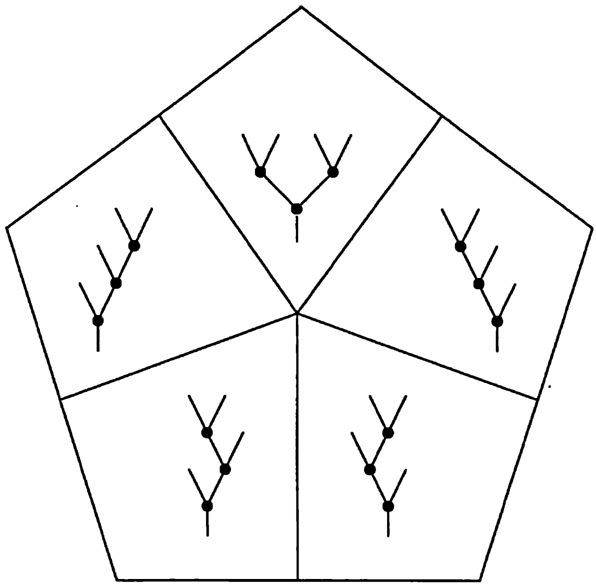


(2)



(3) If in (2) the box "id" is at an input or at the output of the tree we drop it and forget the length of the new input or new output.

We have seen that $W\mathcal{A}(3, 1)$ contains a subdivided interval joining $\mu_2 \circ (\mu_2 \oplus id)$ to $\mu_2 \circ (id \oplus \mu_2)$. The following picture shows the pentagon in $W\mathcal{A}(4, 1)$ subdivided into five squares; each $\bullet$ stands for a μ_2:



We recall that $T\mathcal{B}$ is the free non-Σ operad over $\mathcal{B}$. We obtain $W\mathcal{B}$ from $T\mathcal{B}$ by putting back the relations in $\mathcal{B}$ up to coherent homotopies. The following result makes this statement precise.

2.8 Proposition: The augmentation functor $\varepsilon : W\mathcal{B} \to \mathcal{B}$ obtained by shrinking all edges is a functor of operads and an h-equivalence of underlying morphism spaces.

Proof: $\mathcal{B}(n,1)$ is contained in $W\mathcal{B}(n,1)$ as trees with exactly one box. We deform $W\mathcal{B}(n,1)$ into $\mathcal{B}(n,1)$ by shrinking each internal edge at time t by the factor $(1-t)$. □

$W\mathcal{B}$ is homotopically universal with respect to this property.

2.9 Proposition: Given functors F and G of non-Σ operads such that

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "G"] \\ W\mathcal{B} \arrow[ur, dashed, "\widetilde{F}"] \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

G is an h-equivalence of underlying morphism spaces, then there exists $\widetilde{F} : W\mathcal{B} \to \mathcal{D}$ uniquely up to homotopy through functors of operads such that $G \circ \widetilde{F} \simeq F$ through functors of operads.

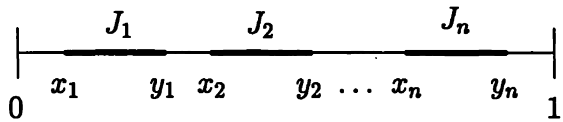
Proof: Construct $\widetilde{F}$ inductively using the obvious skeleton filtration of $W\mathcal{B}$ and the homotopy theoretical properties of an h-equivalence. □

2.10 Corollary: Given an A_∞-operad $\mathcal{B}$ there is a functor $W\mathcal{A} \to \mathcal{B}$ uniquely up to homotopy through operad functors.

3 A_∞-spaces, loop spaces, and monoids

We construct an A_∞-operad which acts on loop spaces.

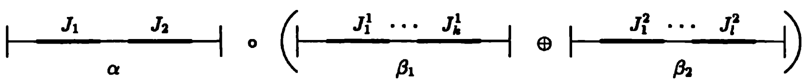
3.1 The operad $\mathcal{Q}$: The morphism space $\mathcal{Q}(n,1)$ is the space of ordered n-tuples of closed subintervals $J_i = [x_i, y_i]$ of I with disjoint interiors



Hence a point in $\mathcal{Q}(n,1)$ is a $2n$-tuple

$$0 \leq x_1 < y_1 \leq x_2 < y_2 \leq \ldots \leq x_n < y_n \leq 1$$

and we topologize $\mathcal{Q}(n,1)$ as subspace of $\mathbb{R}^{2n}$. Composition is insertion. E.g.



is given by inserting the unit intervals of β_1 and of β_2 with their subintervals linearly into J_1 and J_2, respectively.

The action of $\mathcal{Q}$ on a loop space ΩX is given by the maps

$$\begin{array}{ccc} \mathcal{Q}(n,1) \times (\Omega X)^n & \longrightarrow & \Omega X \\ ((J_1, \ldots, J_n), (f_1 \ldots, f_n)) & \mapsto & (g : I \to X) \end{array}$$

where g maps the subinterval J_i by $f_i \circ r_i$, where r_i is the linear expansion of J_i to $[0,1]$. The complement of $\bigcup_{i=1}^n J_i$ is mapped to the base point.

3.2 Lemma: $\mathcal{Q}(n,1)$ is contractible. Hence $\mathcal{Q}$ is an A_∞-operad.

Proof: Deform $(0 \leq x_1 < y_1 \leq \ldots \leq x_n < y_n \leq 1)$ linearly into $(0 = \frac{0}{n} < \frac{1}{n} \leq \frac{1}{n} < \frac{2}{n} \leq \ldots \leq \frac{n-1}{n} < \frac{n}{n} = 1)$. □

This gives a detailed proof of Theorem 1.4, and (2.10) connects the natural A_∞-structure of ΩX to Stasheff's associahedras:

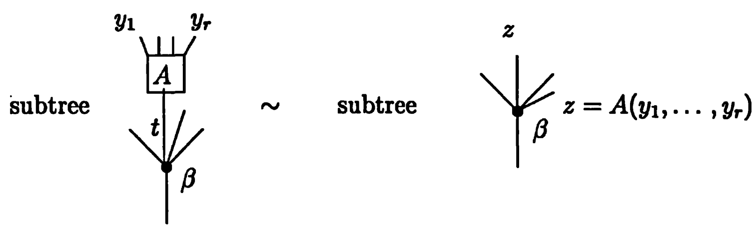
3.3 Theorem: There is a natural A_∞-structure on ΩX codified by the non-Σ operad $\mathcal{Q}$. In particular, ΩX is a $W\mathcal{A}$-space. □

To prove the converse, we first show

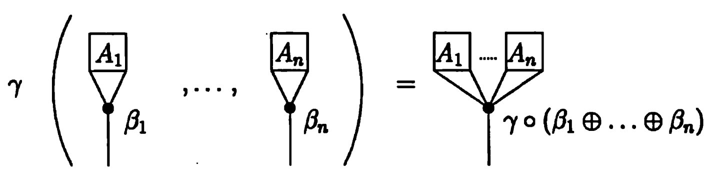
3.4 Proposition: Let $\mathcal{B}$ be any non-Σ operad. Each $W\mathcal{B}$-space X is a strong deformation retract of a $\mathcal{B}$-space $M_{\mathcal{B}}X$. In particular, each A_∞-space is h-equivalent to a monoid.

Proof: $M_{\mathcal{B}}X = \left(\coprod_{n\geq 0} \widetilde{T}\mathcal{B}(n,1) \times X^n\right) / \sim$

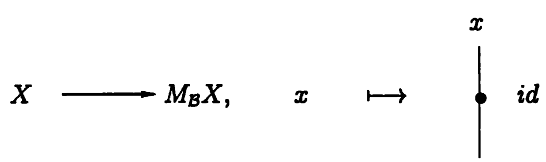
(note $X = X(1)$). We think of an element of $\widetilde{T}\mathcal{B}(n,1) \times X^n$ as a tree with lengths and the entries $(x_1, \ldots, x_n) \in X^n$ assigned to the n inputs. The relations are (2.7, (1), (2)), and (2.7, (3)) if the box is NOT the one at the output. We need the following additional relation: if $t = 1$, then



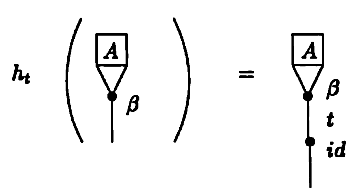
On the right-hand side we evaluated A on the entries $(y_1, \ldots, y_n)$ to obtain z. The $\mathcal{B}$-structure of $M_{\mathcal{B}}X$ is defined by operation at the output box: if $\gamma \in \mathcal{B}(n,1)$, then



The map



includes X as a strong deformation retract with deformation



□

Our structures are homotopy invariant, an important and useful feature of "brave new algebra". Using (2.9) it is not terribly hard to show

3.5 Proposition: If X is a $W\mathcal{B}$-space and Y is h-equivalent to X, then Y is a $W\mathcal{B}$-space, too. □

4 The classifying space construction

Let Δ denote the category of ordered sets $[n] = \{0, 1, \ldots, n\}$ and order preserving maps. The morphisms of Δ are generated by injections

$$d_i = d_i^n : [n-1] \to [n], \quad 0 \le i \le n$$

missing $i \in [n]$ and surjections

$$s_i = s_i^n : [n+1] \to [n], \quad 0 \le i \le n$$

mapping i and $i+1$ to $i \in [n]$.

We have the standard simplex functor

$$\Delta \to \mathcal{T}op,$$

sending $[n]$ to the standard n-simplex Δ^n with vertices $e_0, \ldots, e_n$ and $\alpha : [n] \to [r]$ to the linear map $\alpha : \Delta^n \to \Delta^r$ mapping e_i to $e_{\alpha(i)}$.

A *simplicial space* is a functor

$$X_\bullet : \Delta^{op} \to \mathcal{T}op, \quad [n] \mapsto X_n, \alpha \mapsto \alpha^*.$$

Its *topological realization* is the space

$$|X| = \left(\coprod_{n \ge 0} X_n \times \Delta^n \right) / \sim$$

with the relations

$$(x, d_i(t)) \sim (d_i^*(x), t) \text{ and } (x, s_i(t)) \sim (s_i^*(x), t).$$

Any monoid M with unit e defines a simplicial space

$$M_\bullet : \Delta^{op} \to \mathcal{T}op$$

with $M_n = M_\bullet([n]) = M^n$ and structure maps

$$\begin{array}{llll} s_i^* : M^n \to M^{n+1} & (x_1, \ldots, x_n) \mapsto & (x_1, \ldots, e, x_i \ldots, x_n) & \\ d_i^* : M^n \to M^{n-1} & (x_1, \ldots, x_n) \mapsto & (x_2, \ldots, x_n), & i = 0 \\ & & (x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_n), & 0 < i < n \\ & & (x_1, \ldots, x_{n-1}), & i = n \end{array}$$

4.1 Definition: The topological realization $|M_\bullet|$ is called the *classifying space* of the monoid M and denoted by BM.

Note that BM has a natural filtration B^nM defined by the dimensions of the standard simplices. By construction, B^0M is a point and $B^1M = \Sigma M$, the reduced suspension of (M, e). The inclusion $i : \Sigma M \hookrightarrow BM$ induces a map

$$\hat{\imath} : M \to \Omega BM, \quad x \mapsto (w_x : t \mapsto i(t, x))$$

4.2 Proposition: [48, 1.5] Let M be a monoid with homotopy inverses such that $\{e\} \subset M$ is a closed cofibration. Then $\hat{\imath}$ is an h-equivalence.

4.3 Theorem: X is h-equivalent to a monoid iff X is an A_∞-space.

Proof: A monoid has an $\mathcal{A}$-structure and therefore a $W\mathcal{A}$-structure. If X is h-equivalent to a monoid, it has a $W\mathcal{A}$-structure by (3.5). The converse follows from (3.4). □

4.4 Theorem: X is h-equivalent to a loop space iff X admits a $W\mathcal{A}$-structure with homotopy inverses.

This follows from (4.2) and (4.3). The cofibration condition can always be arranged. □

Chapter II: E_∞-spaces and infinite loop spaces

5 Operads and E_∞-spaces

If a loop space is a good space, an n-fold loop space

$$\Omega^n X = \Omega(\Omega^{n-1}X) = \mathcal{T}op((I^n, \partial I^n), (X, *)) = \mathcal{T}op^*(S^n, X)$$

is an even better space for $n > 1$.

5.1 Definition: An *infinite loop space* is a based space X together with a sequence $\{X_n, \sigma_n; n \in \mathbb{N}\}$ of spaces and h-equivalences $\sigma_n : X_n \simeq \Omega X_{n+1}$ such that $X \simeq X_0$. If $X = X_0$ and each σ_n is a homeomorphism, X is called a *strict infinite loop space.*

5.2 Properties

(1) $\Omega^n X$ is an abelian group up to homotopy for $n \geq 2$.

(2) If X is an infinite loop space, then $Y \mapsto [Y, X_n]^*$ defines a cohomology theory.

(3) If h^* is a cohomology theory, there is an infinite loop space X whose associated cohomology is h^* for $* \geq 0$.

(4) Infinite loop spaces admit homology operations resembling Steenrod operations in cohomology.

Again we are mainly interested in the algebraic structure of an infinite loop space. As with loop spaces, we take property (5.2.1) as a guideline. To go from monoids to commutative monoids we have to impose an additional relation on the μ_n of (2.4), namely

5.3

$$\begin{tikzcd} X^n \arrow[rr, "\tau"] \arrow[dr, "\mu_n"'] & \arrow[d, phantom, "\curvearrowright"] & X^n \arrow[dl, "\mu_n"] \\ & X & \end{tikzcd}$$

commutes for any permutation $\tau \in \Sigma_n$ of the n factors X^n. So we have to add permutations to our non-Σ operads.

5.4 Definition: An *operad* (as opposed to a non-Σ operad) is a non-Σ operad $\mathcal{B}$ with the permutation group Σ_n included in $\mathcal{B}(n,n)$ such that for $\sigma \in \Sigma_k$ and $\rho \in \Sigma_l$

(1) $\sigma \oplus \rho$ is the usual direct sum permutation in Σ_{k+l}

(2) for $f_i \in \mathcal{B}(r_i, 1)$, $i = 1, \ldots, k$, and $n = r_1 + \ldots + r_k$

$$\sigma \circ (f_1 \oplus \ldots \oplus f_k) = (f_{\sigma^{-1}(1)} \oplus \ldots \oplus f_{\sigma^{-1}(k)}) \circ \sigma(r_1, \ldots, r_k)$$

where $\sigma(r_1, \ldots, r_k)$ is the block permutation in Σ_n permuting the blocks $r_1, \ldots, r_k$ according to σ.

(3) Condition $(*)$ of (2.1) is replaced by

$$\begin{array}{rcl} \coprod\limits_{r_1+\ldots+r_k=n} \mathcal{B}(r_1,1) \times \ldots \times \mathcal{B}(r_k,1) \times \Sigma_n/\sim & \longrightarrow & \mathcal{B}(n,k) \\ (f_1, \ldots, f_k; \sigma) & \longmapsto & (f_1 \oplus \ldots \oplus f_k) \circ \sigma \end{array}$$

is a homeomorphism, where the relation is given by

$$(f_1 \circ \pi_1, \ldots, f_k \circ \pi_k; \sigma) \sim (f_1, \ldots, f_k; (\pi_1 \oplus \ldots \oplus \pi_k) \circ \sigma)$$

with $\pi_i \in \Sigma_{r_i}$.

A *$\mathcal{B}$-space* is a continuous functor $\mathcal{B} \to \mathcal{T}op$ mapping $\oplus$ to $\times$ and preserving permutations.

Remark 2.2 applies to operads, too. This time $\mathcal{B}$ is uniquely determined by the spaces $\mathcal{B}(n,1)$, composition, and the action of Σ_n on $\mathcal{B}(n,1)$ by composition.

5.5 Example: Let $\mathcal{E}$ be the operad obtained from $\mathcal{A}$ by imposing the relation (5.3). Then $\mathcal{E}(n,1) = \{\mu_n\}$ necessarily has trivial Σ_n-action and composition is forced. As in (2.4), one shows that $\mathcal{E}$-spaces are exactly the commutative monoids. $\mathcal{E}$ is terminal in the category of operads.

5.6 Definition: An operad $\mathcal{B}$ is called E_∞-*operad* and a $\mathcal{B}$-space an E_∞-*space* if the unique functor $\mathcal{B} \to \mathcal{E}$ is an h-equivalence of underlying morphism spaces, i.e. if $\mathcal{B}(n,1) \simeq *$ for all n.

We again try to construct a homotopy universal E_∞-operad. The ideas are basically the same as in Chapter I, but one has to work Σ_n-equivariantly throughout. Note that any universal operad $\mathcal{B}$ has to have a free Σ_n-action on $\mathcal{B}(n,1)$. We do not go into details and just list the facts we need.

5.7 Proposition: (1) The W-construction extends to operads and $\varepsilon : W\mathcal{B} \to \mathcal{B}$ induces a Σ_n-equivariant h-equivalence $W\mathcal{B}(n,1) \to \mathcal{B}(n,1)$ for each $n \in \mathbb{N}$.

(2) Given a diagram of operads

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "G"] \\ W\mathcal{B} \arrow[ur, dashed] \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

such that $G : \mathcal{D}(n,1) \to \mathcal{C}(n,1)$ is a Σ_n-equivariant h-equivalence for all n, then there exists $\widetilde{F} : W\mathcal{B} \to \mathcal{D}$ such that $G \circ \widetilde{F} \simeq F$ through functors of operads, uniquely up to homotopy through functors of operads.

(3) Property (2) holds if G is an ordinary h-equivalence and $\mathcal{B}(n,1) \to \mathcal{B}(n,1)/\Sigma_n$ is a numerable principal Σ_n-bundle (for a definition see [9]) for all $n \in \mathbb{N}$.

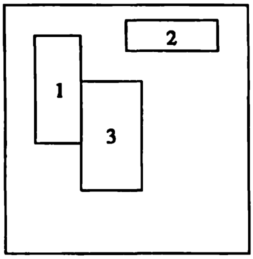
5.8 Proposition: The M-construction (3.4) extends to operads $\mathcal{B}$: each $W\mathcal{B}$-space X is a strong deformation retract of a $\mathcal{B}$-space $M_\mathcal{B}X$.

5.9 Proposition: If X is a $W\mathcal{B}$-space and $Y \simeq X$, then Y admits a $W\mathcal{B}$-structure.

6 The little n-cubes operad $\mathcal{Q}_n$

In this section we construct canonical operads $\mathcal{Q}_n$ acting on $\Omega^n X$ for $1 \leq n \leq \infty$ and show that infinite loop spaces are E_∞-spaces.

6.1 The operad $\mathcal{Q}_n$: The morphism space $\mathcal{Q}_n(k,1)$ is the space of all k-tuples of linear embeddings of I^n into I^n with disjoint interiors and axes parallel to the ones of I^n.



element ω in $Q_2(3,1)$

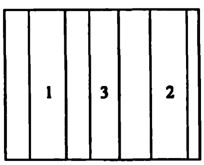
A single embedding is determined by the images of $(0,\dots,0)$ and $(1,\dots,1)$. Hence we can topologize $\mathcal{Q}_n(k,1)$ as subspace of $\mathbb{R}^{2nk}$. The action of Σ_k is given by permuting the labels of the little cubes. Composition $\omega\circ(\omega_1\oplus\dots\oplus w_k)$ is defined by inserting the configuration ω_i linearly into the i-th little cube of ω, producing a new configuration.

$\mathcal{Q}_n$ acts on n-fold loop spaces $\Omega^n X = \mathcal{T}op\,((I^n,\partial I^n),(X,*))$ by

$$\begin{array}{rcl}\mathcal{Q}_n(k,1)\times(\Omega^n X)^k & \longrightarrow & \Omega^n X\\ (\omega,(f_1,\dots,f_k)) & \longmapsto & (g: I^n \to X)\end{array}$$

g maps the i-th little cube J_i of ω via $f_i\circ r_i$, where $r_i : J_i \to I^n$ is inverse to the embedding. The complement of $\bigcup_{i=1}^k J_i$ is mapped to the base point.

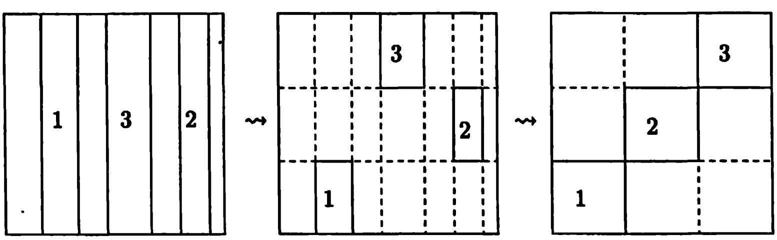
6.2 The operad $\mathcal{Q}_\infty$: Taking the product with I defines an inclusion $\mathcal{Q}_n(k,1)\subset \mathcal{Q}_{n+1}(k,1)$



element of $Q_1(3,1)$ considered as

element of $Q_2(3,1)$

and we define $\mathcal{Q}_\infty(k,1) = \operatorname{colim}_n \mathcal{Q}_n(k,1)$. The following sketch shows that $\mathcal{Q}_n(k,1)$ is contractible in $\mathcal{Q}_{n+1}(k,1)$.



Since the inclusion $\mathcal{Q}_n(k,1)\subset\mathcal{Q}_{n+1}(k,1)$ is a cofibration, we obtain

6.3 Proposition: $\mathcal{Q}_\infty$ is a E_∞-operad.

Σ_k acts freely on $\mathcal{Q}_n(k,1)$ and it is not hard to show that $\mathcal{Q}_\infty(k,1)\to\mathcal{Q}_\infty(k,1)/\Sigma_k$ is a numerable principal Σ_k-bundle. Hence (5.7) implies

6.4 Proposition: $W\mathcal{Q}_\infty$ is a homotopy universal E_∞-operad, i.e. given an E_∞-operad $\mathcal{B}$ there is a functor of operads $W\mathcal{Q}_\infty \to \mathcal{B}$ uniquely up to homotopy through functors of operads.

The operations of the $\mathcal{Q}_n$ on $\Omega^n X$ are natural enough to induce an action of $\mathcal{Q}_\infty$ on any **strict** infinite loop space. By a result of May [30], any infinite loop space is h-equivalent to a strict infinite loop space. Using (5.9) we obtain

6.5 Theorem: Any infinite loop space is a $W\mathcal{Q}_\infty$-space and hence an E_∞-space.

7 Interchange and delooping

We now address the converse of (6.5).

7.1 Definition: Let $\mathcal{B}$ and $\mathcal{C}$ be operads and suppose X has a $\mathcal{B}$- and a $\mathcal{C}$-structure. We say that these structures *interchange* if the diagram

$$\begin{tikzcd} (X^k)^m \arrow[r, "\cong"] \arrow[d, "\beta^m"] & (X^m)^k \arrow[r, "\alpha^k"] & X^k \arrow[d, "\beta"] \\ X^m \arrow[rr, "\alpha"] & & X \end{tikzcd} \qquad (*)$$

commutes for each $\alpha \in \mathcal{B}(m,1)$ and each $\beta \in \mathcal{C}(k,1)$. The morphisms of $\mathcal{B}$ and $\mathcal{C}$ with the relations from the interchange diagrams $(*)$ define a new operad, denoted by $\mathcal{B} \otimes \mathcal{C}$, which acts on X.

7.2 Examples: (1) $\mathcal{A} \otimes \mathcal{A} = \mathcal{E}$. This is the operad version of the well-known fact that two commuting associative structures, which have the same unit, coincide and are also commutative.

(2) $\mathcal{Q} \otimes \mathcal{Q}_n \hookrightarrow \mathcal{Q}_{n+1}$. Here $\mathcal{Q}$ acts on the first and $\mathcal{Q}_n$ on the last n coordinates of I^{n+1}.

We need two properties of the interchange.

7.3 Proposition: If X has an $\mathcal{A}\otimes W\mathcal{B}$-structure (in particular, X is a monoid), then its classifying space BX admits a $W\mathcal{B}$-structure.

7.4 Proposition: If X has a $W\mathcal{A} \otimes W\mathcal{B}$-structure, then the monoid $M_\mathcal{A}X$ of (3.4) has an $\mathcal{A} \otimes W\mathcal{B}$-structure.

We can now prove the converse of (6.5).

7.5 Theorem: An E_∞-space X admitting homotopy inverses is an infinite loop space.

Proof: By (6.4) X is a $W\mathcal{Q}_\infty$-space. Then $Y = M_{\mathcal{Q}_\infty}X$ is a $\mathcal{Q}_\infty$-space equivalent to X and hence a $\mathcal{Q} \otimes \mathcal{Q}_\infty$-space by (7.2). Since $\mathcal{Q}$ is an A_∞-operad, Y is a $W\mathcal{A} \otimes \mathcal{Q}_\infty$-space. By (7.4) $M_{\mathcal{A}}Y$ is an $\mathcal{A} \otimes W\mathcal{Q}_\infty$-space. Hence the classifying space $BM_{\mathcal{A}}Y$ admits a $W\mathcal{Q}_\infty$-structure by (7.3). By assumption, $M_{\mathcal{A}}Y$ has homotopy inverses, and the inclusion of the unit is a cofibration by construction. Hence

$$X \simeq \Omega BM_{\mathcal{A}}Y.$$

One can show that $BM_{\mathcal{A}}Y$ has homotopy inverses and we can proceed inductively. □

8 Applications

Let $\mathcal{J}$ denote the category of real inner product spaces of countable dimensions and linear isometric maps and $\mathcal{J}_{fin}$ the full subcategory of finite dimensional ones. Give each $A \in \mathcal{J}_{fin}$ the norm topology and each infinite dimensional one the colimit topology of its finite dimensional subspaces. The following result is easy to show.

8.1 Lemma: $\mathcal{J}(A, \mathbb{R}^\infty)$ with the function space topology is contractible.

8.2 The *linear isometry operad* $\mathcal{L}$ is defined by $\mathcal{L}(n,1) = \mathcal{J}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty)$ with the obvious Σ_n-action. Composition is the composition of isometic maps. $\mathcal{L}$ is an E_∞-operad by (8.1).

Given a continuous functor $T : \mathcal{J}_{fin} \to \mathcal{T}op$ together with a natural transformation $\omega : TA \times TB \to T(A \oplus B)$ such that
(1) $T(0) = *$
(2) ω is associative, commutative, and unital.
Then $\operatorname{colim}_n T(\mathbb{R}^n)$ is an $\mathcal{L}$-space by direct inspection and hence an E_∞-space.

8.3 Examples: (1) $T(A) = O(A)$, the orthogonal group of A. Then $\operatorname{colim}_n O(\mathbb{R}^n)$ is the stable orthogonal group O.

(2) $T(A) = U(A \otimes \mathbb{C})$, the unitary group of $A \otimes \mathbb{C}$. Then $\operatorname{colim}_n U(\mathbb{R}^n \otimes \mathbb{C}) = \operatorname{colim}_n U(\mathbb{C}^n)$ is the stable unitary group U.

(3) $T(A) = BO(A)$, the classifying space of $O(A)$, and similarly $BU(A \otimes \mathbb{C})$. We obtain the stable spaces BO and BU.

(4) $T(A) = TOP(A)$, the group of homeomorphisms of A. In this case we obtain the stable group TOP.

(5) $T(A) = Sp(A \otimes \mathbb{H})$, the symplective group of $A \otimes \mathbb{H}$. We obtain the stable group Sp.

(6) $T(A) = F(S^A)$, the monoid of self homotopy equivalences of the sphere S^A, which is the one-point compactification of A. Then $\operatorname{colim}_n F(S^{\mathbb{R}^n})$ is the stable group F, the "homotopy units" of the brave new ring $Q(S^0)$ defined in the next chapter.

We obtain

8.4 Theorem: The stable groups O, SO, Sp, U, SU, F, TOP, their coset spaces F/TOP, F/O, TOP/O, etc., and the stable classifying spaces BO, BSO, etc. are all infinite loop spaces.

Chapter III: Brave new rings

So far we have considered A_∞-spaces, loop spaces, E_∞-spaces, and infinite loop spaces and claimed that they are brave new versions of monoids, groups, abelian monoids, and abelian groups. This may be plausible for A_∞-spaces and loop spaces: X is an A_∞-space iff it is h-equivalent to a monoid. Hence an A_∞-structure is the homotopy invariant version of a monoid structure. Similarly, a topological group with reasonable topology is h-equivalent to a loop space and vice versa; and loop spaces are important objects of homotopy theory. Homotopy ring spaces, whose investigation is motivated by the geometry of manifolds, have E_∞-structures as additions. This is why E_∞-spaces ought to replace abelian monoids in brave new algebra.

9 Brave new rings and K-theory

During the sixties topologists became more and more interested in topological groups such as

$$\text{Diff}\,(M, \partial M) = \{\text{diffeomorphisms } f : M \to M \text{ with } f|\partial M = id\},$$

where M is a differentiable manifold with boundary ∂M, possibly empty.

One of the first questions to ask is

9.1 Problem: What are the homotopy groups of Diff $(M, \partial M)$?

This is a difficult problem, and one step in solving it is to consider a supposedly easier variant.

9.2 Problem: Compute $\pi_*(\text{Diff}\,(M \times I, \partial M \times I \cup M \times 0))$.

9.3 Definition: If M is a manifold, $P(M) = \text{Diff}\,(M \times I, \partial M \times I \cup M \times 0)$ is called the *pseudoisotopy space* of M.

The following result is due to Hatcher and Wagner [16] with corrections by Igusa [18].

9.4 Theorem: If M is a differentiable manifold and $dim M \geq 6$, then there is an exact sequence

$$K_3(\mathbb{Z}[G]) \to Wh_1^+(G;(\mathbb{Z}/2) \oplus \pi_2(M)) \to \pi_0(P(M)) \to Wh_2(G) \to 0$$

Here G is the fundamental group $\pi_1(M)$ of M, K_* is Quillen's algebraic K-functor, $Wh_1^+(G;A) \cong A \otimes \mathbb{Z}[G]/G$ with A a G-Modul (take orbits to form the quotient), and $Wh_2(G) = K_2(\mathbb{Z}[G])/\pi_2^s(BG_+)$ is a "higher" Whitehead group.

We need not understand this exact sequence. It serves as a motivation for the things to come. Relations of this kind between algebraic K-theory and the geometry of manifolds made Waldhausen realize that an extension of Quillen's algebraic K-theory [39] from ordinary rings to homotopy ring spaces such as $Q(\Omega X_+)$ defined below should contain much more geometrical information than the algebraic K-theory of the group ring $\mathbb{Z}[\pi_1 X]$.

9.5 The brave new ring $\mathbf{Q(\Omega X_+)}$: If X is a space and may be based, then $X_+ = X \cup \{\text{point}\}$ with the extra point as base point. If X and Y are based spaces with base points $*$, recall the *smash product*

$$X \wedge Y = X \times Y/(X \times \{*\} \cup \{*\} \times Y)$$

and the *reduced suspension* $\Sigma Y = S^1 \wedge Y$. We define

$$Q(Y) = \operatorname{colim}(Y \xrightarrow{\sigma} \Omega\Sigma Y \xrightarrow{\Omega\sigma\Sigma} \Omega^2\Sigma^2 Y \to \dots)$$

with $\sigma(y) : S^1 \to \Sigma Y,\ t \mapsto (t, y)$. Clearly, QY is a strict infinite loop space. If Y is a T_1-space, then $\pi_i(QY) \cong \pi_i(\Omega^n\Sigma^n Y)$ if $n \geq i+3$ by Freudenthal's suspension theorem. Since the group on the left does not depend on n, it is called the i-th *stable homotopy group* of Y. So $Q(Y)$ is the *stable homotopy (space)* of Y.

This infinite loop space structure is the additive structure of $Q(\Omega X_+)$. If Y is an A_∞- or E_∞-space, e.g. a loop space, then $Q(Y_+)$ has an additional multiplicative A_∞- or E_∞-structure, respectively. If $\mu : Y \times Y \to Y$ is an operation and $f, g : S^n \to \Sigma^n(Y_+) = S^n \wedge Y_+$ represent elements in $Q(Y_+)$, then

$$\mu(f,g) : S^{2n} = S^n \wedge S^n \xrightarrow{f \wedge g} S^n \wedge Y_+ \wedge S^n \wedge Y_+ \cong \Sigma^{2n}(Y \times Y)_+ \xrightarrow{\mu} \Sigma^{2n} Y_+$$

defines a multiplication on $Q(Y_+)$ which is related to the additive structure by distributive laws up to homotopy.

Waldhausen's proposal: Replace the ground ring $\mathbb{Z}$ by the E_∞-ring $Q(S^0)$, the stable homotopy of the spheres (note that $S^0 = (\text{Point})_+$), and the group ring $\mathbb{Z}[\pi_1 X]$ by the A_∞-ring $Q(\Omega X_+)$, and extend classical algebraic constructions to these types of "rings". Although E_∞-ring spaces had been introduced earlier by May [32] this was the starting point of "brave new algebra". Note also that $\pi_0 Q(S^0) \cong \mathbb{Z}$, and $\pi_0 Q(\Omega X_+) = \mathbb{Z}[\pi_1 X]$. Hence the classical versions are the rings of path components of these particular brave new rings, and

$$\pi_0 : Q(\Omega X_+) \to \mathbb{Z}[\pi_1 X]$$

is a homomorphism of brave new rings.

To cut things short, let me give a summary of the history of A_∞- and E_∞-ring structures and of some of the early applications.

1977 May defined E_∞-ring spaces using pairs of operads which codify the additive and multiplicative structure and are related via distributive laws. He introduced them to study the various classifying spaces of geometric topology, and, in particular, to construct homology operations. [32]

1978 Waldhausen defined his famous functor $A(X)$, the algebraic K-theory of a topological space X [54]. This is an extension of Quillen's algebraic K-functor to A_∞-ring spaces of the type $Q(\Omega X_+)$.

1978 Motivated by Waldhausen's construction, May introduced more general A_∞-ring spaces using his pairs of operads approach and constructed their K-theory [33]. These structures are very rigid and not well adapted for algebraic constructions. May was very sceptical about good properties of the associated K-theory (see his remarks in [33] and [34].)

1981 Steiner could refute May's scepticism. He proved that the K-theory of an A_∞-ring space in May's sense is an infinite loop space and that for $Q(\Omega X_+)$ its K-theory is equivalent as infinite loop space to Waldhausen's $A(X)$.

1982 (joint work with R. Schwänzl) Using an observation of Steiner we introduced a more flexible homotopy invariant definition of A_∞- and E_∞-ring spaces. May's spaces satisfy our requirements and each homotopy ring space in our sense is h-equivalent to one in May's sense [43].

1984 We showed that matrices and subgadgets like upper triangular matrices over homotopy ring spaces have the properties one expects and show that the algebraic K-theory of an E_∞-ring space is itself an E_∞-ring space. This also clarifies the formal multiplicative properties of classical algebraic K-theory of commutative rings [44], [45], [46].

First applications

In 1979 Waldhausen among other things could prove (we use the notation $K(Q(\Omega X_+))$ instead of $A(X)$):

9.6 Theorem: [55] (1) $K(Q(\Omega X_+)) \simeq Q(X_+) \times Wh^{\mathrm{Diff}}(X)$
(2) The π_0-map induces a rational equivalence $K(Q(S^0)) \to K(\mathbb{Z})$

The first result confirms Waldhausen's conjecture that the algebraic K-theory of $Q(\Omega X_+)$ contains a lot of geometric information about X. For a manifold M

$$\pi_{i+2}(Wh^{\mathrm{Diff}}(M)) \cong \pi_i(\mathcal{P}(M)) \quad i \geq 0$$

where $\mathcal{P}(M) = \operatorname{colim}_n P(M \times I^n)$ is the *stable pseudoisotopy space* of M. Modulo rounding of corners, the colimit is formed by crossing with I. In the "pseudoisotopy stability range" its homotopy groups coincide with the ones of $P(M)$.

9.7 Theorem: (Igusa [19]) For a differentiable manifold M with $\dim M \geq \max(2k+7, 3k+4)$ the homomorphism $\pi_i(P(M)) \to \pi_i(\mathcal{P}(M))$ is surjective for $i \leq k$ and injective for $i < k$.

Waldhausen's result also to some extent explains the intriguing connection between the stable homotopy groups of the spheres and $K(\mathbb{Z})$, and hence also with classical number theory.

The second part of (9.6) is a special case of a more general result of Waldhausen, which Farrell and Hsiang [12] exploited to show (note the pseudo isotopy stability range)

9.8 Theorem: For $n > \max(2i+7, 3i+5)$

$$\pi_i(\operatorname{Diff}(\mathbb{B}^n, \partial\mathbb{B}^n)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i = 4k-1, \quad n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_i(\operatorname{Diff}(S^n)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & i = 4k-1, \quad n \text{ odd} \\ \mathbb{Q} & i = 4k-1, \quad n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

10 Spectra

No matter which definition of homotopy ring space one uses the complexity of its structure is quite formidable. What one ideally wants is a black box which automatically takes care of the coherence structure. Such a black box had been around for some time, but was not sophisticated enough: each strict infinite loop space is the 0-th space of a (connective) spectrum and vice versa.

10.1 Definition: A *naive prespectrum* $E = \{E_n, e_n\}_{n \geq 0}$ consists of based spaces E_n and structure maps $e_n : \Sigma E_n \to E_{n+1}$. If the adjoints

$$\hat{e}_n : E_n \to \Omega E_{n+1}, \quad \hat{e}_n(x) : S^1 \to E_{n+1}, \; t \mapsto e_n(t, x)$$

are homeomorphisms, we call E a *naive spectrum*.

Since each infinite loop space is h-equivalent to a strict one, the category $\mathcal{Sp}$ of spectra could replace the category of infinite loop spaces and hence the category $\mathcal{Ab}$ of abelian groups in brave new algebra. This takes care of the additive structure. To deal with multiplicative structures we need a "brave new tensor product". An indication of what to look for is provided by the Künneth theorem in homology: there is an exact sequence in reduced homology with $\mathbb{Z}$-coefficients

$$0 \to (H_*(X) \otimes H_*(Y))_n \to H_n(X \wedge Y) \to \mathcal{T}or_{n-1}(H_*(X), H_*(Y)) \to 0.$$

Hence the smash product is a kind of totally derived tensor product. Our plan is to extend the smash product to spectra. But here problems arise.

Given spectra $\{E_n, e_n\}$ and $\{F_n, f_n\}$ we can form a bigraded object $\{E_k \wedge F_l\}_{k,l}$. To obtain structure maps we consider the diagram

$$\begin{tikzcd}
\Sigma^2(E_k \wedge F_\ell) \arrow[r, equal] \arrow[d, equal] & \Sigma_1\Sigma_2(E_k \wedge F_\ell) \arrow[r, "\cong"] \arrow[d, "\cong"', "\text{twist}"] & \Sigma_1(E_k \wedge \Sigma_2 F_\ell) \arrow[r, "f_\ell"] & \Sigma_1 E_k \wedge F_{\ell+1} \arrow[ddd, "e_k"] \\
\Sigma^2(E_k \wedge F_\ell) \arrow[r, equal] & \Sigma_2\Sigma_1(E_k \wedge F_\ell) \arrow[d, equal] & & \\
 & \Sigma_2(\Sigma_1 E_k \wedge F_\ell) \arrow[d, "e_k"] & & \\
 & \Sigma_2(E_{k+1} \wedge F_\ell) \arrow[r, "\cong"] & E_{k+1} \wedge \Sigma_2 F_\ell \arrow[r, "f_\ell"] & E_{k+1} \wedge F_{\ell+1}
\end{tikzcd}$$

The big square commutes, but the twist is homotopic to $-id$. Hence structure maps for $\{E_k \wedge F_l\}$ involve substantial choices so that the resulting smash product cannot have good properties.

A first solution was discovered by Boardman in 1964 [2]. In Example 8.3.6 the natural transformation $\omega : F(S^A) \times F(S^B) \to F(S^{A\oplus B})$ is the smash product of homotopy equivalences $S^A \to S^A$ with homotopy equivalences $S^B \to S^B$ using the fact that $S^{A\oplus B} \cong S^A \wedge S^B$. Boardman's construction is based on this observation and uses real inner product spaces and linear isometries. We describe a variant of his construction due to Lewis, May, Steinberger [27], which is more widely used.

10.2 Definition: A *universe* $\mathcal{U}$ is a real inner product space $\mathcal{U} \cong \mathbb{R}^\infty$. A *prespectrum* E assigns a based space EV to each finite dimensional subspace V of $\mathcal{U}$ and a map

$$\overline{\sigma}_{V,W} : EV \to \Omega^{W-V} EW$$

for each $V \subset W$. Here $W - V$ is the orthogonal complement of V in W and

$$\Omega^V X = \mathcal{T}op^*(S^V, X)$$

If the $\overline{\sigma}_{V,W}$ are homeomorphisms, E is a *spectrum*.
A *map of (pre-) spectra* $f : E \to F$ is a collection of maps $f_V : EV \to FV$ such that

$$\begin{array}{ccc} EV & \xrightarrow{f_V} & FV \\ {\scriptstyle \overline{\sigma}_{V,W}}\downarrow & & \downarrow{\scriptstyle \overline{\sigma}_{V,W}} \\ \Omega^{W-V}EW & \xrightarrow{\Omega^{W-V}f_W} & \Omega^{W-V}FW \end{array}$$

commutes. Let $\mathcal{SU}$ and $\mathcal{PSU}$ denote the categories of spectra and prespectra over the universe $\mathcal{U}$, respectively.

10.3 The inclusion $\mathcal{SU} \hookrightarrow \mathcal{PSU}$ has a left adjoint $L : \mathcal{PSU} \to \mathcal{SU}$ called *spectrification.*

10.4 *The external smash product* $-\wedge- : \mathcal{SU}\times\mathcal{SV} \longrightarrow \mathcal{S}(\mathcal{U}\oplus\mathcal{V})$ is the spectrification of the functor

$$-\wedge- : \mathcal{SU}\times\mathcal{SV} \longrightarrow \mathcal{PS}(\mathcal{U}\oplus\mathcal{V})$$

with $(E\wedge F)(U\oplus V) = EU\wedge FV$.

To get an internal smash product Boardman showed that any linear isometry $f : \mathcal{U} \to \mathcal{V}$ induces a functor $f_* : \mathcal{SU} \to \mathcal{SV}$. So an isometric isomorphism $f : \mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ induces an internal smash product

$$\mathcal{S}(\mathbb{R}^\infty)\times\mathcal{S}(\mathbb{R}^\infty) \longrightarrow \mathcal{S}(\mathbb{R}^\infty).$$

Using the fact that the space $\mathcal{J}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty)$ of linear isometries is contractible Boardman proved

10.5 Proposition: Any two internal smash products defined in this way are h-equivalent and each is coherently homotopy-associative, -commutative, and -unital.

While this was good enough for the problems considered at that time, such as the investigation of generalized cohomology theories, it is not good enough for our purposes. We put the additive E_∞-structure into a black box at the expense of replacing an associative, commutative and unital smash product by an E_∞-smash product. We have gained nothing!

The breakthrough came with an observation by Hopkins in 1992 [17], which was subsequently exploited by Elmendorf, Kriz, Mandell, and May [11]. For its description we need an extension of Boardman's functor f_* from [27]. Let $\mathcal{U}$ and $\mathcal{U}'$ be universes and $\mathcal{T}op/\mathcal{J}(\mathcal{U},\mathcal{U}')$ the category of spaces over the space $\mathcal{J}(\mathcal{U},\mathcal{U}')$ of linear isometries.

10.6 Proposition: There is a functor

$$\mathcal{T}op/\mathcal{J}(\mathcal{U},\mathcal{U}')\times\mathcal{SU} \to \mathcal{SU}', \quad (A,E)\mapsto A\ltimes E$$

such that
(1) $\{id_{\mathcal{U}}\} \ltimes E \cong E$ $\quad$ ($id_{\mathcal{U}}$ is the space $* \to \mathcal{J}(\mathcal{U},\mathcal{U}), * \mapsto id_{\mathcal{U}}$)

(2) Given $B \times A \xrightarrow{g\times f} \mathcal{J}(\mathcal{U}',\mathcal{U}'') \times \mathcal{J}(\mathcal{U},\mathcal{U}') \xrightarrow{\text{comp.}} \mathcal{J}(\mathcal{U},\mathcal{U}'')$, then

$$(B \times A) \ltimes E \cong B \ltimes (A \ltimes E).$$

Once $\mathcal{U}$ is fixed we use the notation $\mathcal{L}(n,1) = \mathcal{J}(\mathcal{U}^{\oplus n},\mathcal{U})$ of §8.

The proposition makes sense for the non-universe 0. Since a 0-indexed spectrum is just a single space and $\mathcal{U}^0 = 0$ we obtain a functor

$$\Sigma^\infty : \mathcal{T}op \to \mathcal{SU}, \quad X \mapsto \mathcal{L}(0,1) \ltimes X.$$

The *sphere spectrum* S is defined to be $\Sigma^\infty(S^0)$.

10.7 Definition: An $\mathbb{L}$-*spectrum* is a spectrum $E \in \mathcal{SU}$ equipped with an action $\mathcal{L}(1,1) \ltimes E \to E$ of the monoid $\mathcal{L}(1,1)$.

For $\mathbb{L}$-spectra we can define an even better smash product. Note that $\mathcal{L}(1,1)$ acts on the left of $\mathcal{L}(2,1)$ and $\mathcal{L}(1,1) \times \mathcal{L}(1,1)$ on its right by composition. Mimicking the usual tensor product construction we define

10.8 Definition: On the category of $\mathbb{L}$-spectra we have a smash product

$$E \wedge_{\mathcal{L}} F = \mathcal{L}(2,1) \ltimes_{\mathcal{L}(1,1)\times\mathcal{L}(1,1)} (E \wedge F)$$

where $E \wedge F$ is the external smash product.

The proof of the following result now reduces to an analysis of linear isometries.

10.9 Proposition: This smash product is associative and commutative (up to coherent isomorphisms) but it is NOT unital, but there is a natural weak equivalence $\lambda_E : \mathrm{S} \wedge_{\mathcal{L}} E \to E$.

10.10 Definition: An S-*module* is an $\mathbb{L}$-spectrum E for which λ_E is an isomorphism.

10.11 Proposition: The smash product $\wedge_{\mathcal{L}}$ on the category of $\mathbb{L}$-spectra restricts to an associative, commutative, and unital smash product $\wedge_{\mathrm{S}}$ on the subcategory $\mathcal{M}od_{\mathrm{S}}$ of S-module spectra.

10.12 Definition: A (*commutative*) S-*algebra* is a (commutative) monoid in the symmetric monoidal category $(\mathcal{M}od_{\mathrm{S}}, \wedge_{\mathrm{S}})$.

As in classical algebra we can extend the smash product to a smash product $\wedge_R$ over a commutative S-algebra R. We then can talk about R-modules and R-algebras, etc.

10.13 Proposition: $\wedge_R$ enjoys the formal properties of the classical tensor product such as associativity, commutativity, and cancellation $R \wedge_R M \cong M$.

$Mod_{\mathbb{S}}$ and the categories of module and algebra spectra obtained from it are the black boxes we were looking for because we have

10.14 Proposition: Each spectrum is weakly equivalent to an S-module. The 0-th space of a (commutative) S-algebra is an (E_∞-) A_∞-ring space. Conversely, the spectrum associated to an (E_∞-) A_∞-ring space, which is obtained from its underlying additive infinite loop space, is weakly equivalent to a (commutative) S-algebra.

Chapter IV: Algebraic applications

Practically all known algebraic applications of "brave new algebra" originate from functors defined to gain information about K-theory. At a conference in Evanston in 1976, K. Dennis constructed trace maps from the K-theory $K(R)$ of a ring to its Hochschild homology $HH(R)$. This triggered Waldhausen to construct a more general trace map for simplicial rings [55]. He then conjectured that a type of Hochschild homology over the homotopy ring space $Q(S^0)$ would provide more information for the K-theory even of classical rings. This "topological" Hochschild homology $THH(R)$ was constructed by Bökstedt for FSPs, which stands for Functors with Smash Products, in 1985 [6]. Simplicial rings give rise to such FSPs and Bökstedt showed that Waldhausen's trace factors through THH. $THH(R)$ is equipped with an S^1-action and Goodwillie suggested a program for a "topological" cyclic homology $TC(R)$ through which Bökstedt's trace should factor. The details were worked out by Bökstedt, Hsiang, Madsen [7], and independently by Goodwillie [14]. So we have the following h-commutative diagram of trace maps

$$\begin{tikzcd} K(R) \arrow[r, "tr"] \arrow[d, "trc"'] \arrow[dr, "trt"] & HH(R) \\ TC(R) \arrow[r] & THH(R) \arrow[u] \end{tikzcd}$$

where trc is the cyclotomic trace of [7] and trt the topological trace of [6].

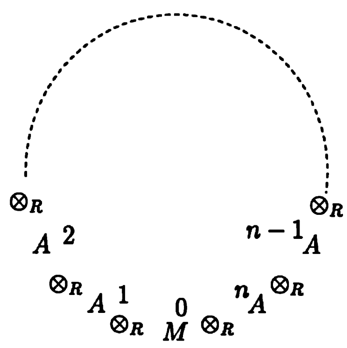
11 Topological Hochschild homology

We start with the topologist's definition of Hochschild homology. Let R be a commutative ring, A an R-algebra, and M an A-bimodule. We define a simplicial R-module by

$$[n] \longmapsto A \otimes_R A \otimes_R \ldots A \otimes_R M \quad (n \text{ copies } A)$$

For the structure maps arrange the tensor product in a circle

11.1



The i-th boundary map d_i^* is given by the multiplication of the i-th and $(i+1)$-st factor, the i-th degeneracy inserts 1 between the i-th and $(i+1)$-st factor.

11.2 Definition: The *Hochschild homology space* $HH^R(A;M)$ is the topological realization of this simplicial R-module. The Hochschild homology groups are

$$HH_i^R(A;M) = \pi_i HH^R(A;M).$$

We note that $HH^R(A;M)$ is a topological R-module. For simplicity we denote $HH^R(A;A)$ by $HH^R(A)$ and observe that we have a cyclic structure on $HH^R(A)$ induced by the obvious rotations of the Diagrams 11.1.

11.3 Definition: Let R be a commutative S-algebra spectrum, A an R-algebra spectrum, and M an A-bimodule spectrum. *Topological Hochschild homology* $THH^R(A;M)$ is defined to be the realization in the category $\mathcal{M}od_R$ of R-module spectra of the simplicial R-module spectrum

$$[n] \longmapsto A \wedge_A A \wedge_A \ldots \wedge_A A \wedge_A M \quad (n \text{ copies } A)$$

with structure maps as in 11.1. Again we write $THH^R(A)$ for $THH^R(A;A)$ and $THH_i^R(A;M)$ for $\pi_i(THH^R(A;M))$..

11.4 Remark: In 1985 when Bökstedt defined $THH^{\mathbf{S}}$ there was no known associative, commutative and unital smash product. To get around this problem Bökstedt used FSPs instead of spectra and went through a stabilization process. Since a (commutative) FSP defines a (commutative) S-algebra, Definition 11.3 covers the case of FSPs.

Even for classical rings R represented by their Eilenberg-MacLane ring spectra topological Hochschild homology is a new invariant. In [5] Bökstedt showed

11.5

$$THH_k^{\mathbf{S}}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/i & k=2i-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \qquad \begin{array}{l} THH_*^{\mathbf{S}}(\mathbb{F}_p) \cong \mathbb{F}_p[x] \\ \text{with degree } (x)=2 \end{array}$$

A first surprise was that topological Hochschild homology could be exploited to compute homology groups of $Gl_\infty(R)$ with coefficients in the adjoint representation $M_\infty(R)$ for classical rings. Here $Gl_\infty(R) = \mathrm{colim}_n\, Gl_n(R)$ and $M_\infty(R) = \mathrm{colim}_n\, M_n(R)$ where $M_n(R)$ is the set of all $(n \times n)$-matrices over R, and $Gl_\infty(R)$ acts on $M_\infty(R)$ by conjugation. Work of Waldhausen [54] and Dundas-McCarthy [10] implies

11.6 Proposition: $H_n(Gl_\infty(R), M_\infty(R)) \cong \bigoplus_{i+j=n} H_i(K(R); THH_j^{\mathbf{S}}(R))$.

From Quillen's calculations of $K(\mathbb{F}_p)$ and Bökstedt's calculations 11.5 we obtain

11.7 Proposition: $H_n(Gl_\infty(\mathbb{F}_p), M_\infty(\mathbb{F}_p)) \cong \begin{cases} \mathbb{Z}/p & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

The next surprise was that $THH_*^{\mathbf{S}}(R; M)$ for classical rings and bimodules coincides with MacLane homology $H_*^{ML}(R; M)$ defined 30 years earlier [29]. The equivalence of the two functors was proved by Pirashvili and Waldhausen [38] by characterizing both of them as derived functors of the same functor. There is a short heuristic argument for this equivalence using brave new algebra [13]. So 11.5 also constitutes a first complete calculation of the MacLane homology for two examples.

Some years after Bökstedt's result appeared in preprint form, Lindenstrauss calculated $THH^{\mathbf{S}}(R)$ for $R = \mathbb{Z}[X[/(X^n), R = \mathbb{Z}[X]/(X^n - 1)$, and $R = \mathbb{F}_p[X]/(f), f$ a monic polynomial [23], and Pirashvili for group rings $R[G]$ [37], both using spectral sequence arguments and 11.5. Their calculations follow much more easily from a general result proved by using brave new algebra [47]:

11.8 Proposition: Let K be a classical commutative ring and R a flat K-algebra. Let HK and HR denote their associated Eilenberg-MacLane ring and algebra spectra. Assume there is a commutative $\mathbf{S}$-algebra spectrum A and an A-algebra spectrum E such that
(1) HK is a commutative algebra over A
(2) there is a weak equivalence of HK-algebras $E \wedge_A HK \to HR$.
Then (modulo technical cofibrancy conditions)

$$THH_*^A(HR) \cong HH_*^K(R) \overset{L}{\otimes}_K THH_*^A(HK),$$

as graded K-modules (as graded K-algebras in the commutative case), where $\overset{L}{\otimes}$ stands for the total left derived of $\otimes$.

Proof sketch: The symbol $\simeq_w$ will denote a weak equivalence.

$THH^A(HR) \simeq_w THH^A(E \wedge_A HK) \cong THH^A(E) \wedge_A THH^A(HK)$
$\cong THH^A(E) \wedge_A HK \wedge_{HK} THH^A(HK)$

Now $THH^A(E) \wedge_A HK$ is the realization of the simplicial spectrum

$$[n] \mapsto (E \wedge_A E \wedge_A \ldots \wedge_A E) \wedge_A HK \cong (E \wedge_A HK) \wedge_{HK} \ldots \wedge_{HK} (E \wedge_A HK)$$

Hence

$$\begin{aligned} THH^A(HR) &\simeq_w THH^{HK}(E \wedge_A HK) \wedge_{HK} THH^A(HK) \\ &\simeq_w THH^{HK}(HR) \wedge_{HK} THH^A(HK). \end{aligned}$$

Our assumptions imply that $THH_*^{HK}(HR) \cong HH_*^K(R)$, and we have already pointed out that (modulo cofibrancy conditions, which we have to assume throughout the proof) $\wedge_{HK}$ corresponds to $\overset{L}{\otimes}_K$ by passage to homotopy groups. □

11.9 Example: Let G be a group, $R = \mathbb{Z}[G]$ the group ring. Take $K = \mathbb{Z}$ and A to be the spere spectrum $\mathbb{S}$ in Proposition 11.8. Then $E = \Sigma^\infty(G_+)$ satisfies the assumptions. Hence

$$THH_*^{\mathbb{S}}(\mathbb{Z}[G]) \cong HH_*^{\mathbb{Z}}(\mathbb{Z}[G]) \overset{L}{\otimes}_{\mathbb{Z}} THH_*^{\mathbb{S}}(\mathbb{Z})$$

The right-hand side is known in terms of group homology. E.g. if G is a finite abelian group of order n, then

$$THH_k^{\mathbb{S}}(\mathbb{Z}[G]) \cong \bigoplus_{p+q=k} H_p(G;\mathbb{Z})^n \otimes_{\mathbb{Z}} THH_q^{\mathbb{S}}(\mathbb{Z}) \oplus \bigoplus_{p+q=k-1} Tor^{\mathbb{Z}}(H_p(G;\mathbb{Z})^n, THH_q^{\mathbb{S}}(\mathbb{Z})).$$

For further examples see [47].

Not every classical ring satisfies the assumptions of the proposition. An example is the ring $\mathbb{Z}[i]$ of Gaussian numbers, whose topological Hochschild homology has been determined by Lindenstrauss [24].

12 Topological cyclic homology

Today the main interest in topological Hochschild homology comes from the fact that its cyclic structure given by cyclicly rotating the Diagrams 11.1 gives rise to topological cyclic homology. Let C_n denote the cyclic group of order n and $T^{C_n}(R)$ the C_n-fixed point set of $THH^{\mathbb{S}}(R)$. We have the obvious inclusion functors

$$D_k : T^{C_{k\cdot l}} \to T^{C_l}$$

and there are k-th power maps

$$\Phi_k : T^{C_{k\cdot l}} \to T^{C_l}.$$

The D_k and Φ_k satisfy the following relations

$$\begin{array}{ll} \Phi_k \circ \Phi_l = \Phi_{k\cdot l} & D_k \circ D_l = D_{k\cdot l} \\ \Phi_k \circ D_l = D_l \circ \Phi_k & D_1 = \Phi_1 = id \end{array}$$

Let $\mathcal{B}$ be the category with objects $n \in \mathbb{N}, n \geq 1$, and morphisms $D_k, \Phi_k : k \cdot l \to l$ satisfying these relations.

12.1 Definition: *Topological cyclic homology* $TC(R)$ of a classical ring R is

$$TC(R) = holim_B T^{C_n}$$

12.2 Remark: The formal properties of the cyclic structure of THH are well understood for FSPs and, so far, topological cyclic homology is just defined for FSPs; the extension to ring spectra is still work in progress.

As we pointed out at the beginning of this chapter, $TC(R)$ is much closer to $K(R)$ than $THH(R)$. We have the following result due to McCarthy [28].

12.3 Proposition: Let $f : A \to B$ be a ring epimorphism with nilpotent kernel, and let p be a prime. Then the cyclotomic trace is an isomorphism of relative groups

$$trc : K(A \to B)^\wedge_p \cong TC(A \to B)^\wedge_p$$

after p-completion.

So in some cases the calculation of algebraic K-theory reduces to the more easily accessible calculation of topological cyclic homology. E.g. Hesselholt and Madsen could show [15]

12.4 Proposition: Let K be a perfect field of characteristic $p > 0$, and let A be a finitely generated algebra over the Witt vectors $W(K)$. Then

$$K_*(A)^\wedge_p \cong TC_*(A)^\wedge_p \ .$$

Explicit calculations of $K(\mathbb{Z}^\wedge_p)^\wedge_p$ were obtained by Bökstedt and Madsen for $p > 2$ [8] and by Rognes for $p = 2$ [41].

13 The K-theory of the integers

The algebraic K-theory $K(\mathbb{Z})$ of the integers is of great interest for various reasons. We have already pointed out its connection with the stable homotopy groups of the spheres. There are also various number theoretic conjectures related to $K(\mathbb{Z})$. Let me mention just one

13.1 Weak Quillen-Lichtenbaum Conjecture: The ζ-function satisfies

$$|\zeta(-n)| = 2\frac{\#K_{2n}(\mathbb{Z})}{\#K_{2n-1}(\mathbb{Z})}$$

where $\#$ denotes the number of elements of the group.

13.2 Calculations of $K_*(\mathbb{Z})$:

$K_0(\mathbb{Z}) \cong \mathbb{Z}$	classical
$K_1(\mathbb{Z}) \cong \mathbb{Z}/2$	classical
$K_2(\mathbb{Z}) \cong \mathbb{Z}/2$	Milnor (1971)
$\#K_3(\mathbb{Z})$ is divisible by 48	Karoubi (1974)
$K_3(\mathbb{Z}) \cong \mathbb{Z}/48$	Lee, Szczarba (1976)
$K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$ have no p-torsion for $p \geq 5$	Lee, Szczarba Soulé (1978)
$K_4(\mathbb{Z})$ has no 3-torsion	Rognes (1994)
$K_4(\mathbb{Z}) \cong 0$	Rognes, Weibel (1996)

A considerable advance in the calculation of $K(\mathbb{Z})$, including the final part of the proof that $K_4(\mathbb{Z}) = 0$, was brought about by Voevodsky's solution of the Milnor Conjecture [52]. Although the published version does not explicitly use "brave new algebra", parts of the proof are definitively inspired by it. Using this result and the Bloch-Lichtenbaum spectral sequence [1], Weibel calculated the 2-primary part of $K(\mathbb{Z})$ [56]. He tacitly assumed that the spectral sequence has certain multiplicative properties, which have not yet been verified. This gap was closed by Rognes using topological cyclic homology [42].

13.3 The 2-primary part of $K_*(\mathbb{Z})$

n	$K_n(\mathbb{Z})$
5	$\mathbb{Z} \oplus (3 - \text{torsion})$
6	(odd torsion)
7	$\mathbb{Z}/240 \oplus$ (odd torsion)
$k \geq 1$	
$8k$	(odd torsion)
$8k+1$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus$ (odd torsion)
$8k+2$	$\mathbb{Z}/2 \oplus$ (odd torsion)
$8k+3$	$\mathbb{Z}/16 \oplus$ (odd torsion)
$8k+4$	(odd torsion)
$8k+5$	$\mathbb{Z} \oplus$ (odd torsion)
$8k+6$	(odd torsion)
$8k+7$	$\mathbb{Z}/w_i \oplus$ (odd torsion), $i = 4(k+1)$
	$w_i =$ largest power of 2 in $4i$

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