Bogdan Balcerzak

Classification of endomorphisms of some Lie algebroids up to homotopy and the fundamental group of a Lie algebroid

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 18th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1999. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 59. pp. 89–101.

Persistent URL: http://dml.cz/dmlcz/701628

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 59 (1999) pp. 89-101

CLASSIFICATION OF ENDOMORPHISMS OF SOME LIE ALGEBROIDS UP TO HOMOTOPY AND THE FUNDAMENTAL GROUP OF A LIE ALGEBROID

BOGDAN BALCERZAK

ABSTRACT. The notion of a homotopy joining two homomorphisms of Lie algebroids comes from J. Kubarski [3]. Firstly, in the present paper we investigate this notion in the case of endomorphisms of the trivial Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$ with the isotropy algebra $\mathbb{R}$ and characterize its homotopic endomorphisms. Secondly, for any regular Lie algebroid A, we introduce the notion of the fundamental group $\pi_1(A)$ as the group of classes of homotopic automorphisms of A and, finally, obtain that $\pi_1(T\mathbb{R}^n \times \mathbb{R}) \cong \mathrm{GL}(\mathbb{R})$.

1. INTRODUCTION

We begin by recalling the notions of a regular Lie algebroid and of a homomorphism of Lie algebroids. These are fundamental notions in this work.

1.1. **Definition of a regular Lie algebroid on a foliated manifold.** Let F be a smooth, constant-dimensional and involutive distribution on a smooth, paracompact, connected and Hausdorff manifold M. The pair (M, F) is called a *foliated manifold.*

Definition 1.1. [6], [7] By a *regular Lie algebroid on a foliated manifold* (M, F) we mean a system

$$(A, [\![\cdot,\cdot]\!], \gamma)$$

where A is a vector bundle over the manifold M, $[\![\cdot,\cdot]\!] : \mathrm{Sec}\,A \times \mathrm{Sec}\,A \to \mathrm{Sec}\,A$ is a Lie algebra product on the module $\mathrm{Sec}\,A$ of global cross-sections of a vector bundle A and $\gamma : A \to TM$is a vector bundle map (called an *anchor*) such that

1. $\mathrm{Im}\,\gamma = F$,
2. the mapping $\mathrm{Sec}\,\gamma : \mathrm{Sec}\,A \to \mathfrak{X}(M)$, $\xi \mapsto \gamma\circ\xi$, is a homomorphism of Lie algebras,
3. $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$ for any $\xi,\eta \in \mathrm{Sec}\,A$ and $f \in C^\infty(M)$.

In the case when $F = TM$ (i.e. $\gamma : A \to TM$ is a surjective homomorphism of vector bundles), the algebroid $(A, [\![\cdot,\cdot]\!], \gamma)$ is called a *transitive Lie algebroid.*

Example 1.1. Let M be a smooth manifold. Any smooth, constant-dimensional and involutive distribution $F \subset TM$ is an example of a nontransitivie Lie algebroid with Lie bracket $[X, Y]$ of vector fields as a commutator $[\![X, Y]\!]$ and the inclusion $\iota : F \hookrightarrow TM$ as an anchor.

Example 1.2. [5] Let M be a smooth manifold and $\mathfrak{g}$ a finite-dimensional $\mathbb{R}$-Lie algebra. Then $TM \times \mathfrak{g}$ is a transitive Lie algebroid with the canonical projection $\mathrm{pr}_1 : TM \times \mathfrak{g} \to TM$ as an anchor and with the bracket

$$[\![\cdot,\cdot]\!] : \mathrm{Sec}\,(TM \times \mathfrak{g}) \times \mathrm{Sec}\,(TM \times \mathfrak{g}) \to \mathrm{Sec}\,(TM \times \mathfrak{g})$$

satisfying the relation

$$[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - \mathcal{L}_Y\sigma + [\sigma,\eta])$$

for all X, $Y \in \mathfrak{X}(M)$, σ, $\eta \in C^\infty(M;\mathfrak{g})$.

1.2. The notion of a homomorphism of Lie algebroids.

Definition 1.2. [7], [5] Let $(A,[\![\cdot,\cdot]\!],\gamma)$ and $(A',[\![\cdot,\cdot]\!]',\gamma')$ be two regular Lie algebroids on the same foliated manifold (M,F) and let $H : A' \to A$ be a vector bundle map (over $\mathrm{id}_M : M \to M$). Then H is said to be a *strong homomorphism of Lie algebroids* if the following relations hold:

1. $\gamma \circ H = \gamma'$,
2. the mapping $\mathrm{Sec}\, H$: Sec is a homomorphism of Lie algebras.

Definition 1.3. [1], [2] Let $(A',[\![\cdot,\cdot]\!]',\gamma')$ and $(A,[\![\cdot,\cdot]\!],\gamma)$ be two Lie algebroids on manifolds $M'$and M, respectively. By a *homomorphism between them*

$$H : (A',[\![\cdot,\cdot]\!]',\gamma') \longrightarrow (A,[\![\cdot,\cdot]\!],\gamma)$$

we mean a homomorphism of vector bundles $H : A' \to A$ (over $f : M' \longrightarrow M$) such that:

1. $\gamma \circ H = f_* \circ \gamma'$,
2. for arbitrary cross-sections ξ, $\xi' \in \mathrm{Sec}\, A'$ with H-decompositions

$$\begin{aligned} H\circ\xi &= \sum_i f^i\cdot(\eta_i\circ f), \\ H\circ\xi' &= \sum_j g^j\cdot(\eta_j\circ f) \end{aligned}$$

where f^i, $g^j \in C^\infty(M')$, η_i, $\eta_j \in \mathrm{Sec}\, A$, we have relation

$$\begin{aligned} H\circ[\![\xi,\xi']\!]' &= \sum_{i,j} f^i\cdot g^j\cdot([\![\eta_i,\eta_j]\!]\circ f) + \\ &\quad + \sum_j (\gamma'\circ\xi)\left(g^j\right)\cdot(\eta_j\circ f) - \sum_i (\gamma'\circ\xi')\left(f^i\right)\cdot(\eta_i\circ f). \end{aligned}$$

Remark 1.1. In the case of Lie algebroids A and A' on the same manifold M, the notion of a homomorphism $H : A' \to A$ (over the identity mapping $\mathrm{id}_M : M \to M$) is equivalent to the one given in definition 1.2.

1.3. The inverse image of a regular Lie algebroid.

Definition 1.4. [2] Let $(A,[\![\cdot,\cdot]\!],\gamma)$ be a regular Lie algebroid on a foliated manifold (M,F) and let $f : (M',F') \to (M,F)$ be a morphism of the category of foliated manifolds. The *inverse image of A by f* is a regular Lie algebroid on (M',F')

$$(f^\wedge A, [\![\cdot,\cdot]\!]^\wedge, \mathrm{pr}_1)$$

where we have

1. $f^\wedge A = \{(\mathrm{v},\mathrm{w}) \in F' \times A : f_*(\mathrm{v}) = \gamma(\mathrm{w})\} \subset F' \oplus f^*A$,

2. the bracket $[\![\cdot,\cdot]\!]^\wedge$ in $\operatorname{Sec} f^\wedge A$ is defined in the following way: let $\left(X_1,\overline{\xi}_1\right)$, $\left(X_2,\overline{\xi}_2\right) \in \operatorname{Sec} f^\wedge A$ be two cross-sections of $f^\wedge A$, where $X_i \in \operatorname{Sec} F'$, $\overline{\xi}_i \in \operatorname{Sec} f^* A$ and $i \in \{1,2\}$. Then, for each point $x \in M'$, there exists an open subset $U \subset M'$ such that $x \in U$ and $\left(\overline{\xi}_i\right)_{|U}$ is of the form $\sum_j g_i^j \cdot \left(\xi_i^j \circ f\right)$ for some $g_i^j \in C^\infty(M')$ and $\xi_i^j \in \operatorname{Sec} A$. Then we put

$$[\![\left(X_1,\overline{\xi}_1\right), \left(X_2,\overline{\xi}_2\right)]\!]^\wedge_{|U} = \Big([X_1,X_2], \sum_{j,k} g_1^j \cdot g_2^k \cdot \left([\![\xi_1^j,\xi_2^k]\!] \circ f\right) + \sum_k X_1\left(g_2^k\right)\cdot\left(\xi_2^k\circ f\right) - \sum_j X_2\left(g_1^j\right)\cdot\left(\xi_1^j\circ f\right)\Big)_{|U}.$$

Theorem 1.1. [2] *Any homomorphism of regular Lie algebroids $H : A' \to A$ over $f : (M',F') \to (M,F)$ may be represented as a superposition*

$$\begin{tikzcd} & A' \arrow[dl, "\overline{H}"'] \arrow[dr, "H"] & \\ f^\wedge A \arrow[rr, "\mathrm{pr}_2"'] & & A \end{tikzcd}$$

of a homomorphism $\overline{H} : A' \to f^\wedge A$ defined by

$$\overline{H}(v) = \left(\gamma'(v), H(v)\right) \text{ for each } v \in A' \tag{1.1}$$

with the canonical one $\mathrm{pr}_2 : f^\wedge A \to A$. ∎

Theorem 1.2. [2] *Let A and A' be two regular Lie algebroids on foliated manifolds (M',F') and (M,F), respectively. Let $H : A' \to A$ be a homomorphism of vector bundles over $f : (M',F') \to (M,F)$. Then H is a homomorphism of Lie algebroids if and only if*

1. $\gamma \circ H = f_* \circ \gamma'$,
2. *the mapping $\overline{H} : A' \to f^\wedge A$ defined by $v \mapsto (\gamma'(v), H(v))$ is a homomorphism of Lie algebroids.* ∎

1.4. **The Cartesian product of regular Lie algebroids.** By a *Cartesian product of two regular Lie algebroids $(A', [\![\cdot,\cdot]\!]', \gamma')$ and $(A, [\![\cdot,\cdot]\!], \gamma)$* on foliated manifolds (M',F') and (M,F), respectively, we mean the Lie algebroid

$$\left(A \times A', [\![\cdot,\cdot]\!]^\times, \gamma \times \gamma'\right)$$

over the foliated manifold $(M \times M', F \times F')$, and, for $\overline{\xi} = \left(\overline{\xi}^1, \overline{\xi}^2\right)$, $\overline{\eta} = (\overline{\eta}^1, \overline{\eta}^2) \in \operatorname{Sec}(A \times A')$ and $(x,y) \in M \times M'$, we define

$$[\![\overline{\xi},\overline{\eta}]\!]^\times_{(x,y)} = \left([\![\overline{\xi},\overline{\eta}]\!]^{\times 1}_{(x,y)}, [\![\overline{\xi},\overline{\eta}]\!]^{\times 2}_{(x,y)}\right)$$

where

$$[\bar{\xi},\bar{\eta}]^{\times 1}_{(x,y)} = [\bar{\xi}^1(\cdot,y),\bar{\eta}^1(\cdot,y)]_x + \left(\gamma'\circ\bar{\xi}^2\right)_{(x,y)}\left(\bar{\eta}^1(x,\cdot)\right) - \left(\gamma'\circ\bar{\eta}^2\right)_{(x,y)}\left(\bar{\xi}^1(x,\cdot)\right),$$
$$[\bar{\xi},\bar{\eta}]^{\times 2}_{(x,y)} = [\bar{\xi}^2(x,\cdot),\bar{\eta}^2(x,\cdot)]'_y + \left(\gamma\circ\bar{\xi}^1\right)_{(x,y)}\left(\bar{\eta}^2(\cdot,y)\right) - \left(\gamma\circ\bar{\eta}^1\right)_{(x,y)}\left(\bar{\xi}^2(\cdot,y)\right).$$

2. CHARACTERIZATION OF ENDOMORPHISMS OF THE LIE ALGEBROID $T\mathbb{R}^n \times \mathbb{R}$

We shall consider a strong endomorphism $H : T\mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n \times \mathbb{R}$ of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$.

Remark 2.1. An element of the tangent bundle $T\mathbb{R}^n$ we identified with a point of $\mathbb{R}^n \times \mathbb{R}^n$ by the isomorphism

$$\omega : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow T\mathbb{R}^n\ , \quad (x,y) \longmapsto \sum_{i=1}^{n} y_i \cdot \left.\frac{\partial}{\partial x_i}\right|_x$$

for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$, where the system $\left(\left.\frac{\partial}{\partial x_i}\right|_x\right)_{i=1}^n$ forms the base of the tangent space of $\mathbb{R}^n$ at x induced by the identity map on $\mathbb{R}^n$.

Theorem 2.1. *An endomorphism* $H : T\mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n \times \mathbb{R}$ *of the vector bundle* $T\mathbb{R}^n \times \mathbb{R}$ *is an endomorphism of the Lie algebroid* $T\mathbb{R}^n \times \mathbb{R}$ *if and only if, for any* $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$ *and* $r \in \mathbb{R}$, H *is of the form*

$$H(x,y,r) = \left(x, y, \sum_{i=1}^{n} A^i(x)\cdot y_i + B\cdot r\right)$$

where $B \in \mathbb{R}$, $A^i \in C^\infty(\mathbb{R}^n)$ *and, for all* $i, j \in \{1,2,...,n\}$ *such that* $i \neq j$, *the relations*

$$\left.\frac{\partial A^i}{\partial x^j}\right|_{(x_1,...,x_n)} = \left.\frac{\partial A^j}{\partial x^i}\right|_{(x_1,...,x_n)} \tag{2.1}$$

hold.

Proof. " $\Longrightarrow$ " Assume that $H : T\mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n \times \mathbb{R}$ is an endomorphism of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$ (over $\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$). Since the following diagram

$$\begin{tikzcd} T\mathbb{R}^n \times \mathbb{R} \arrow[r, "H"] \arrow[d, "\gamma"'] & T\mathbb{R}^n \times \mathbb{R} \arrow[dl, "\gamma"] \\ T\mathbb{R}^n & \end{tikzcd}$$

commutes, H is of the form

$$H(x,y,r) = (x, y, \lambda(x,y,r)) \quad \text{for } x,\ y \in \mathbb{R}^n \text{ and } r \in \mathbb{R},$$

where $\lambda : (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}$ is a smooth function. Moreover, since the restrictions $H_{|x} = H_{|T_x\mathbb{R}^n\times\mathbb{R}} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ are linear mappings for each $x \in \mathbb{R}^n$, therefore

$H_{|x}$ is of the form

$$H_{|x}(y,r)=\left(y,\sum_{i=1}^{n}A^{i}(x)\cdot y_i+B(x)\cdot r\right)$$

for all $y=(y_1,...,y_n)\in\mathbb{R}^n$, $r\in\mathbb{R}$ and for some smooth functions A^i, $B\in C^\infty(\mathbb{R}^n)$. Thus

$$\lambda(x,y,r)=\sum_{i=1}^{n}A^{i}(x)\cdot y_i+B(x)\cdot r \text{ for } y\in\mathbb{R}^n \text{ and } r\in\mathbb{R}.$$

Let $X\in\mathfrak{X}(\mathbb{R}^n)$ and $\eta\in C^\infty(\mathbb{R})$ be arbitrary, whereas $Y=0\in\mathfrak{X}(\mathbb{R}^n)$ – the zero vector field on the manifold $\mathbb{R}^n$ and $\sigma=0$ – the zero function on $\mathbb{R}^n$. Observe that

$$H\circ[\![(X,\sigma),(Y,\eta)]\!]=H\circ[\![(X,0),(0,\eta)]\!]=H(0,X(\eta))=(0,B\cdot X(\eta))$$

and

$$\begin{aligned}[\![H\circ(X,\sigma),H\circ(Y,\eta)]\!] &= [\![H\circ(X,0),H\circ(0,\eta)]\!]\\ &= [\![\left(X,\sum_{i=1}^{n}A^i\cdot X^i\right),(0,B\cdot\eta)]\!]\\ &= (0,X(B\cdot\eta))=(0,B\cdot X(\eta)+X(B)\cdot\eta).\end{aligned}$$

Since $\operatorname{Sec}H:\operatorname{Sec}(T\mathbb{R}^n\times\mathbb{R})\to\operatorname{Sec}(T\mathbb{R}^n\times\mathbb{R})$ is a homomorphism of Lie algebras, we have the equality

$$H\circ[\![(X,0),(0,\eta)]\!]=[\![H\circ(X,0),H\circ(0,\eta)]\!].$$

Hence we obtain that $X(B)\cdot\eta=0$ for each $\eta\in C^\infty(\mathbb{R})$. For a non-zero function on $\mathbb{R}$, we have

$$X(B)=0.$$

But $X\in\mathfrak{X}(\mathbb{R}^n)$ was an arbitrarily taken vector field, therefore B is constant.

Now, let X, $Y\in\mathfrak{X}(\mathbb{R}^n)$ be two arbitrary vector fields on $\mathbb{R}^n$. Then

$$H\circ[\![(X,0),(Y,0)]\!]=H([X,Y],0)=\left([X,Y],\sum_{i=1}^{n}A^i\cdot[X,Y]^i\right)$$

and

$$\begin{aligned}[\![H\circ(X,0),H\circ(Y,0)]\!] &= [\![(X,\sum_{i=1}^{n}A^i\cdot X^i),(Y,\sum_{i=1}^{n}A^i\cdot Y^i)]\!]\\ &= \left([X,Y],X\left(\sum_{i=1}^{n}A^i\cdot Y^i\right)-Y\left(\sum_{i=1}^{n}A^i\cdot X^i\right)\right),\end{aligned}$$

where X^i, Y^i, $[X,Y]^i\in C^\infty(\mathbb{R}^n)$ are coordinates of the vector fields X, Y, $[X,Y]$, respectively. Since $\operatorname{Sec}H$ is a homomorphism of Lie algebras, we have

$$H\circ[\![(X,0),(Y,0)]\!]=[\![H\circ(X,0),H\circ(Y,0)]\!],$$

whence

$$\sum_{i=1}^{n}A^i\cdot[X,Y]^i=\sum_{i=1}^{n}X\left(A^i\cdot Y^i\right)-\sum_{i=1}^{n}Y\left(A^i\cdot X^i\right). \tag{2.2}$$

Concider vector fields $X = \sum_{i=1}^{n} X^i \cdot \frac{\partial}{\partial x_i}$, $Y = \sum_{j=1}^{n} Y^j \cdot \frac{\partial}{\partial x_j} \in \mathfrak{X}(\mathbb{R}^n)$ where X^i, $Y^j \in C^\infty(\mathbb{R}^n)$ and $\left(\frac{\partial}{\partial x_i}\right)_{i=1}^{n}$ forms the base of the module $\mathfrak{X}(\mathbb{R}^n)$, induced by the identity map on $\mathbb{R}^n$. In view of the properities of the Lie bracket $[\cdot,\cdot]$ of vector fields on $\mathbb{R}^n$, we obtain

$$[X,Y] = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} X^j \cdot \frac{\partial}{\partial x_j}\left(Y^i\right) - \sum_{j=1}^{n} Y^j \cdot \frac{\partial}{\partial x_j}\left(X^i\right)\right) \cdot \frac{\partial}{\partial x_i}.$$

Hence (2.2) implies that

$$\sum_{i=1}^{n} A^i \cdot \left(\sum_{j=1}^{n} X^j \cdot \frac{\partial}{\partial x_j}\left(Y^i\right) - \sum_{j=1}^{n} Y^j \cdot \frac{\partial}{\partial x_j}\left(X^i\right)\right) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} X^j \cdot \frac{\partial}{\partial x_j}\left(A^i \cdot Y^i\right) - \sum_{i=1}^{n}\sum_{j=1}^{n} Y^j \cdot \frac{\partial}{\partial x_j}\left(A^i \cdot X^i\right),$$

whence

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(X^j \cdot Y^i - Y^j \cdot X^i\right) \cdot \frac{\partial}{\partial x_j}\left(A^i\right) = 0.$$

Let $i_0 \neq j_0$ and $X = \frac{\partial}{\partial x_{i_0}}$, $Y = \frac{\partial}{\partial x_{j_0}}$, i.e. $X^i = \delta_i^{i_0}$ and $Y^j = \delta_j^{j_0}$ for i, $j \in \{1, ..., n\}$. From the above it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\delta_j^{i_0} \cdot \delta_i^{j_0} - \delta_j^{j_0} \cdot \delta_i^{i_0}\right) \cdot \frac{\partial}{\partial x_j}\left(A^i\right) = 0;$$

consequently,

$$\frac{\partial}{\partial x_{i_0}}\left(A^{j_0}\right) = \frac{\partial}{\partial x_{j_0}}\left(A^{i_0}\right).$$

On account of the arbitrariness of $i_0 \neq j_0$, we have (2.1).

" $\Longleftarrow$ " Let $H : T\mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n \times \mathbb{R}$ be an endomorphism of the vector bundle $T\mathbb{R}^n \times \mathbb{R}$, such that

$$H(x,y,r) = \left(x, y, \sum_i A^i(x) \cdot y_i + B \cdot r\right)$$

for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$, $r \in \mathbb{R}$, where $B \in \mathbb{R}$, and $A^i \in C^\infty(\mathbb{R}^n)$ satisfy condition (2.1).

Consider (X, σ), $(Y, \eta) \in \mathrm{Sec}(T\mathbb{R}^n \times \mathbb{R})$ where σ, $\eta \in C^\infty(\mathbb{R}^n)$ and $X = \sum_{i=1}^{n} X^i \cdot \frac{\partial}{\partial x_i}$, $Y = \sum_{j=1}^{n} Y^j \cdot \frac{\partial}{\partial x_j} \in \mathfrak{X}(\mathbb{R}^n)$, X^i, $Y^j \in C^\infty(\mathbb{R}^n)$. Observe that

$$\sum_{i=1}^{n} X\left(A^i \cdot Y^i\right) - \sum_{i=1}^{n} Y\left(A^i \cdot X^i\right) = \sum_{i=1}^{n} A^i \cdot [X,Y]^i + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} X^j \cdot Y^i \cdot \left(\frac{\partial A^i}{\partial x_j} - \frac{\partial A^j}{\partial x_i}\right).$$

Thus (2.1) implies that

$$\sum_{i=1}^{n} X\left(A^i \cdot Y^i\right) - \sum_{i=1}^{n} Y\left(A^i \cdot X^i\right) = \sum_{i=1}^{n} A^i \cdot [X,Y]^i .$$

Then we obtain

$$\begin{aligned}
&[H \circ (X,\sigma), H \circ (Y,\eta)] = \\
&= [\left(X, \sum_{i=1}^{n} A^i \cdot X^i + B \cdot \sigma\right), \left(Y, \sum_{i=1}^{n} A^i \cdot Y^i + B \cdot \eta\right)] \\
&= \left([X,Y], X\left(\sum_{i=1}^{n} A^i \cdot Y^i + B \cdot \eta\right) - Y\left(\sum_{i=1}^{n} A^i \cdot X^i + B \cdot \sigma\right)\right) \\
&= \left([X,Y], \sum_{i=1}^{n} A^i \cdot [X,Y]^i + B \cdot (X(\eta) - Y(\sigma))\right) \\
&= H \circ ([X,Y], X(\eta) - Y(\sigma)) = H \circ [(X,\sigma),(Y,\eta)].
\end{aligned}$$

Therefore the mapping $\operatorname{Sec} H$ is a homomorphism of Lie algebras. It follows that H is a strong endomorphism of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$. ■

Corollary 2.2. ($n = 2$) An endomorphism $H : T\mathbb{R}^2 \times \mathbb{R} \to T\mathbb{R}^2 \times \mathbb{R}$ of the vector bundle $T\mathbb{R}^2 \times \mathbb{R}$ is an endomorphism of the Lie algebroid $T\mathbb{R}^2 \times \mathbb{R}$ if and only if H is of the form

$$\begin{aligned}
&H((x_1,x_2),(y_1,y_2),r) = \\
&= \left((x_1,x_2),(y_1,y_2), A^1(x_1,x_2) \cdot y_1 + A^2(x_1,x_2) \cdot y_2 + B \cdot r\right)
\end{aligned} \tag{2.3}$$

for all $((x_1,x_2),(y_1,y_2),r) \in (\mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R}$, where $B \in \mathbb{R}$, $A^1 \in C^\infty(\mathbb{R}^2)$, and $A^2 \in C^\infty(\mathbb{R}^2)$ is given by

$$A^2(x_1,x_2) = \frac{\partial}{\partial x_2} \int_0^{x_1} A^1(t,x_2)\, dt + \varphi(x_2) \tag{2.4}$$

for a certain function $\varphi \in C^\infty(\mathbb{R})$ depending on x_2 only.

Proof. " $\Longrightarrow$ " Suppose that H is an endomorphism of the Lie algebroid $T\mathbb{R}^2 \times \mathbb{R}$ and $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$, $r \in \mathbb{R}$. By theorem 2.1,

$$H((x_1,x_2),(y_1,y_2),r) = \left((x_1,x_2),(y_1,y_2), A^1(x_1,x_2) \cdot y_1 + A^2(x_1,x_2) \cdot y_2 + B \cdot r\right)$$

where $B \in \mathbb{R}$ and

$$\left.\frac{\partial A^1}{\partial x_2}\right|_{(x_1,x_2)} = \left.\frac{\partial A^2}{\partial x_1}\right|_{(x_1,x_2)} \quad \text{for any} \quad (x_1,x_2) \in \mathbb{R}^2. \tag{2.5}$$

Since there exists a function $\varphi \in C^\infty(\mathbb{R})$ dependent on x_2 only, such that

$$A^2(x_1,x_2) = \int_0^{x_1} \left.\frac{\partial A^2}{\partial x_1}\right|_{(t,x_2)} dt + \varphi(x_2)$$

and (2.5) holds, therefore

$$A^2(x_1,x_2) = \int_0^{x_1} \frac{\partial A^1}{\partial x_2}\Big|_{(t,x_2)} dt + \varphi(x_2),$$

whence we obtain

$$A^2(x_1,x_2) = \frac{\partial}{\partial x_2}\int_0^{x_1} A^1(t,x_2)\, dt + \varphi(x_2)$$

for all $(x_1,x_2) \in \mathbb{R}^2$.

" $\Longleftarrow$ " Let now the endomorphism $H : T\mathbb{R}^2 \times \mathbb{R} \to T\mathbb{R}^2 \times \mathbb{R}$ be defined by (2.3). A^1 is any function of $C^\infty(\mathbb{R}^2)$ and $A^2 \in C^\infty(\mathbb{R}^2)$ is given by (2.4). Then

$$\frac{\partial A^2}{\partial x_1}\Big|_{(x_1,x_2)} = \frac{\partial A^1}{\partial x_2}\Big|_{(x_1,x_2)} \quad \text{for any} \quad (x_1,x_2) \in \mathbb{R}^2.$$

On account of theorem 2.1, we have that H is an endomorphism of the Lie algebroid $T\mathbb{R}^2 \times \mathbb{R}$. ■

3. HOMOTOPY

3.1. Definition of a homotopy joining two homomorphisms of Lie algebroids. Let A and A' be regular Lie algebroids on manifolds M and M', respectively, and let H_0, $H_1 : A' \to A$ be homomorphisms of Lie algebroids. By a *homotopy joining H_0 to H_1* we mean a homomorphism of Lie algebroids

$$H : T\mathbb{R} \times A' \longrightarrow A$$

such that

$$H(\theta_0,\cdot) = H_0 \text{ and } H(\theta_1,\cdot) = H_1,$$

where θ_0 and θ_1 are null vectors tangent to $\mathbb{R}$ at 0 and 1, respectively. We then say that the endomorphism H_0 is homotopic to H_1 and write $H_0 \sim H_1$.

This definition comes from J. Kubarski [3].

Since we are interested in strong endomorphisms of a Lie algebroid A, we modify the above definition assuming that H is over the projection $\mathrm{pr}_2 : \mathbb{R} \times M \to M$. Then H is said to be a *strong homotopy*.

3.2. Characterization of a homotopy joining two endomorphisms of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$. Let $\mathrm{pr}_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be given by $\mathrm{pr}_n(x_0,x_1,...,x_n) = (x_1,...,x_n)$ for all $(x_0,x_1,...,x_n) \in \mathbb{R}^{n+1}$.

Lemma 3.1. *The mapping $\Lambda : T\mathbb{R}^{n+1} \times \mathbb{R} \to \mathrm{pr}_n^\wedge(T\mathbb{R}^n \times \mathbb{R})$ defined by*

$$\begin{aligned} &\Lambda\left(\left(\left(x_0,x_1,...,x_n\right),\left(y_0,y_1,...,y_n\right)\right),s\right) = \\ &= \left(\left(\left(x_0,x_1,...,x_n\right),\left(y_0,y_1,...,y_n\right)\right),\left(\left(\left(x_1,...,x_n\right),\left(y_1,...,y_n\right),s\right)\right)\right) \end{aligned} \tag{3.1}$$

for any $(x_0,x_1,...,x_n)$, $(y_0,y_1,...,y_n) \in \mathbb{R}^{n+1}$ and $s \in \mathbb{R}$ is an isomorphism of Lie algebroids.

Proof. The proof is standard. ■

The following lemma is preparatory to the main theorem of our paper – theorem 3.3.

Lemma 3.2. *Let H_0, $H_1 : T\mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n \times \mathbb{R}$ be two endomorphisms of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$ and let, for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$,*

$$\begin{aligned} H_0((x,y),r) &= \left((x,y), \sum_{i=1}^{n} A_0^i(x) \cdot y_i + B_0 \cdot r\right), \\ H_1((x,y),r) &= \left((x,y), \sum_{i=1}^{n} A_1^i(x) \cdot y_i + B_1 \cdot r\right) \end{aligned}$$

(according to theorem 2.1, each endomorphism of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$ is of this form), where $B_0, B_1 \in \mathbb{R}$, and A_0^i, $A_1^i \in C^\infty(\mathbb{R}^n)$ satisfy relation (2.1). There exists a strong homotopy joinig H_0 to H_1 if and only if $B_0 = B_1$ and there exist functions $G^i \in C^\infty(\mathbb{R}^{n+1})$ $(i \in \{0, 1, ..., n\})$ such that

$$\begin{aligned} G^k(0, x_1, ..., x_n) &= A_0^k(x_1, ..., x_n), \\ G^k(1, x_1, ..., x_n) &= A_1^k(x_1, ..., x_n) \end{aligned} \tag{3.2}$$

$(k \in \{1, 2, ..., n\})$ and

$$\left.\frac{\partial G^i}{\partial x_j}\right|_{(x_0,x_1,...,x_n)} = \left.\frac{\partial G^j}{\partial x_i}\right|_{(x_0,x_1,...,x_n)} \tag{3.3}$$

for all $(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1}$ and i, $j \in \{0, 1, ..., n\}$ such that $i \neq j$.

Proof. " $\Longrightarrow$ " Assume that there exists a strong homotopy $H : T\mathbb{R}\times (T\mathbb{R}^n \times \mathbb{R}) \to T\mathbb{R}^n \times \mathbb{R}$ joining H_0 to H_1. Then we have

$$H((0,0),((x,y),r)) = H_0((x,y),r), \tag{3.4}$$
$$H((1,0),((x,y),r)) = H_1((x,y),r) \tag{3.5}$$

for any x, $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$.

Let $\overline{H} : T\mathbb{R}\times (T\mathbb{R}^n \times \mathbb{R}) \to \mathrm{pr}_n^{\wedge}(T\mathbb{R}^n \times \mathbb{R})$ denote the homomorphism of Lie algebroids, determined by H via formula (1.1). Since the homomorphism $\Lambda : T\mathbb{R}^{n+1} \times \mathbb{R} \to \mathrm{pr}_n^{\wedge}(T\mathbb{R}^n \times \mathbb{R})$ defined by (3.1) is an isomorphism of Lie algebroids, we see at once, after the identification of $T\mathbb{R}\times (T\mathbb{R}^n \times \mathbb{R})$ with $T\mathbb{R}^{n+1} \times \mathbb{R}$, that $\Lambda^{-1} \circ \overline{H}$ is an endomorphism of the Lie algebroid $T\mathbb{R}^{n+1} \times \mathbb{R}$. Thus and by theorem 2.1, there exist functions $G^i \in C^\infty(\mathbb{R}^{n+1})$ and a real number B, such that $\Lambda^{-1} \circ \overline{H}$ is defined by

$$\left(\Lambda^{-1} \circ \overline{H}\right)((x,y),r) = \left((x,y), \sum_{i=0}^{n} G^i(x) \cdot y_i + B \cdot r\right)$$

for any $x = (x_0, x_1, ..., x_n)$, $y = (y_0, y_1, ..., y_n) \in \mathbb{R}^{n+1}$, $r \in \mathbb{R}$, and the following condition is satisfied

$$\frac{\partial G^i}{\partial x_j} = \frac{\partial G^j}{\partial x_i} \quad \text{for } i,\ j \in \{0, 1, ..., n\} \text{ and } i \neq j.$$

Hence we obtain that $\overline{H}$ is of the form

$$\overline{H}((x,y),r) = \left((x,y), \left(((x_1, ..., x_n), (y_1, ..., y_n)), \sum_{i=0}^{n} G^i(x) \cdot y_i + B \cdot r\right)\right).$$

From the definition of $\overline{H}$ and from the above it follows that H is given by

$$H\left((x_0,y_0),\left(((x_1,...,x_n),(y_1,...,y_n)),r\right)\right) = \tag{3.6}$$
$$= \left(((x_1,...,x_n),(y_1,...,y_n)),\sum_{i=0}^{n} G^i(x_0,x_1,...,x_n)\cdot y_i + B\cdot r\right)$$

for all $(x_0,y_0)\in\mathbb{R}\times\mathbb{R}$, $(x_1,...,x_n)$, $(y_1,...,y_n)\in\mathbb{R}^n$ and $r\in\mathbb{R}$. We deduce from (3.4) and (3.5) that $B_0=B_1$ and

$$\begin{aligned} G^i(0,\cdot) &= A_0^i, \\ G^i(1,\cdot) &= A_1^i \end{aligned}$$

for any $i,\, j\in\{1,...,n\}$.

" $\Longleftarrow$ " Suppose that $B=B_0=B_1$ and there exist functions $G^i\in C^\infty(\mathbb{R}^{n+1})$ ($i\in\{0,1,...,n\}$) satisfying conditions (3.2) and (3.3). Then the mapping

$$H: T\mathbb{R}\times(T\mathbb{R}^n\times\mathbb{R})\to T\mathbb{R}^n\times\mathbb{R}$$

given by (3.6) is a strong homotopy joinig H_0 to H_1. ■

Finally, we shall prove the main theorem of this work.

Theorem 3.3. *Let H_0, $H_1: T\mathbb{R}^n\times\mathbb{R}\to T\mathbb{R}^n\times\mathbb{R}$ be two endomorphisms of the Lie algebroid $T\mathbb{R}^n\times\mathbb{R}$ defined (in view of theorem 2.1) by*

$$\begin{aligned} H_0((x,y),r) &= \left((x,y),\sum_{i=1}^{n} A_0^i(x)\cdot y_i + B_0\cdot r\right), \\ H_1((x,y),r) &= \left((x,y),\sum_{i=1}^{n} A_1^i(x)\cdot y_i + B_1\cdot r\right) \end{aligned}$$

for all $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)\in\mathbb{R}^n$, where B_0, $B_1\in\mathbb{R}$, and A_0^i, $A_1^i\in C^\infty(\mathbb{R}^n)$ satisfy relation (2.1). There exists a strong homotopy joinig H_0 to H_1 if and only if $B_0=B_1$.

Proof. " $\Longrightarrow$ " Assume that the endomorphisms H_0, $H_1: T\mathbb{R}^n\times\mathbb{R}\to T\mathbb{R}^n\times\mathbb{R}$ are homotopic. Now, lemma 3.2 shows that $B_0=B_1$.

" $\Longleftarrow$ " Let now $B_0=B_1$. Take G^0, $G^i\in C^\infty(\mathbb{R}^{n+1})$ ($i\in\{1,2,...,n\}$) defined by

$$\begin{aligned} G^0(x) &= \sum_{j=1}^{n}\int_0^{x_j}\left(A_1^j-A_0^j\right)(\underbrace{0,...,0}_{j-1},t_j,...,x_n)\,dt_j, \\ G^i(x) &= x_0\cdot A_1^i(x_1,...,x_n)+(1-x_0)\cdot A_0^i(x_1,...,x_n) \end{aligned}$$

for any $x=(x_0,x_1,...,x_n)\in\mathbb{R}^{n+1}$ and $i\in\{1,2,...,n\}$.

Then

$$G^i(0,\cdot)=A_0^i \text{ and } G^i(1,\cdot)=A_1^i \text{ for } i\in\{1,2,...,n\}.$$

Since H_0 and H_1 are endomorphisms of Lie algebroid $T\mathbb{R}^n\times\mathbb{R}$, therefore theorem 2.1 implies the equalities

$$\frac{\partial A_k^j}{\partial x_i}=\frac{\partial A_k^i}{\partial x_j}$$

for $k \in \{0,1\}$, $i,j \in \{1,2,...,n\}$ and $i \neq j$. Let $x = (x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1}$. Hence, for $i,j \in \{1,2,...,n\}$ such that $i \neq j$, we have, of cource, that

$$\left.\frac{\partial G^i}{\partial x_j}\right|_x = \left.\frac{\partial G^j}{\partial x_i}\right|_x .$$

Moreover,

$$\begin{aligned}
\left.\frac{\partial G^0}{\partial x_1}\right|_x &= \frac{\partial}{\partial x_1}\left(\sum_{j=1}^{n}\int_0^{x_j}\left(A_1^j - A_0^j\right)(\underbrace{0,...,0}_{j-1}, t_j, ..., x_n)\,dt_j\right) \\
&= \frac{\partial}{\partial x_1}\int_0^{x_1}\left(A_1^1 - A_0^1\right)(t_1, ..., x_n)\,dt_1 = \left(A_1^1 - A_0^1\right)(x_1, ..., x_n) = \left.\frac{\partial G^1}{\partial x_0}\right|_x
\end{aligned}$$

and

$$\begin{aligned}
\left.\frac{\partial G^0}{\partial x_i}\right|_x &= \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\int_0^{x_j}\left(A_1^j - A_0^j\right)(\underbrace{0,...,0}_{j-1}, t_j, ..., x_n)\,dt_j\right) = \\
&= \sum_{j=1}^{i-1}\int_0^{x_j}\frac{\partial\left(A_1^j - A_0^j\right)}{\partial x_i}(\underbrace{0,...,0}_{j-1}, t_j, ..., x_n)\,dt_j + \\
&\quad + \frac{\partial}{\partial x_i}\int_0^{x_i}\left(A_1^i - A_0^i\right)(\underbrace{0,...,0}_{i-1}, t_i, ..., x_n)\,dt_i \\
&= \sum_{j=1}^{i-1}\int_0^{x_j}\frac{\partial(A_1^i - A_0^i)}{\partial x_j}(\underbrace{0,...,0}_{j-1}, t_j, ..., x_n)\,dt_j + \left(A_1^i - A_0^i\right)(\underbrace{0,...,0}_{i-1}, x_i, ..., x_n) \\
&= \left(A_1^i - A_0^i\right)(x_1, x_2, ..., x_n) - \left(A_1^i - A_0^i\right)(0, x_2, ..., x_n) + \\
&\quad + \sum_{1<j<i}\left(A_1^i - A_0^i\right)(\underbrace{0,...,0}_{j-1}, x_j, ..., x_n) - \sum_{1<j<i}\left(A_1^i - A_0^i\right)(\underbrace{0,...,0}_{j}, x_{j+1}, ..., x_n) \\
&\quad + \left(A_1^i - A_0^i\right)(\underbrace{0,...,0}_{i-1}, x_i, ..., x_n) \\
&= \left(A_1^i - A_0^i\right)(x_1, x_2, ..., x_n) - \left(A_1^i - A_0^i\right)(0, x_2, ..., x_n) + \\
&\quad + \left(A_1^i - A_0^i\right)(0, x_2, ..., x_n) + \sum_{2<j\leq i}\left(A_1^i - A_0^i\right)(\underbrace{0,...,0}_{j-1}, x_j, ..., x_n) + \\
&\quad - \sum_{1<j<i}\left(A_1^i - A_0^i\right)(\underbrace{0,...,0}_{j}, x_{j+1}, ..., x_n) \\
&= \left(A_1^i - A_0^i\right)(x_1, ..., x_n) = \left.\frac{\partial G^i}{\partial x_0}\right|_x
\end{aligned}$$

for $i \in \{2, ..., n\}$. From this and theorem 3.2 we conclude that the endomorphism H_0 is homotopic to H_1. The proof is completed. ∎

4. FUNDAMENTAL GROUP OF A REGULAR LIE ALGEBROID

Let A be a regular Lie algedroid on a smooth manifold M. Consider the set

$$\pi_1(A) = \{[f]\,;\, f : A \to A\}$$

where $[f]$ denotes a class of strong automorphisms of the Lie algebroid A, strong homotopic to the automorphism $f : A \to A$, and define the product of two classes $[f]$, $[g] \in \pi_1(A)$ by

$$[f] \cdot [g] = [f \circ g]\,.$$

If $f \sim f' : A \longrightarrow A$ via a homotopy H_1, and $g \sim g' : A \longrightarrow A$ via a homotopy H_2, then $f \circ g \sim f' \circ g'$ via the homotopy $H = H_1 \circ (\mathrm{pr}_1, H_2)$ where $\mathrm{pr}_1 : T\mathbb{R} \times \mathbf{A} \longrightarrow \mathbb{TR}$ is the canonical projection. This observation gives the correctness of above definition.

In this way, $\pi_1(A)$ becomes a group, called the *fundamental group of the Lie algebroid* A.

Theorem 4.1. *The fundamental group* $\pi_1(T\mathbb{R}^n \times \mathbb{R})$ *is isomorphic to the linear group* $\mathrm{GL}(\mathbb{R})$.

Proof. Let $f : T\mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n \times \mathbb{R}$ be an automorphism of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$. On account of theorem 2.1, it is, for any $x,\ y \in \mathbb{R}^n$, $r \in \mathbb{R}$, of the form

$$f((x,y),r) = \left(x, y, \sum_{i=1}^{n} A_f^i(x) \cdot y_i + B_f \cdot r\right)$$

with $B_f \in \mathbb{R} \setminus \{0\}$ and functions $A_f^i \in C^\infty(\mathbb{R}^n)$ satisfying condition (2.1). It is clear that f defines a linear automorphism $a_f : \mathbb{R} \to \mathbb{R}$ by the formula $a_f(r) = B_f \cdot r$. Now, we define an isomorphism of groups $\Omega : \pi_1(T\mathbb{R}^n \times \mathbb{R}) \to \mathrm{GL}(\mathbb{R})$ by setting

$$[f] \mapsto a_f.$$

It is evident that Ω is an isomorphism. Let g be another automorphism of the Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$ and let, for any $x,\ y \in \mathbb{R}^n$, $r \in \mathbb{R}$,

$$g((x,y),r) = \left(x, y, \sum_{i=1}^{n} A_g^i(x) \cdot y_i + B_g \cdot r\right)$$

with $B_g \in \mathbb{R} \setminus \{0\}$ and $A_g^i \in C^\infty(\mathbb{R}^n)$ satisfying (2.1). Then

$$(f \circ g)((x,y),r) = \left((x,y), \sum_{i=1}^{n} \left(A_f^i(x) + B_f \cdot A_g^i(x)\right) \cdot y_i + B_f \cdot B_g \cdot r\right).$$

From this we obtain

$$\Omega([f] \cdot [g]) = \Omega([f \circ g]) = a_f \circ a_g = \Omega([f]) \circ \Omega([g])\,.$$

For this reason the mapping Ω is an isomorphism of the groups $\pi_1(T\mathbb{R}^n \times \mathbb{R})$ and $\mathrm{GL}(\mathbb{R})$. ■

Finally, we raise an open problem: investigate the fundamental group $\pi_1(A)$ for an arbitrary Lie algebroid A, first for $A = TM \times \mathfrak{g}$ (where $\mathfrak{g}$ is an arbitrary Lie algebra).

References

[1] P. J. Higgins, K. C. H. Mackenzie, *Algebraic constructions in the category of Lie algebroids*, J. Algebra 129 (1990), 194-230.

[2] J. Kubarski, *The Chern-Weil homomorphism of regular Lie algebroids*, Publ. Dép. Math. Univ. de Lyon 1, 1991.

[3] J. Kubarski, *Invariant cohomology of regular Lie algebroids*, Proceedings of the VII International Colloquium on Differential Geometry, "Analysis and Geometry in Foliated Manifolds", Santiago de Compostella, Spain, 26-30 July 1994, World Scientific Publishing Singapore-New Yersey-London-Hong Kong, 1995, 137-151.

[4] K. C. H. Mackenzie, *Lie grupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series 124, Cambridge University Press, 1987.

[5] Ngô van Quê, *Sur l'espace de prolongement différentiable*, Journal of Differential Geometry 2 (1968), 33-40.

[6] J. Pradines, *Théorie de Lie pour les grupoïdes différientiables. Calcul différentiel dans la catégorie des grupoïdes infinitésimaux*, C. R. Acad. Sc. Paris, Série A, t. 264 (1967), 245-248.

[7] J. Pradines, *Théorie de Lie pour les grupoïdes différientiables*, Atti del Convegano Internazionale di Geometria Differenziale (Bologna, 28-30, IX, 1967), Bologna-Amsterdam, 1967.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
Al. Politechniki 11
90-924 Łódź
Poland
bogdan@ck-sg.p.lodz.pl