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ON A CLASS OF POLYNOMIAL LAGRANGIANS

MARCELLA PALESE* AND RAFFAELE VITOLO

ABSTRACT. In the framework of finite order variational sequences a new class of Lagrangians arises, namely, *special* Lagrangians. These Lagrangians are the horizontalization of forms on a jet space of lower order. We describe their properties together with properties of related objects, such as Poincaré–Cartan and Euler–Lagrange forms, momenta and momenta of generating forms, a new geometric object arising in variational sequences. Finally, we provide a simple but important example of special Lagrangian, namely the Hilbert–Einstein Lagrangian.

1. INTRODUCTION

The theory of variational sequences provides a geometric framework for the calculus of variations. In this theory the Euler–Lagrange operator is just a morphism in an exact sequence of vector spaces (or sheaves of vector spaces). Geometric objects like Lagrangians, momenta, Poincaré–Cartan forms, Helmholtz conditions, find a nice interpretation in the vector spaces of the sequence.

We are concerned with some aspects of the theory of variational sequences in finite order jet spaces (see [17, 18, 20] for the basics on this subject), as was mainly developed in [14, 15] (see also [21, 22] for further developments). In this theory a subset of r–th order Lagrangians is selected in a natural way by the geometric structure of finite order jets. Namely, this distinguished subset is made by r–th order Lagrangians which are the horizontalization [14, 21, 22] of n–forms on the jet space of order $r-1$. Such Lagrangians are said to be *special*.

The aim of this paper is to study in detail the properties of special Lagrangians and related geometric objects.

In the second section, we review the main results on the geometry of spaces of jets J_rY of a fibration $Y \to X$. We recall that the tangent space TJ_rY has a natural splitting when pulled back to the higher order jet space $J_{r+1}Y$. Namely, it splits into

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the (pullback of the) vertical subbundle plus a bundle which is fiberwise isomorphic to TX. Then, we introduce horizontalization as the projection of forms on J_rY, or r–th order forms, on forms on $J_{r+1}Y$ having the highest exterior factor $\overset{k}{\wedge}T^*X$ in their target space. We then recall Krupka's theory of finite order variational sequences [14]. A variational sequence on J_rY is produced by taking the quotient of de Rham sequence on J_rY with respect to a sequence defined through the kernel of horizontalization. The commutative diagram built by the three sequences is said to be the (finite order) variational bicomplex.

In the third section, we describe the horizontalization of k–forms, with $k \leq n$ (here n is the dimension of the base manifold). Horizontal n–forms of order $r+1$ are usually interpreted as $(r+1)$–th order Lagrangians, [3, 5, 6, 9, 17, 19], but we prove that not any horizontal form of order $r+1$ is the horizontalization of some form of order r. We see that the components of horizontalized $(r+1)$–th order k–forms have polynomial coefficients of degree k in the derivatives of order $r+1$. Then, we define special Lagrangians of order $r+1$ to be n–forms coming from the horizontalization of a r–th order n–form. We see that horizontalization provides an isomorphism of the quotient space of n–forms in the variational sequence on J_rY with the space of $(r+1)$–th order special Lagrangians.

The fourth section is devoted to Euler–Lagrange forms. We recall that Euler–Lagrange forms are representatives of classes of $(n+1)$–forms in the variational sequence [22], through horizontalization and a geometric version of Green's formula [9]. In particular, we are able to split any horizontalized $(n+1)$–form, which we call generating form, into an Euler–Lagrange form (not necessarily induced by a Lagrangian) and the horizontal differential (i.e. the total divergence) of a form, which is said to be a momentum for the generating form. These momenta were first introduced in [22], but here we study their properties in detail. Then, we prove that it is possible to compute the Euler–Lagrange form for special Lagrangians both in the standard way and by using the commutativity of the variational bicomplex. Finally, we describe the polynomial structure of the Euler–Lagrange forms induced by special Lagrangians.

The fifth section contains a description of properties of momenta of generating forms and their relationship with standard momenta of (special) Lagrangians. We give a detailed analysis of their uniqueness properties. Namely, we prove that such momenta are uniquely determined either for $\dim X = 1$ or for generating forms of order 2. We show that such a momentum can be naturally determined for generating forms of order 3. We think that momenta for generating forms could play an important role in *multisymplectic theories* (see [7, 8] and their rich bibliography). These theories are a generalization of symplectic formalism to field theory. They all involve a closed $(n+1)$–form Ω on J_1Y as the main geometric object. An analysis of these theories with the powerful tool of variational sequences has never been attempted. Indeed, field equations can be easily recovered via the Euler–Lagrange form induced by the generating form $h(\Omega)$. Here, momentum should play an essential role. This will be the subject of further studies. This is also a good motivation for introducing and studying such objects.

In the sixth section, we give a characterization of Poincaré–Cartan forms for both special and general Lagrangians. Namely, we prove that a form θ is a Poincaré–Cartan form for a given Lagrangian if the Lagrangian is the horizontalization of θ, the vertical part of θ is in the space of momenta and the momentum of the generating form $h(d\theta)$ can be chosen to be zero. Of course, this can also be taken as a definition of Poincaré–Cartan form inspired by the variational sequences (and close to the theory of Lepagian equivalents [13, 15, 16]).

In the last section, we will show a relevant example of special Lagrangian, namely the Hilbert–Einstein Lagrangian. We provide also the related objects, such as the Poincaré–Cartan form, the momentum, the Euler–Lagrange form and the momentum of the natural generating form.

Here, manifolds and maps between manifolds are assumed to be C^∞.

2. Jet spaces and variational sequences

In this section we recall some basic facts about jet spaces [2, 18, 20] and Krupka's formulation of the finite order variational sequence [14, 22].

Our framework is a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

For $r \geq 0$ we are concerned with the r–jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^r_{r-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

Charts on $\boldsymbol{Y}$ adapted to π are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \dots$ run from 1 to n and they label base coordinates, while Latin indices $i, j, \dots$ run from 1 to m and label fiber coordinates, unless otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i) the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart, respectively.

We denote multi–indices of dimension n by the boldface Greek letters $\boldsymbol{\gamma}, \boldsymbol{\delta}$. We have $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ with $0 \leq \gamma_\mu$, $\mu = 1, \dots, n$; by an abuse of notation, we denote by λ the multi–index such that $\gamma_\mu = 0$ if $\mu \neq \lambda$, $\gamma_\mu = 1$ if $\mu = \lambda$. We also set $|\boldsymbol{\gamma}| := \gamma_1 + \cdots + \gamma_n$ and $\boldsymbol{\gamma}! := \gamma_1! \dots \gamma_n!$.

The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^\lambda, y^i_{\boldsymbol{\gamma}})$, with $0 \leq |\boldsymbol{\gamma}| \leq r$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\gamma}}_i)$ and $(d^i_{\boldsymbol{\gamma}})$, respectively.

In the theory of variational sequences a fundamental role is played by the *contact maps* on jet spaces (see [2, 18, 20]). Namely, for $r \geq 1$, we consider the natural complementary fibered morphisms over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$\mathcal{D} : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y}\,, \qquad \vartheta : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y}\,,$$

with coordinate expressions, for $0 \leq |\boldsymbol{\gamma}| \leq r - 1$, given by

$$\mathcal{D} = d^\lambda \otimes \mathcal{D}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\gamma}+\lambda} \partial^{\boldsymbol{\gamma}}_j)\,, \qquad \vartheta = \vartheta^j_{\boldsymbol{\gamma}} \otimes \partial^{\boldsymbol{\gamma}}_j = (d^j_{\boldsymbol{\gamma}} - y^j_{\boldsymbol{\gamma}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\gamma}}_j\,.$$

We have

$$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \left(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \right) \oplus \overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}]\,, \tag{1}$$

where $\overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}] := \operatorname{im} \vartheta^*_r$.

Now, we introduce some distinguished sheaves of forms on jet spaces [22]. Let $k \geq 0$.

1. For $r \geq 0$, we consider the standard sheaf $\overset{k}{\Lambda}_r$ of *k-forms* on $J_r\boldsymbol{Y}$. We have the coordinate expression

$$\alpha = \alpha^{\gamma_1 \dots \gamma_h}_{i_1 \dots i_h}{}_{\lambda_{h+1} \dots \lambda_k} d^{i_1}_{\gamma_1} \wedge \dots \wedge d^{i_h}_{\gamma_h} \wedge d^{\lambda_{h+1}} \wedge \dots \wedge d^{\lambda_k} .$$

2. For $0 \leq s \leq r$, we consider the sheaves $\overset{k}{\mathcal{H}}_{(r,s)}$ and $\overset{k}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibered morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ and $J_r\boldsymbol{Y} \to \boldsymbol{X}$ of the type, respectively,

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* J_s\boldsymbol{Y} \qquad \textit{and} \qquad \lambda : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* \boldsymbol{X} ;$$

   in coordinates $\lambda = \lambda_{\lambda_1 \dots \lambda_k} d^{\lambda_1} \wedge \dots \wedge d^{\lambda_k}$.
3. Furthermore, we consider the subsheaf $\overset{k}{\mathcal{H}}{}^P_r \subset \overset{k}{\mathcal{H}}_r$ of local fibered morphisms $\alpha \in \overset{k}{\mathcal{H}}_r$ such that α is a *polynomial* fibered morphism over $J_{r-1}\boldsymbol{Y} \to \boldsymbol{X}$ of degree k. In coordinates, the components $\lambda_{\lambda_1 \dots \lambda_n}$ are polynomials in y^i_γ of degree k, where $|\gamma| = r$.
4. For $0 \leq s < r$, we consider the subsheaf $\overset{k}{\mathcal{C}}_{(r,s)} \subset \overset{k}{\mathcal{H}}_{(r,s)}$ of *contact forms*, i.e. of local fibered morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ of the type

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} \overset{*}{\mathcal{C}}_s[\boldsymbol{Y}] \subset \overset{k}{\wedge} T^* J_s\boldsymbol{Y} ,$$

   and the subsheaf $\overset{k}{\mathcal{C}}_r \subset \overset{k}{\mathcal{C}}_{(r+1,r)}$ of local fibered morphisms $\alpha \in \overset{k}{\mathcal{C}}_{(r+1,r)}$ such that $\alpha = \overset{p}{\wedge} \vartheta^*_{r+1} \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of the fibration $J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} \overset{p}{\wedge} V^* J_r\boldsymbol{Y} \to J_{r+1}\boldsymbol{Y}$ which projects down onto $J_r\boldsymbol{Y}$.

The fibered splitting (1) yields the sheaf splitting

$$\overset{k}{\mathcal{H}}_{(r+1,r)} = \oplus_{l=0}^k \overset{k-l}{\mathcal{C}}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1} \tag{2}$$

[20, 22]. We set h to be the restriction to $\overset{k}{\Lambda}_r$ of the projection of the above splitting on the term with the highest degree of the horizontal factor. We set also v to be the complementary projection $v := \mathrm{id} - h$. We say h to be the *horizontalization* of forms on jet spaces.

The splitting (1) induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$, where d_H and d_V are defined to be the *horizontal* and the *vertical differential* [20].

We recall now Krupka's variational sequence on finite order jet spaces [14].

Let us denote by $\overline{d \ker h}$ the sheaf generated by the presheaf $d \ker h$ (see [23]). We set $\overset{*}{\Theta}_r := \ker h + \overline{d \ker h}$. In [14] it is proved that the following diagram is commutative and that its rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Theta}_r \arrow[r, "d"] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r \arrow[r, "\mathcal{E}_2"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_{I-1}"] & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r \arrow[r, "\mathcal{E}_I"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & &
\end{tikzcd}$$

The top row of the above diagram is said to be the r–th order *contact sequence* and the bottom row is said to be the r–th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$ (see [14, 22] for the relationship with calculus of variations).

The variational sequence can be read through some intrinsic isomorphisms of quotient sheaves with sheaves of forms on jets [22]. This shows the connection of the variational sequence with the geometric formulations of the calculus of variations [3, 5, 6, 9, 17, 19]. Here, we are concerned with the columns of n and $n+1$ forms.

3. Special Lagrangians

In this section, we introduce special Lagrangians as distinguished representatives of equivalence classes in $\overset{n}{\Lambda}_r/\overset{n}{\Theta}_r$. More precisely, this representative will be obtained through horizontalization.

For $k \leq n$, let us set

$$\overset{k}{\mathcal{H}}{}^h_{r+1} := h(\overset{k}{\Lambda}_r)\,.$$

We say $\overset{k}{\mathcal{H}}{}^h_{r+1}$ to be the sheaf of *special horizontal forms* of order $r+1$.

Special horizontal k–forms are k–th degree polynomial in higher order derivatives, i.e. $\overset{k}{\mathcal{H}}{}^h_{r+1} \subset \overset{k}{\mathcal{H}}{}^P_{r+1}$. In fact, if $\alpha \in \overset{k}{\Lambda}_r$, then

$$h(\alpha) = y^{i_1}_{\gamma_1+\lambda_1} \cdots y^{i_h}_{\gamma_h+\lambda_h} \alpha^{\gamma_1\ldots\gamma_h}_{i_1\ldots i_h}{}_{\lambda_{h+1}\ldots\lambda_k} d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_k}\,,$$

with $0 \leq h \leq k$.

Remark 3.1. The sheaf $\overset{k}{\mathcal{H}}{}^h_{r+1}$ admits the following characterization [22]: a section $\alpha \in \overset{k}{\mathcal{H}}{}^P_{r+1}$ is a section of the subsheaf $\overset{k}{\mathcal{H}}{}^h_{r+1}$ if and only if there exists a section $\beta \in \overset{k}{\Lambda}_r$ such that

$$(j_r\sigma)^*\beta = (j_{r+1}\sigma)^*\alpha$$

for each section $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$.

If $\dim \boldsymbol{X} = 1$ then the inclusion $\overset{k}{\mathcal{H}}{}^h_{r+1} \subset \overset{k}{\mathcal{H}}{}^P_{r+1}$ is an equality. In fact, in this case the above coordinate expression turns out to be the general coordinate expression for a section of $\overset{1}{\mathcal{H}}{}^P_{r+1}$.

If $\dim \boldsymbol{X} \neq 1$, then the inclusion $\overset{k}{\mathcal{H}}{}^h_{r+1} \subset \overset{k}{\mathcal{H}}{}^P_{r+1}$ *is not* an equality, in general, due to the above characterization. We can check it via the following example. Consider a 1–form $\beta \in \overset{1}{\Lambda}_0$. Then we have the coordinate expressions $\beta = \beta_\lambda d^\lambda + \beta_i d^i$, $h(\beta) = (\beta_\lambda + y^i_\lambda \beta_i) d^\lambda$. If $\alpha \in \overset{1}{\mathcal{H}}{}^P_1$, then we have the coordinate expression $\alpha = (\alpha_\lambda + y^i_\mu \alpha^\mu_i{}_\lambda) d^\lambda$. It is evident that, in general, there does not exist $\beta \in \overset{1}{\Lambda}_r$ such that $h(\beta) = \alpha$.

Let us recall that, according to the standard definition, an r–th order *Lagrangian* is defined to be a form $\lambda \in \overset{n}{\mathcal{H}}_r$ [3, 5, 6, 9, 17, 19].

The horizontalization induces a natural sheaf isomorphism between $\overset{n}{\Lambda}_r/\overset{n}{\Theta}_r$ and $\overset{n}{\mathcal{H}}{}^h_{r+1}$. This motivates the following definition.

Definition 3.2. We say forms in $\overset{n}{\mathcal{H}}{}^h_{r+1}$ to be *special Lagrangians* of order $r+1$.

We also say a Lagrangian $\lambda \in \overset{n}{\mathcal{H}}_r$ to be *general* if it is not special. Equivalently, λ is general either if it is not the horizontalization of a form in $\overset{n}{\Lambda}_{r-1}$, or if $\lambda \notin \overset{n}{\mathcal{H}}{}^h_r$.

Remark 3.3. Special Lagrangians of order $r+1$ differs from both general and polynomial Lagrangians of order $r+1$ for one essential feature: they come from a form in $\overset{n}{\Lambda}_r$ through horizontalization.

4. Euler–Lagrange forms and special Lagrangians

Here we describe the properties of Euler–Lagrange forms induced by special Lagrangians. We see that any Euler–Lagrange form (even not induced by a Lagrangian) is obtained from a horizontalized $(n+1)$–form by adding a suitable form which is an exact horizontal differential. The horizontalized $(n+1)$–form is said to be a generating form, while a (horizontal) potential of the exact form is said to be a momentum for the Euler–Lagrange form. Then, we prove that it is possible to compute the Euler–Lagrange form for special Lagrangians both in the standard way and by using the commutativity of the variational bicomplex. Finally, we describe the structure of Euler–Lagrange forms of special Lagrangians.

The horizontalization induces the natural injective sheaf morphism

$$\left(\overset{n+1}{\Lambda}{}_r/\overset{n+1}{\Theta}{}_r\right) \to \left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right)/h(\overline{d\ker h}) : [\alpha] \mapsto [h(\alpha)] .$$

Then, it can be proved that $h(\overline{d\ker h}) \subset \overline{d_H(\overset{1}{\mathcal{C}}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1})}$ [22]. So, we can use Kolář's geometric version of Green's formula to provide an isomorphism of the above quotient sheaf with a sheaf of forms on jet spaces. Namely, Let us consider $h(\alpha) \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$;

such a form is said to be a *generating form*. It is proved in [9] that for any generating form $h(\alpha)$ then there is a unique pair of sheaf morphisms

$$E_{h(\alpha)} \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}{}^{h}_{2r+1}\,, \qquad F_{h(\alpha)} \in \overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n}{\mathcal{H}}{}^{h}_{2r}\,, \tag{3}$$

such that $h(\alpha) = E_{h(\alpha)} + F_{h(\alpha)}$ and $F_{h(\alpha)}$ is locally of the form $F_{h(\alpha)} = d_H p_{h(\alpha)}$, with $p_{h(\alpha)} \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}^{h}_{2r}$. Note that a *global* section $p_{h(\alpha)}$ such that $F_{h(\alpha)} = d_H p_{h(\alpha)}$ always exists [2, 3, 5, 9], essentially due to the fact that d_H has zero cohomology when restricted on certain subsequences (see [1] for a deeper discussion).

Definition 4.1. Let $\alpha \in \overset{n+1}{\Lambda}_r$. Then any form $p_{h(\alpha)}$ is said to be a *momentum* of the generating form $h(\alpha)$.

Notice that we are able to consider momentum also for Euler–Lagrange forms which are not variational, i.e. which do not come from any Lagrangian.

Remark 4.2. We think that momenta of this kind could play an important role in the study of *multisymplectic theories* (see [7, 8] and their rich bibliography). These theories are a generalization of symplectic formalism to field theory and all of them involve a closed $(n+1)$–form Ω on $J_1\boldsymbol{Y}$ as the main geometric object. An analysis of these theories with the powerful tool of variational sequences has never been attempted. Indeed, field equations can be easily recovered via the Euler–Lagrange form induced by the generating form $h(\Omega)$. Here, momentum could play an essential role.

The above yields [22] the sheaf isomorphism

$$\left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^{h}_{r+1}\right) / h(\overline{d \ker h}) \to \overset{n+1}{\mathcal{V}}_r : [h(\alpha)] \mapsto E_{h(\alpha)}\,, \tag{4}$$

where $\overset{n+1}{\mathcal{V}}_r := \left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^{h}_{r+1} + d_H(\overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r})\right) \cap \left(\overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1}\right)$. It is now clear that generating forms of order $r+1$ provide all Euler–Lagrange forms in the quotient space of $(n+1)$–forms in the variational sequence of order r.

Let us recall the standard definition of Euler–Lagrange form and momentum for a Lagrangian $\lambda \in \overset{n}{\mathcal{H}}_r$ [2, 3, 5, 9, 17]. We apply (3) to obtain $d\lambda = E_{d\lambda} + d_H p_{d\lambda}$ for any choice of $p_{d\lambda}$. We say

- $E_{d\lambda}$ to be the *Euler–Lagrange form* of the Lagrangian λ;
- $p_{d\lambda}$ to be a *momentum* of the Lagrangian λ.

The momentum of a Lagrangian is uniquely defined only in some special cases [2, 3, 5, 9]. Namely, either if $\dim \boldsymbol{X} = 1$ or if $r = 1$. If $r = 2$ then we are able to naturally determine $p_{d\lambda}$ through a further assumption [9]. If $r = 3$ then there does not exist, in the general situation, a natural $p_{d\lambda}$ [10]. Anyway, an intrinsic choice of $p_{d\lambda}$ is always possible [9].

We show that the operator $\mathcal{E}_n$ of the variational sequence associates to any Lagrangian its Euler–Lagrange form through the above isomorphism (4).

Proposition 4.3. *Let $\lambda \in \overset{n}{\mathcal{H}}{}^{h}_{r+1}$ and $\beta \in \overset{n}{\Lambda}_r$ such that $h(\beta) = \lambda$. Then we have $\mathcal{E}_n(\lambda) = E_{h(d\beta)}$. Moreover, $E_{h(d\beta)} = E_{d\lambda}$.*

Proof. By the above decomposition formula, $h(d\beta) = E_{h(d\beta)} + d_H p_{h(d\beta)}$ for any choice of $p_{h(d\beta)}$. But the commutativity of the diagram

$$\begin{array}{ccc} \overset{n}{\Lambda}_r & \xrightarrow{d} & \overset{n+1}{\Lambda}_r \\ \downarrow & & \downarrow \\ \overset{n}{\mathcal{H}}{}^h_r & \xrightarrow{\mathcal{E}_n} & \overset{n+1}{\mathcal{V}}_r \end{array}$$

yields $\mathcal{E}_n(\lambda) = E_{h(d\beta)}$. As for the second result, we consider λ as being a form $\lambda \in \overset{n}{\Lambda}_{r+1}$. In this case, $\mathcal{E}_n(\lambda) = E_{d\lambda}$. By the inclusion of the r-th variational bicomplex into the $(r+1)$-th one [14, 22], we obtain $E_{h(d\beta)} = E_{d\lambda}$. □

If $\lambda \in \overset{n}{\mathcal{H}}_r$ is general, then the form $E_{d\lambda}$ is defined on $J_{2r}\boldsymbol{Y}$, and has a peculiar structure with respect to the derivative coordinates of order greater than r. In fact, if we assign to the variables y^i_γ with $|\gamma| = r + s$ the weight s, then it is easily seen that $E_{d\lambda}$ is a polynomial with weighted degree r with respect to y^i_γ, with $|\gamma| > r$ [12].

Corollary 4.4. *If $\lambda \in \overset{n}{\mathcal{H}}_r$ is special, then the form $E_{d\lambda}$ is defined on $J_{2r-1}\boldsymbol{Y}$, and the coefficients of the polynomials in $E_{d\lambda}$ are polynomials of (standard) degree $n+1$ with respect to the coordinates y^i_γ, with $|\gamma| = r+1$.*

5. Momentum and special Lagrangians

Now, we describe general properties of momentum for generating forms $h(\alpha) \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$. Then, we see the relationship with momenta for special Lagrangians.

We recall the coordinate expression $h(\alpha) = \tilde{\alpha}^\gamma_i \vartheta^i_\gamma \wedge \omega$, where $\tilde{\alpha}^\gamma_i$ are polynomials of (standard) degree $n+1$ with respect to the coordinates y^i_γ, with $|\gamma| = r+1$, with coefficients the components of α.

As we already said, *global momenta* $p_{h(\alpha)}$ for any generating form $h(\alpha)$ exist. This is essentially due to the fact that d_H has zero cohomology. A proof of this can be found in an early work by Kolář (see references in [9]). See also [1] for a cohomological proof.

Then, we check uniqueness properties of $p_{h(\alpha)}$. Of course, if $\dim \boldsymbol{X} = 1$ then $p_{h(\alpha)}$ is unique. This is because $d_H p_{h(\alpha)} = 0$ implies $p_{h(\alpha)} = 0$, as it is easily seen in coordinates.

Remark 5.1. There exists a natural sheaf morphism [10, 19, 20, 22]

$$p : \overset{1}{\mathcal{C}}_{(r,1)} \wedge \overset{n}{\mathcal{H}}_r \to \overset{1}{\mathcal{C}}_{(r,0)} \wedge \overset{n-1}{\mathcal{H}}_r \,.$$

If $\phi \in \overset{1}{\mathcal{C}}_{(r,1)} \wedge \overset{n}{\mathcal{H}}_r$ has the coordinate expression $\phi = \phi_i \vartheta^i \wedge \omega + \phi^\lambda_i \vartheta^i_\lambda \wedge \omega$, then we have the coordinate expression $p_\phi = \phi^\lambda_i \vartheta^i \wedge \omega_\lambda$.

Theorem 5.2. (Uniqueness I). *Let $\alpha \in \overset{n}{\Lambda}_1$. Then, the momentum $p_{h(\alpha)}$ of $h(\alpha)$ is unique. We have the coordinate expression*

$$p_{h(\alpha)} = \tilde{\alpha}^\lambda_i \vartheta^i \wedge \omega_\lambda \,.$$

Proof. In fact, we deduce the above coordinate expression from (3). Then, it is clear that $p_{h(\alpha)}$ is defined up a n–form whose horizontal differential vanish. It is easy to see in coordinates that such a form must be zero. □

Remark 5.3. It is easy to verify that if we start with $\alpha \in \overset{n}{\Lambda}_2$ we obtain $h(\alpha) \in \overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{n}{\mathcal{H}}{}^h_3$, so $h(\alpha)$ *is not* in the domain of p.

In the case $r = 2$ there is not a unique choice of momentum for the generating form $h(\alpha)$. But we are able to choose it in a natural way.

Remark 5.4. There exists a natural sheaf morphism [10, 22]

$$s : \overset{1}{\mathcal{C}}_{(r,1)} \wedge \overset{n-1}{\mathcal{H}}_r \to \overset{1}{\mathcal{C}}_{(r,0)} \wedge \overset{n-2}{\mathcal{H}}_r .$$

If $p \in \overset{1}{\mathcal{C}}_{(r,1)} \wedge \overset{n-1}{\mathcal{H}}_r$ has the coordinate expression $p = p_i{}^{\mu}\, \vartheta^i \wedge \omega_\mu + p_i^{\lambda\mu}\, \vartheta^i_\lambda \wedge \omega_\mu$, then we have the coordinate expression $s(p) = p_i^{\lambda\mu}\, \vartheta^i \wedge \omega_{\lambda\mu}$.

Theorem 5.5. (Uniqueness II). *Let $\alpha \in \overset{n}{\Lambda}_2$. Then, there exists a unique momentum $p_{h(\alpha)}$ of $h(\alpha)$ such that $s(p_{h(\alpha)}) = 0$. We have the coordinate expression*

$$p_{h(\alpha)} = (\tilde{\alpha}_i^\lambda - D_\mu \tilde{\alpha}_i^{\mu+\lambda})\, \vartheta^i \wedge \omega_\lambda + \tilde{\alpha}_i^{\mu+\lambda}\, \vartheta^i_\mu \wedge \omega_\lambda .$$

Proof. Suppose that $p_{h(d\beta)} = p_i^\lambda \vartheta^i \wedge \omega_\lambda + p_i^{\lambda\mu} \vartheta^i_\mu \wedge \omega_\lambda$. Then $s(p_{h(\alpha)}) = 0$ yields $p_i^{\lambda\mu} = -p_i^{\mu\lambda}$. By (3) one obtains the above $p_{h(d\beta)}$ as the unique momentum fulfilling the above requirement. □

Remark 5.6. It is easy to verify that if we start with $h(\alpha) \in \overset{n}{\Lambda}_3$ then we obtain $h(\alpha) \in \overset{1}{\mathcal{C}}_{(4,2)} \wedge \overset{n}{\mathcal{H}}{}^h_4$, hence $p_{h(\alpha)} \in \overset{1}{\mathcal{C}}_{(5,2)} \wedge \overset{n}{\mathcal{H}}{}^h_5$, so that $p_{h(\alpha)}$ *is not* in the domain of s.

Remark 5.7. The reader could have realized that the above proofs go in the same way as in the case of general Lagrangians λ [9]. The difference is that here we used generating forms $h(\alpha)$ instead. This means that, even if results refer to orders 1 and 2 as in the case of Lagrangians, generating forms are of order 2 and 3, respectively.

Now, we deal with the interplay between the two kind of momenta that we introduced: momenta of (special) Lagrangians and momenta of generating forms. Let $\lambda \in \overset{n}{\mathcal{H}}{}^h_{r+1}$ be a special Lagrangian. Then, there exists $\beta \in \overset{n}{\Lambda}_r$ such that $h(\beta) = \lambda$. So, we can consider the generating form $h(d\beta)$ and evaluate its momentum $p_{h(d\beta)}$. It is natural to ask the relationship between the momentum $p_{d\lambda}$ of λ and the momentum $p_{h(d\beta)}$ of $h(d\beta)$.

First of all, we note that β is not unique, hence all uniqueness results referring to $p_{h(d\beta)}$ that we evaluated above cannot be related to λ.

Theorem 5.8. *We have $h(d\beta) = h(d_H\, v(\beta)) + d\lambda$, hence the momenta $p_{h(d\beta)}$ and $p_{d\lambda}$ can be chosen to be equal if and only if $h(d_H\, v(\beta)) = 0$.*

Proof. In fact,

$$\begin{aligned} h(d\beta) &= h((d_H + d_V)(\lambda + v(\beta))) \\ &= h(d_H\, v(\beta) + d_V\lambda + d_V\, v(\beta)) \\ &= h(d_H\, v(\beta)) + d_V\lambda\,, \end{aligned}$$

where, in this case, $d_V\lambda = d\lambda$.

Corollary 5.9. *Let $\lambda \in \overset{n}{\mathcal{H}}_r \subset \overset{n}{\mathcal{H}}{}^h_{r+1}$ be a general Lagrangian. Then, the momenta $p_{h(d\beta)}$ and $p_{d\lambda}$ can be chosen to be equal.*

Proof. In fact, in this case $\beta = \lambda$ hence $v(\beta) = 0$ and we can choose $p_{h(d\beta)} = p_{d\lambda}$. □

6. Poincaré–Cartan forms and special Lagrangians

Here, we give a characterization of Poincaré–Cartan forms in the framework of variational sequences. This characterization is inspired by and formulated through special $(r+1)$–th order Lagrangians, but obviously it holds also for general Lagrangians of any order.

We recall that, given a Lagrangian $\lambda \in \overset{n}{\mathcal{H}}_r$, we define the form $\theta_\lambda := \lambda + p_{d\lambda} \in \overset{n}{\Lambda}_{2r-1}$ to be a *Poincaré–Cartan form* [2, 3, 5, 9, 17, 19, 20]. Such a definition is motivated by the fact that the differential of the Poincaré–Cartan form splits into the sum of the Euler–Lagrange form for λ plus a contact form, namely $d\theta_\lambda = E_{d\lambda} + d_V p_{d\lambda}$. Uniqueness consideration for the Poincaré–Cartan form are the same as momentum (3).

Our characterization of Poincaré–Cartan forms is inspired by the fact that we can choose zero momentum for the generating form $h(d\theta_\lambda)$!

Theorem 6.1. *Let $\lambda \in \overset{n}{\mathcal{H}}{}^h_{r+1}$ be a special Lagrangian. Then there exists a unique class of forms $\theta \in \overset{n}{\Lambda}_{2r}$ fulfilling*

1 - $h(\theta) = \lambda$;

2 - $v(\theta) \in \overset{1}{\mathcal{C}}_{2r} \wedge \overset{n-1}{\mathcal{H}}_{2r}$;

3 - $h(d\theta) = E_{h(d\theta)}$, or we can choose zero momentum for the generating form $h(d\theta)$.

Namely, $\theta = \theta_\lambda$.

Proof. In fact, requirements 1 and 2 imply that θ should be of the form $\theta = \lambda + p$, with $p \in \overset{1}{\mathcal{C}}_{2r} \wedge \overset{n-1}{\mathcal{H}}_{2r}$. Now,

$$h(d\theta) = h(d_H p) + d_V\lambda = h(d_H p) + E_{d\lambda} - d_H p_{d\lambda}$$

But $h(d\theta) = E_{h(d\theta)} = E_{d\lambda}$ due to proposition 4.3. Moreover, requirement 2 implies $h(d_H p) = d_H p$. Summing up, $d_H(p - p_{d\lambda}) = 0$, hence p is also a momentum for λ.

Conversely, it is trivial to see that Poincaré–Cartan forms fulfill the requirements of the theorem. □

Remark 6.2. We would like to justify the requirements of the above theorem. The first requirement is obviously necessary. The second requirement is a requirement of 'minimality' of the vertical part of θ with respect to the splitting 2. The third requirement is inspired by the main property of Poincaré–Cartan forms that we recalled at the beginning of the section.

Remark 6.3. Of course, the requirements of the above theorem could be taken as a definition of Poincaré–Cartan form naturally provided by variational sequences. This in the same spirit as definitions of Lagrangians, Euler–Lagrange forms and momenta in the above framework. Moreover, we stress that the structure of the variational sequence, via Proposition 4.3, characterizes the Poincaré–Cartan form as a Lepagean equivalent of λ (see e.g. [13, 15, 16]). The last requirement of the above theorem explicitely expresses that the generating form of the Poincaré–Cartan form coincides with the associated Euler–Lagrange form.

7. The Hilbert–Einstein Lagrangian

In this brief section we show an important and simple example of special Lagrangian, namely the Hilbert–Einstein Lagrangian. We also derive all related geometric objects like the momentum of the Hilbert–Einstein Lagrangian, its Euler–Lagrange form and the momentum of the Euler–Lagrange form.

Let $\dim \boldsymbol{X} = 4$ and $\boldsymbol{X}$ be orientable. Let $Lor(\boldsymbol{X})$ be the bundle of Lorenzian metrics on $\boldsymbol{X}$ (provided that it exists). Local fibered coordinates on $J_2(Lor(\boldsymbol{X}))$ are $(x^\lambda; g_{\mu\nu}, g_{\mu\nu,\sigma}, g_{\mu\nu,\sigma\rho})$.

The Hilbert–Einstein Lagrangian is the form $\lambda_{HE} \in \overset{4}{\mathcal{H}}_2$ defined by $\lambda_{HE} = L_{HE}\omega$, were $L_{HE} = r\sqrt{\boldsymbol{g}}$. Here $r : J_2(Lor(\boldsymbol{X})) \to \mathbb{R}$ is the function such that, for any Lorenz metric g, we have $r \circ j_2 g = s$, being s the scalar curvature associated with g, and $\boldsymbol{g}$ is the determinant of g.

The function L_{HE} is a *linear* function in the second derivatives of g. In fact, let us set $G^{\alpha\beta\epsilon\gamma} := g^{\alpha\epsilon}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\epsilon} - 2g^{\alpha\beta}g^{\epsilon\gamma}$; then we have [4]

$$r = \frac{1}{2}G^{\alpha\beta\epsilon\gamma}\left(g_{\epsilon\gamma,\alpha\beta} + g_{\mu\nu}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\epsilon\gamma}\right) .$$

We can prove even more. Indeed, $\lambda_{HE} \in \overset{4}{\mathcal{H}}{}^h_2$. In fact, the momentum for the second order Lagrangian λ_{HE} (in the sense of [9]) turns out to be [4]

$$\begin{aligned} p_{\lambda_{HE}} = {} & \frac{1}{2}\left(G^{\alpha\beta\epsilon\gamma}g_{\mu\nu}\partial^{\mu\nu,\lambda}\left(\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\epsilon\gamma}\right) - \mathcal{D}_\rho(G^{\lambda\rho\mu\nu}\sqrt{\boldsymbol{g}})\right)\vartheta_{\mu\nu}\wedge\omega_\lambda + \\ & \frac{1}{2}G^{\lambda\rho\mu\nu}\sqrt{\boldsymbol{g}}\vartheta_{\mu\nu,\rho}\wedge\omega_\lambda , \end{aligned}$$

and the Poincaré–Cartan form

$$\begin{aligned} \theta_{\lambda_{HE}} = {} & \frac{1}{2}G^{\alpha\beta\epsilon\gamma}g_{\mu\nu}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\epsilon\gamma}\sqrt{\boldsymbol{g}}\,\omega + \\ & \frac{1}{2}\left(G^{\alpha\beta\epsilon\gamma}g_{\mu\nu}\partial^{\mu\nu,\lambda}\left(\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\epsilon\gamma}\right) - \mathcal{D}_\rho(G^{\lambda\rho\mu\nu}\sqrt{\boldsymbol{g}})\right)\vartheta_{\mu\nu}\wedge\omega_\lambda + \\ & \frac{1}{2}G^{\lambda\rho\mu\nu}\sqrt{\boldsymbol{g}}\vartheta_{\mu\nu,\rho}\wedge\omega_\lambda . \end{aligned}$$

Of course, $\theta_{\lambda_{HE}} \in \overset{4}{\Lambda}_1$. Moreover, a direct computation shows that

$$h(\theta_{\lambda_{HE}}) = \lambda_{HE} .$$

So, λ_{HE} *is a special Lagrangian* $(r = 1)$.

In view of the previous results, its Euler–Lagrange form should be an element $E_{d\lambda_{HE}} \in \overset{1}{\mathcal{C}}_{(2,0)} \wedge \overset{4}{\mathcal{H}}_3$. But, due to a property of λ_{HE} [4], we have $E_{d\lambda_{HE}} \in \overset{1}{\mathcal{C}}_{(2,0)} \wedge \overset{4}{\mathcal{H}}_2$. Of course, a direct computation shows that $E_{d\lambda_{HE}} = G := R - \frac{1}{2}\, s\, g$, R being the Ricci tensor of the metric g.

Another important consideration is that we can also compute $E_{d\lambda_{HE}}$ through proposition 4.3, namely as $E_{d\lambda_{HE}} = E_{h(d\beta)}$. In this case, we have a natural candidate of β, namely we can take $\beta = \theta_{\lambda_{HE}} \in \overset{4}{\Lambda}_1$. So,

$$d\theta_{\lambda_{HE}} = E_{d\lambda_{HE}} + d_V p_{d\lambda_{HE}}$$

(see the above section), which yields the natural generating form $h(d\theta_{\lambda_{HE}}) = E_{d\lambda_{HE}} = E_{h(d\theta_{\lambda_{HE}})}$. So, by theorem 5.2, the unique momentum of the generating form $h(d\theta_{\lambda_{HE}})$ is the zero form. This very peculiar behaviour is due to the geometric structure of general relativity. It is also an example of a special Lagrangian with a non trivial momentum and whose momentum of the natural generating form vanishes.

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