Piotr Multarzyński; Wiesław Sasin

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ALGEBRAIC CHARACTERIZATION OF THE DIMENSION OF DIFFERENTIAL SPACES

Piotr Multarzyński, Wiesław Sasin

This work is a continuation of our previous investigations of the dimension problem for the tangent space to a differential space at a point [1]. Here we present a full characterization of the tangent space dimension basing on algebraic properties of the linear ring of all smooth functions on a differential space in the sense of Sikorski [7],[8].

1. PRELIMINARIES. Let M be any set and let C be any non-empty set of real functions on M. By τ_C we shall denote the weakest topology on M in which all functions from C are continuous. For any subset $A \subset M$, let C_A be the set of all real functions β on A such that, for any $p \in A$, there exist an open neighbourhood $U \in \tau_C$ of p and a function $\alpha \in C$ such that $\beta | A \cap U = \alpha | A \cap U$. By scC we shall denote the family of all real functions on M of the form $\omega \circ (\alpha_1, \dots, \alpha_n)$, where $\omega \in \mathcal{E}_n$, $\alpha_1, \dots, \alpha_n \in C$, $n \in \mathbb{N}$, and $\mathcal{E}_n = C^\infty(\mathbb{R}^n)$.

A family C of real functions on M is called the differential structure (shortly a d-structure) on M if $C = C_M = \mathrm{sc}C$ [8]. The pair (M,C) is said to be a differential space (shortly a d-space); the family C is then a linear ring [8] and its elements are called smooth functions on M. For an arbitrary set C_0 of real functions on M, the set $(\mathrm{sc}C_0)_M$ is the smallest differential structure on M containing C_0. A differential structure C is said to be generated by C_0 if $C = (\mathrm{sc}C_0)_M$.

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By a tangent vector to a d-space (M,C) at a point $p\in M$ we shall mean any linear mapping $v\colon C \longrightarrow \mathbb{R}$ which satisfies the condition

$$v(\alpha\cdot\beta) = v(\alpha)\cdot\beta(p) + \alpha(p)\cdot v(\beta)$$

for $\alpha,\beta\in C$. By T_pM we shall denote the linear space of all tangent vectors to (M,C) at $p\in M$, called the tangent space to (M,C) at $p\in M$. The C-module of all derivations of the linear ring C will be denoted by $\mathfrak{X}(M)$. In the pointwise interpretation $\mathfrak{X}(M)$ is the C-module of all smooth vector fields tangent to (M,C) [7],[8]. A sequence $W_1,\dots,W_n\in\mathfrak{X}(M)$ is said to be a vector basis of the C-module $\mathfrak{X}(M)$ if for every point $p\in M$ the sequence $W_1(p),\dots,W_n(p)$ is a basis of T_pM. We say that the differential space (M,C) is of constant differential dimension n if every point $p\in M$ has a neighbourhood $U\in\tau_C$ such that there is a vector basis of $\mathfrak{X}(U)$ composed of n vector fields.

2. MAIN RESULTS. Let (M,C) be a differential space. For any $p\in M$ we shall denote by $\mathfrak{a}_p$ the set of all smooth functions $f\in C$ for which there exists an open neighbourhood $U\in\tau_C$ of p and functions $f_1,\dots,f_n\in C$, $\omega\in\mathcal{E}_n$, for some $n\in\mathbb{N}$, such that

$$f|U = \omega\circ(f_1,\dots,f_n)|U$$

and $\qquad \omega'_{|j}(f_1(p),\dots,f_n(p)) = 0 \quad$ for $j=1,\dots,n$.

It can easily be seen that $\mathfrak{a}_p$ is a linear subspace of C.

Let $C/\mathfrak{a}_p$ be the quotient linear space and $[f]_p$ the equivalence class of $f\in C$.

LEMMA 1. Let (M,C) be a d-space, $p\in M$ an arbitrary point. Then

1° $$[\theta\cdot(\alpha_1,\dots,\alpha_n)]_p = \sum_{i=1}^{n}\theta'_{|i}(\alpha_1(p),\dots,\alpha_n(p))[\alpha_i]_p$$

for any $\alpha_1,\dots,\alpha_n\in C$, $\theta\in\mathcal{E}_n$, $n\in\mathbb{N}$.

2° $[\alpha\cdot\beta]_p = \alpha(p)\cdot[\beta]_p + [\alpha]_p\cdot\beta(p)$ for any $\alpha,\beta\in C$.

3° If $f,g\in C$ and $f|U = g|U$ for a neighbourhood $U\in\tau_C$ of p, then $[f]_p = [g]_p$.

4° If $k\in C$ is a constant function then $[k]_p = 0$.

Proof. 1° It is enough to show that

$$\theta\circ(\alpha_1,\dots,\alpha_n) - \sum_{i=1}^{n}\theta'_{|i}(\alpha_1(p),\dots,\alpha_n(p))\cdot\alpha_i \in \mathfrak{a}_p.$$

Let $\omega \in \mathcal{E}_n$ be a function given by the formula

$$\omega(x_1,\dots,x_n) = \theta(x_1,\dots,x_n) - \sum_{i=1}^{n}\theta'_{|i}(\alpha_1(p),\dots,\alpha_n(p))\cdot x_i$$

for $(x_1,\dots,x_n) \in \mathbb{R}^n$. We see that

$$\omega\circ(\alpha_1,\dots,\alpha_n) = \theta(\alpha_1,\dots,\alpha_n) - \sum_{i=1}^{n}\theta'_{|i}(\alpha_1(p),\dots,\alpha_n(p))\cdot\alpha_i$$

and

$$\omega'_{|i}(\alpha_1(p),\dots,\alpha_n(p)) = 0 \quad \text{for } i = 1,\dots,n.$$

Hence

$$\theta\circ(\alpha_1,\dots,\alpha_n) - \sum_{i=1}^{n}\theta'_{|i}(\alpha_1(p),\dots,\alpha_n(p))\cdot\alpha_i \in \mathfrak{a}_p.$$

2° follows from 1° if we take $\theta \in \mathcal{E}_2$, given by $\theta(x_1,x_2) = x_1\cdot x_2$ for $(x_1,x_2) \in \mathbb{R}^2$. 3° and 4° are obvious.

Let $v \in T_pM$ be any vector tangent to (M,C) at $p \in M$. Note that $v|\mathfrak{a}_p = 0$. Hence v induces a linear functional $l_v \in (C/\mathfrak{a}_p)^*$ defined by

(1) $\qquad l_v([f]_p) := v(f) \quad$ for any $f \in C$.

<u>PROPOSITION 1</u>. The mapping $I\colon T_pM \longrightarrow (C/\mathfrak{a}_p)^*$ defined by

(2) $\qquad I(v) := l_v \quad$ for any $v \in T_pM$

is an isomorphism of linear spaces.

<u>Proof</u>. The linearity of the mapping I is clear. Obviously if $l_v = 0$ for some $v \in T_pM$, then $v = 0$. Hence I is a monomorphism. Now we shall show that I is an epimorphism. For any $l \in (C/\mathfrak{a}_p)^*$, let $v_l\colon C \longrightarrow \mathbb{R}$ be the mapping defined by

(3) $\qquad v_l(f) := l([f]_p) \quad$ for $f \in C$.

It follows from condition 2° of Lemma 1 that v_l is a tangent vector to (M,C) at p such that $I(v_l) = l$.

COROLLARY 1. Let (M,C) be a d-space and $p \in M$. Then for any $n \in \mathbb{N}$, $\dim T_pM = n$ if and only if $\dim C/\mathfrak{a}_p = n$. In particular $\dim T_pM = 0$ iff $C = \mathfrak{a}_p$.

COROLLARY 2. Let (M,C) be a d-space and $p \in M$. If $f \in C$ satisfies $v(f) = 0$ for each $v \in T_pM$, then $f \in \mathfrak{a}_p$.

<u>Proof</u>. If $v(f) = 0$ for any $v \in T_pM$, then for an arbitrary

linear functional $l \in (C/\mathfrak{a}_p)^*$, $l([f]_p) = v_l(f) = 0$. Hence we get $[f]_p = 0$ or equivalently $f \in \mathfrak{a}_p$.

DEFINITION 1. A set $\mathcal{F} \subset C$ is said to be a local basis (l-basis for short) of the differential structure C on M at $p \in M$ if any function $f \in C$ can be uniquely expressed in the form

$$f = \lambda^1 \cdot f_1 + \dots + \lambda^n \cdot f_n + g,$$

where $f_1, \dots, f_n \in \mathcal{F}$, $\lambda^1, \dots, \lambda^n \in \mathbb{R} \setminus \{0\}$, $g \in \mathfrak{a}_p$.

PROPOSITION 2. Let (M,C) be a d-space with the differential structure C generated by a set C_o. Then, for any $p \in M$, there exists an l-basis $\mathcal{F}$ of C at p such that $\mathcal{F} \subset C_o$.

Proof. Consider the quotient space $C/\mathfrak{a}_p$. It can easily be seen that the set $\{[f]_p : f \in C_o\}$ generates the linear space $C/\mathfrak{a}_p$. Let $B := \{[f_s]_p : f_s \in C_o, s \in S\}$, where S is a set of indices, be a basis of $C/\mathfrak{a}_p$. Then the set $\mathcal{F} := \{f_s : s \in S\}$ is clearly an l-basis of the differential structure C at p.

LEMMA 2. Let (M,C) be a d-space with C generated by C_o. Then, for any $p \in M$, in the definition of $\mathfrak{a}_p$ we can take f_i to belong to C_o (see the beginning of this section).

The proof of this lemma is obvious.

LEMMA 3. The set $\mathfrak{a}_p$ is a differential structure on M such that $\tau_{\mathfrak{a}_p} = \tau_C$.

Proof. Let $f_1, \dots, f_n \in \mathfrak{a}_p$. We shall show that $\omega \circ (f_1, \dots, f_n) \in \mathfrak{a}_p$. Indeed, from condition 1^o of Lemma 1 it follows that

$$[\omega \circ (f_1, \dots, f_n)]_p = \sum_{i=1}^{n} \omega'_{|i}(f_1(p), \dots, f_n(p)) \cdot [f_i]_p = 0,$$

or equivalently $\omega \circ (f_1, \dots, f_n) \in \mathfrak{a}_p$.

In order to show that $\tau_{\mathfrak{a}_p} = \tau_C$ observe that

$$A := \{(f - f(p))^3 : f \in C\} \subset \mathfrak{a}_p \subset C.$$

It is trivial that $\tau_A = \tau_C$. Since $A \subset \mathfrak{a}_p \subset C$ implies $\tau_A \subset \tau_{\mathfrak{a}_p} \subset \tau_C$, we see that $\tau_{\mathfrak{a}_p} = \tau_C$.

LEMMA 4. Let (M,C) be a d-dpace and let $\mathcal{F}$ be an l-basis of the d-structure C at $p \in M$. For any function $u_o : \mathcal{F} \longrightarrow \mathbb{R}$ there exists exactly one tangent vector $u: C \longrightarrow \mathbb{R}$ at p such that $u|\mathcal{F} = u_o$.

Proof. Let $u\colon C \longrightarrow \mathbb{R}$ be a mapping given by the formula

$$u(f) = \sum_{i=1}^{n} \lambda^i \cdot u_o(f_i) \quad \text{for } f \in C, \tag{5}$$

where $f_1,\dots,f_n \in \mathcal{F}$, $\lambda^1,\dots,\lambda^n \in \mathbb{R}$ are elements such that $f = \sum_{i=1}^{n} \lambda^i \cdot f_i + g$, where $g \in \mathfrak{a}_p$. It can easily be noticed that u is a linear mapping and $u \mid \mathfrak{a}_p = 0$, hence $u \in T_pM$, and $u \mid \mathcal{F} = u_o$. The uniqueness of u is clear.

LEMMA 5. All 1-bases of a differential structure C at $p \in M$ are of the same cardinality. If C_o generates C then, for any 1-basis $\mathcal{F}$ of C at $p \in M$, $\operatorname{Card} \mathcal{F} \leq \operatorname{Card} C_o$.

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two 1-basis of C at p. Then the sets $[\mathcal{F}_1]_p := \{[f]_p : f \in \mathcal{F}_1\}$ and $[\mathcal{F}_2]_p := \{[f]_p : f \in \mathcal{F}_2\}$ are bases of the linear space $C/\mathfrak{a}_p$, and $\operatorname{Card} \mathcal{F}_i = \operatorname{Card} [\mathcal{F}_i]_p$ for $i = 1,2$. Obviously, $\operatorname{Card} [\mathcal{F}_1]_p = \operatorname{Card} [\mathcal{F}_2]_p$. Hence $\operatorname{Card} \mathcal{F}_1 = \operatorname{Card} \mathcal{F}_2$. The second assertion follows from the first and Proposition 2.

PROPOSITION 3. Let (M,C) be a d-space and let $\mathcal{F} \subset C$ be an 1-basis of C at $p \in M$. Then the mapping $\Phi\colon T_pM \longrightarrow \mathbb{R}$ defined by

$$\Phi(u) := u \mid \mathcal{F} \quad \text{for } u \in T_pM \tag{6}$$

is an isomorphism of linear spaces.

Proof. This follows immediately from Lemma 4.

COROLLARY 3. Let (M,C) be a d-space and let $\mathcal{F}$ be an 1-basis of C at $p \in M$. Then

(a) $\operatorname{Card} \mathcal{F} < \infty \implies \operatorname{Card} \mathcal{F} = \dim T_pM$

(b) $\operatorname{Card} \mathcal{F} = \infty \implies 2^{\operatorname{Card} \mathcal{F}} = \dim T_pM$.

PROPOSITION 4. A d-space (M,C) is of constant differential dimension n if and only if, for any $p \in M$, there exist a neighbourhood $U \in \tau_C$ of p and a subset $\{f_1,\dots,f_n\} \subset C$ which forms an 1-basis of C at any point of U.

Proof. "$\Longrightarrow$" Assume that (M,C) is of constant dimension n. Then for any point p there exist an open neighbourhood $V \in \tau_C$ of p and a vector basis $\{W_1,\dots,W_n\} \subset \mathfrak{X}(V)$ of the C_V-module $\mathfrak{X}(V)$ [7],[8]. It can easily be seen [8] that there exist an open subset $U \subset V$ containing p and functions $f_1,\dots,f_n \in C$ such that

(7) $W_i(q)(f_j) = \delta_{ij}$ for $q \in U$, $i,j = 1,\dots,n$.

We shall show that the set $\{f_1,\dots,f_n\}$ is an 1-basis at any $q \in U$. Since $\{W_1(q),\dots,W_n(q)\}$ is a basis of the linear space T_qM, $I(\{W_1(q),\dots,W_n(q)\})$ is a basis of the linear space $(C/\mathfrak{a}_q)^*$, where I is the isomorphism given by (2). From (1) and (7) we obtain $I(W_i(q)) = [f_i]_q^*$ for $q \in U$, $i = 1,\dots,n$. Hence $\{[f_1]_q,\dots,[f_n]_q\}$ is a basis of the linear space $C/\mathfrak{a}_q$ for $q \in U$. Let $f \in C$. Then, for $q \in U$, the element $[f]_q$ has a unique decomposition $[f]_q = \lambda^1\cdot[f_1]_q +\dots+ \lambda^n\cdot[f_n]_q$, where $\lambda^1,\dots,\lambda^n \in \mathbb{R}\setminus\{0\}$ or equivalently $f = \lambda^1\cdot f_1 +\dots+ \lambda^n\cdot f_n + g$, where $g \in \mathfrak{a}_q$. Thus the set $\{f_1,\dots,f_n\}$ is an 1-basis of the d-structure C at any point of U.

"$\Longleftarrow$" Let $p \in M$ and let $U \in \tau_C$ be a neighbourhood of p such that the set $\{f_1,\dots,f_n\} \subset C$ is an 1-basis of C at all $q \in U$. Let W_i, for $i = 1,\dots,n$, be a vector field on U satisfying the condition $W_i(q)(f_j) = \delta_{ij}$ for $q \in U$, $j = 1,\dots,n$. The uniqueness of the fields $W_1,\dots,W_n$ follows from Lemma 4. We shall show that the vector fields $W_1,\dots,W_n$ are smooth. Each function $f \in C$ has a unique decomposition in the form

$$f = \lambda^1\cdot f_1 +\dots+ \lambda^n\cdot f_n + g,$$

where $\lambda^1,\dots,\lambda^n \in \mathbb{R}\setminus\{0\}$ and $g \in \mathfrak{a}_p$. One can easily see that $W_i(f|U) = \lambda^i$, for $i = 1,\dots,n$. This demonstrates the smoothness of the vector fields $W_1,\dots,W_n$. It can easily be seen that $\{W_1(q),\dots,W_n(q)\}$ is a basis of the linear space T_qM, for $q \in U$. Thus the d-space (M,C) is of constant differential dimension n.

EXAMPLE. Let C be the d-structure on $\mathbb{R}$ generated by the set of real functions $C_0 := \{f_n\colon n \in \mathbb{N}\}$, where $f_n(x) := x^{1/(2n-1)}$. Then $\mathfrak{a}_0 = C$ and $\dim T_x\mathbb{R} = 1$ for $x \in \mathbb{R}\setminus\{0\}$, $\dim T_xM = 0$ for $x=0$.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
PL. JEDNOSCI ROBOTNICZEJ 1, 00-661 WARSZAWA, POLAND.