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NATURAL TRANSFORMATIONS OF FOLIATIONS INTO FOLIATIONS ON THE COTANGENT BUNDLE

Włodzimierz M. Mikulski

In this paper a classification of natural transformations of foliations into foliations on the cotangent bundle is given. All manifolds, foliations and maps are assumed to be of class C^∞ . Foliation are assumed to be without singularities.

1. Let n be a natural number. Let M be an n -dimensional manifold. The vector bundle $(\pi_M : T^*M \rightarrow M) = T^*M := (T^*M)$ (dual to the tangent bundle TM of M) is called the cotangent bundle of M . Every embedding $f : M \rightarrow N$ of n -manifolds induces a vector bundle embedding $T^*f := (T(f^{-1}))^* : T^*M \rightarrow T^*N$ covering f , where Tf denotes the differential of f . One can verify easily that the rule $M \rightarrow T^*M$, $f \rightarrow T^*f$, is a natural bundle in the sense of [4].

From now on we fix two natural numbers n and p such that $1 \leq p \leq n - 1$. We identify a foliation with its tangent distribution (see [5]). A *natural transformation of foliations into foliations on the cotangent bundle* is a system of foliations $Q(M, F)$ on T^*M , for every n -manifold M and every p -dimensional foliation F on M , satisfying the following naturality condition: for any n -manifolds M, N , p -dimensional foliations F_1 on M and F_2 on N and every embedding $f : M \rightarrow N$ the assumption $Tf \circ F_1 = F_2 \circ f$ implies $TT^*f \circ Q(M, F_1) = Q(N, F_2) \circ T^*f$. (This definition is similar to the definition of natural base-extending operators (see [2]).

We have the following five natural transformations of foliations into foliations on the cotangent bundle. Let F be a p -dimensional foliation on an n -manifold M . Then we define the following distributions on T^*M :

$$\begin{aligned} {}^1Q(M, F)_\omega &= \{0\}, \\ {}^2Q(M, F)_\omega &= \left\{ \frac{d}{dt}(\omega + t\sigma)_{t=0} \in T_\omega T^*M : \sigma \in \text{Anih}(F_{\pi_M(\omega)}) \right\}, \end{aligned}$$

⁰This paper is in final form and no version of it will be submitted for publication elsewhere.

$${}^3Q(M, F)_\omega = \ker(T_\omega \pi_M),$$

$${}^4Q(M, F)_\omega = \{T^*X|_\omega : X \text{ is a } F\text{-vector field}\} + \ker(T_\omega \pi_M),$$

$${}^5Q(M, F)_\omega = T_\omega T^*M,$$

where $\omega \in T^*M$, T^*X is the complete lift of X to the cotangent bundle (see [1], [6]) and $\text{Anih}(F_y) = \{\sigma \in T_y^*M : \sigma(v) = 0 \text{ for all } v \in F_y\}$. If $(x^1, \dots, x^n)$ are F -adapted coordinates on M and $(x^1, \dots, x^n, v^1, \dots, v^n)$ are the induced coordinates on T^*M , then

$${}^2Q(M, F) \text{ is spanned by } \frac{\partial}{\partial v^{p+1}}, \dots, \frac{\partial}{\partial v^n},$$

$${}^3Q(M, F) \text{ is spanned by } \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n},$$

$${}^4Q(M, F) \text{ is spanned by } \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}.$$

Therefore ${}^iQ(M, F)$ is of class C^∞ and involutive. It is easy to verify that the system ${}^iQ = \{{}^iQ(M, F)\}$ is a natural transformation of foliations into foliations on the cotangent bundle.

The main result in this paper is the following theorem.

Theorem 1.1. *Any natural transformation of foliations into foliations on the cotangent bundle belongs to the set $\{{}^1Q, {}^2Q, {}^3Q, {}^4Q, {}^5Q\}$ defined above.*

The proof of this theorem will occupy the rest of the paper.

2. From now on we denote by $\mathcal{F}^p$ the standard p -dimensional foliation on $\mathbf{R}^n$ spanned by $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}$. By $dx^1, \dots, dx^n$ we denote the canonical forms on $\mathbf{R}^n$ dual to $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$.

The following lemma plays an essential role in the proof of the main theorem.

Lemma 2.1. *Let Q_1 and Q_2 be two natural transformations of foliations into foliations to the cotangent bundle. Let us assume that $Q_1(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset Q_2(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Then $Q_1(M, F)_\omega \subset Q_2(M, F)_\omega$ for any p -dimensional foliation F on an n -manifold M . In particular, the equality $Q_1(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = Q_2(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ implies $Q_1 = Q_2$.*

Proof. Consider $\omega \in T_y^*M - \text{Anih}(F_y)$. By the Frobenius theorem there exists an embedding $f : \mathbf{R}^n \rightarrow M$ such that $Tf \circ \mathcal{F}^p = F \circ f$ on some open neighbourhood V of $0 \in \mathbf{R}^n$ and $T^*f(dx^1|0) = \omega$. Let $\tilde{\mathcal{F}}^p$ be a foliation on V such that $Tj \circ \tilde{\mathcal{F}}^p = \mathcal{F}^p \circ j$, where $j : V \rightarrow \mathbf{R}^n$ is the inclusion. Let $\omega_o \in T^*V$ be such that $T^*j(\omega_o) = dx^1|0$. Then by the naturality condition we obtain $Q_i(M, F)_\omega = Q_i(M, F) \circ T^*(f \circ j)(\omega_o) = TT^*(f \circ j)(Q_i(V, \tilde{\mathcal{F}}^p)_{\omega_o}) = TT^*f(Q_i(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0})$ for $i = 1, 2$. Hence $Q_1(M, F)_\omega \subset$

$Q_2(M, F)_\omega$. Since $T_y^*M - \text{Anih}(F_y)$ is dense in T_y^*M , we obtain the inclusion for all $\omega \in T^*M$. $\square$

3. Let $\omega \in T_0^*\mathbb{R}^n$. A diffeomorphism $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called ω -admissible iff $T^*\varphi(\omega) = \omega$ and $T\varphi \circ \mathcal{F}^p = \mathcal{F}^p \circ \varphi$. A subspace $W \subset T_\omega T^*\mathbb{R}^n$ is called ω -admissible iff $TT^*\varphi(W) = W$ for any ω -admissible diffeomorphism φ .

We have the following corollary of the naturality condition.

Corollary 3.1. *If Q is a natural transformation of foliations into foliations on the cotangent bundle, then $Q(\mathbb{R}^n, \mathcal{F}^p)_\omega$ is ω -admissible for any $\omega \in T_0^*\mathbb{R}^n$.*

In particular, ${}^1Q(\mathbb{R}^n, \mathcal{F}^p)_\omega, \dots, {}^5Q(\mathbb{R}^n, \mathcal{F}^p)_\omega$ are ω -admissible, where iQ is defined in Item 1.

We have also the following corollary.

Corollary 3.2. *The vector spaces*

$${}^3Q(\mathbb{R}^n, \mathcal{F}^p)_{dx^1|0} + \text{span}\left\{T^*\left(\frac{\partial}{\partial x^i}\right)_{dx^1|0} : i = 2, \dots, n\right\}$$

and $\text{span}\left\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\right\}$ are $(dx^1|0)$ -admissible.

Proof. It is easy to verify that the second space is $(dx^1|0)$ -admissible. Of course, the first space is equal to $\ker\{(dx^1|0) \circ T_{i, dx^1|0} \pi_{\mathbb{R}^n}\}$ i.e. $(dx^1|0)$ -admissible. $\square$

4. In the proof of Theorem 1.1 we use the following lemmas.

Lemma 4.1. *If $W \subset {}^3Q(\mathbb{R}^n, \mathcal{F}^p)_0$ is a 0-admissible subspace such that $W \neq \{0\}$ and $W \neq {}^3Q(\mathbb{R}^n, \mathcal{F}^p)_0$, then $W = {}^2Q(\mathbb{R}^n, \mathcal{F}^p)_0$.*

Lemma 4.2. *Let $W \subset {}^3Q(\mathbb{R}^n, \mathcal{F}^p)_{dx^1|0}$ be a $(dx^1|0)$ -admissible subspace such that $\dim W = n - p$. Then we have the following implications:*

- (a) *If $n - p \geq 2$, then $W = {}^2Q(\mathbb{R}^n, \mathcal{F}^p)_{dx^1|0}$.*
- (b) *If $n - p = 1$, then $W = {}^2Q(\mathbb{R}^n, \mathcal{F}^p)_{dx^1|0}$ or $W = \text{span}\left\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\right\}$.*

Lemma 4.3. *Let W be a $(dx^1|0)$ -admissible subspace. If $W - {}^3Q(\mathbb{R}^n, \mathcal{F}^p)_{dx^1|0} \neq \emptyset$, then ${}^3Q(\mathbb{R}^n, \mathcal{F}^p)_{dx^1|0} \subset W$.*

Lemma 4.4. *Let W be a 0-admissible subspace. Assume that $W \neq {}^3Q(\mathbb{R}^n, \mathcal{F}^p)_0$. ${}^3Q(\mathbb{R}^n, \mathcal{F}^p)_0 \subset W$ and $W \neq {}^5Q(\mathbb{R}^n, \mathcal{F}^p)_0$. Then $W = {}^4Q(\mathbb{R}^n, \mathcal{F}^p)_0$.*

Lemma 4.5. *Let W be a $(dx^1|0)$ -admissible subspace such that $\dim W = n + p$ and ${}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset W$. Then we have the following implications:*

- (a) *If $n + p < 2n - 1$, then $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$.*
- (b) *If $n + p = 2n - 1$, then $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} + \text{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \dots, n\}$ or $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$.*

Proof of Lemma 4.1. Consider two cases:

(I) $W \not\subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Then there exists $\sigma \in T_0^*\mathbf{R}^n - \text{Anih}(\mathcal{F}_0^p)$ such that $\frac{d}{dt}[t\sigma]_{t=0} \in W$. Consider $\mu \in T_0^*\mathbf{R}^n - \text{Anih}(\mathcal{F}_0^p)$. There exists an 0-admissible linear isomorphism φ such that $T^*\varphi(\sigma) = \mu$. Then $\frac{d}{dt}[t\mu]_{t=0} = TT^*\varphi(\frac{d}{dt}[t\sigma]_{t=0}) \in W$. Therefore $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Contradiction.

(II) $W \subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Let $\sigma \in \text{Anih}(\mathcal{F}_0^p) - \{0\}$ be such that $\frac{d}{dt}[t\sigma]_{t=0} \in W$. Consider $\mu \in \text{Anih}(\mathcal{F}_0^p) - \{0\}$. There exists an 0-admissible linear isomorphism φ such that $T^*\varphi(\sigma) = \mu$. Then $\frac{d}{dt}[t\mu]_{t=0} \in W$. That is why $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$. $\square$

Proof of Lemma 4.2. Consider two cases:

(I) $W \subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Then $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ because of the dimension argument.

(II) $W \not\subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. We can assume that $\text{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\} \not\subset W$. Consider two subcases:

(a) $W \not\subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \oplus \text{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$. Then there exists $\sigma \in T_0^*\mathbf{R}^n - \text{Anih}(\mathcal{F}_0^p) \oplus \text{span}\{dx^1|0\}$ such that $\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0} \in W$. It is clear that $p \geq 2$. Consider two subsubcases:

(1) $\sigma(\frac{\partial}{\partial x^1}|0) \neq 0$. If $\mu \in T_0^*\mathbf{R}^n - \text{Anih}(\mathcal{F}_0^p) \oplus \text{span}\{dx^1|0\}$ and $\mu(\frac{\partial}{\partial x^1}|0) \neq 0$, then there exist $\lambda \in \mathbf{R}$ and a $(dx^1|0)$ -admissible linear isomorphism φ such that $T^*\varphi(\lambda\sigma) = \mu$, and then $\frac{d}{dt}[(dx^1|0) + t\mu]_{t=0} = \lambda TT^*\varphi(\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0}) \in W$. Therefore $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ i.e. $\dim W = n > n - p$. Contradiction.

(2) $\sigma(\frac{\partial}{\partial x^1}|0) = 0$. If $\mu \in T_0^*\mathbf{R}^n - \text{Anih}(\mathcal{F}_0^p) \oplus \text{span}\{dx^1|0\}$ and $\mu(\frac{\partial}{\partial x^1}|0) = 0$, then there exists a $(dx^1|0)$ -admissible linear isomorphism φ such that $T^*\varphi(\sigma) = \mu$, and then $\frac{d}{dt}[(dx^1|0) + t\mu]_{t=0} = TT^*\varphi(\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0}) \in W$. Therefore $\text{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^i|0)]_{t=0} : i = 2, \dots, n\} \subset W$ i.e. $\dim W \geq n - 1 > n - p$. Contradiction.

(b) $W \subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \oplus \text{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$. Then there exists $\sigma \in \text{Anih}(\mathcal{F}_0^p) \oplus \text{span}\{dx^1|0\} - \text{Anih}(\mathcal{F}_0^p) \cup \text{span}\{dx^1|0\}$ such that $\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0} \in W$. If $\mu \in \text{Anih}(\mathcal{F}_0^p) \oplus \text{span}\{dx^1|0\} - \text{Anih}(\mathcal{F}_0^p) \cup \text{span}\{dx^1|0\}$, then there exist $\lambda \in \mathbf{R}$ and a $(dx^1|0)$ -admissible linear isomorphism φ such that $T^*\varphi(\lambda\sigma) = \mu$, and then $\frac{d}{dt}[(dx^1|0) + t\mu]_{t=0} \in W$. Therefore $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \oplus \text{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$ i.e. $\dim W = n - p + 1$. Contradiction. $\square$

Proof of Lemma 4.3. There exist real numbers $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{R}$ such that $Y := a_1 \frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0} + \dots + a_n \frac{d}{dt}[(dx^1|0) + t(dx^n|0)]_{t=0} + b_1 T^*(\frac{\partial}{\partial x^1})_{dx^1|0} + \dots + b_n T^*(\frac{\partial}{\partial x^n})_{dx^1|0} \in W$ and $b_q \neq 0$ for some $q \in \{1, \dots, n\}$.

Consider $k \in \{1, \dots, n\}$. Let $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a diffeomorphism such that $\varphi^{-1}(y^1, \dots, y^n) = (y^1 + y^q y^k, y^2, \dots, y^n)$ on some open neighbourhood of $0 \in \mathbf{R}^n$. Then φ is $(dx^1|0)$ -admissible. By a standard verification (see [6]) one can show that $TT^*\varphi(Y) = Y + b_q \frac{d}{dt}[(dx^1|0) + t(dx^k|0)]_{t=0} + b_k \frac{d}{dt}[(dx^1|0) + t(dx^q|0)]_{t=0}$. Since W is $(dx^1|0)$ -admissible and $Y \in W$, then $TT^*\varphi(Y) \in W$, and then $b_q \frac{d}{dt}[(dx^1|0) + t(dx^k|0)]_{t=0} + b_k \frac{d}{dt}[(dx^1|0) + t(dx^q|0)]_{t=0} \in W$. Putting $k = q$ we find $\frac{d}{dt}[(dx^1|0) + t(dx^q|0)]_{t=0} \in W$, and then $\frac{d}{dt}[(dx^1|0) + t(dx^k|0)]_{t=0} \in W$. $\square$

Proof of Lemma 4.4. Consider two cases:

(I) $W \not\subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Then there exists real numbers $a_1, \dots, a_n \in \mathbf{R}$ such that $T^*(a_1 \frac{\partial}{\partial x^1} + \dots + a_n \frac{\partial}{\partial x^n})_0 \in W$ and $a_i \neq 0$ for some $i = p+1, \dots, n$. Consider $b_1, \dots, b_n \in \mathbf{R}$ such that $b_j \neq 0$ for some $j = p+1, \dots, n$. There exists an 0-admissible linear isomorphism φ such that $T\varphi(a_1(\frac{\partial}{\partial x^1})_0 + \dots + a_n(\frac{\partial}{\partial x^n})_0) = b_1(\frac{\partial}{\partial x^1})_0 + \dots + b_n(\frac{\partial}{\partial x^n})_0$. Then $T^*(b_1 \frac{\partial}{\partial x^1} + \dots + b_n \frac{\partial}{\partial x^n})_0 = TT^*\varphi(T^*(a_1 \frac{\partial}{\partial x^1} + \dots + a_n \frac{\partial}{\partial x^n})_0) \in W$. Therefore $W = {}^5Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Contradiction.

(II) $W \subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Let $a_1, \dots, a_p \in \mathbf{R}$ be such that $T^*(a_1 \frac{\partial}{\partial x^1} + \dots + a_p \frac{\partial}{\partial x^p})_0 \in W$ and $a_i \neq 0$ for some $i = 1, \dots, p$. Consider $b_1, \dots, b_p \in \mathbf{R}$ such that $b_j \neq 0$ for some $j = 1, \dots, p$. There exists an 0-admissible linear isomorphism φ such that $T\varphi(a_1(\frac{\partial}{\partial x^1})_0 + \dots + a_p(\frac{\partial}{\partial x^p})_0) = b_1(\frac{\partial}{\partial x^1})_0 + \dots + b_p(\frac{\partial}{\partial x^p})_0$. Then $T^*(b_1 \frac{\partial}{\partial x^1} + \dots + b_p \frac{\partial}{\partial x^p})_0 = TT^*\varphi(T^*(a_1 \frac{\partial}{\partial x^1} + \dots + a_p \frac{\partial}{\partial x^p})_0) \in W$. That is why $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$. $\square$

Proof of Lemma 4.5. Consider two cases:

(I) $W \subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Then $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ because of the dimension argument.

(II) $W \not\subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Consider two subcases;

(a) At first we assume that there exist $a_2, \dots, a_n \in \mathbf{R}$ such that $T^*(a_2 \frac{\partial}{\partial x^2} + \dots + a_n \frac{\partial}{\partial x^n})_0 \in W$ and $a_j \neq 0$ for some $j = p+1, \dots, n$. Consider $b_2, \dots, b_n \in \mathbf{R}$ such that $b_q \neq 0$ for some $q = p+1, \dots, n$. There exists a $(dx^1|0)$ -admissible linear isomorphism φ such that $T\varphi(a_2(\frac{\partial}{\partial x^2})_0 + \dots + a_n(\frac{\partial}{\partial x^n})_0) = b_2(\frac{\partial}{\partial x^2})_0 + \dots + b_n(\frac{\partial}{\partial x^n})_0$. Then $T^*(b_2 \frac{\partial}{\partial x^2} + \dots + b_n \frac{\partial}{\partial x^n})_{dx^1|0} = TT^*\varphi(T^*(a_2 \frac{\partial}{\partial x^2} + \dots + a_n \frac{\partial}{\partial x^n})_{dx^1|0}) \in W$. Hence $\text{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \dots, n\} \subset W$ i.e. $\dim W \geq 2n - 1$. Therefore $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} + \text{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \dots, n\}$, provided $\dim W = n + p = 2n - 1$.

(b) Now, we suppose that there exist $a_1, \dots, a_n \in \mathbf{R}$ such that $T^*(a_1 \frac{\partial}{\partial x^1} + \dots + a_n \frac{\partial}{\partial x^n})_0 \in W$, $a_1 \neq 0$ and $a_j \neq 0$ for some $j = p+1, \dots, n$. Consider $b_1, \dots, b_n \in \mathbf{R}$ such that $b_1 \neq 0$ and $b_q \neq 0$ for some $q = p+1, \dots, n$. Then there exists a $(dx^1|0)$ -admissible

linear isomorphism φ such that $T\varphi(a_1(\frac{\partial}{\partial x^1})_0 + \dots + a_n(\frac{\partial}{\partial x^n})_0) = \frac{a_1}{b_1}\{b_1(\frac{\partial}{\partial x^1})_0 + \dots + b_n(\frac{\partial}{\partial x^n})_0\}$. Then $T^*(b_1\frac{\partial}{\partial x^1} + \dots + b_n\frac{\partial}{\partial x^n})_{dx^1|0} = \frac{a_1}{b_1}TT^*(a_1\frac{\partial}{\partial x^1} + \dots + a_n\frac{\partial}{\partial x^n})_{dx^1|0} \in W$. We have proved that $\text{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 1, \dots, n\} \subset W$. Hence $\dim W = 2n$. Contradiction. $\square$

5. We are now in position to prove Theorem 1.1. Let Q be a natural transformation of foliations into foliations on the cotangent bundle such that $Q \neq {}^1Q$, $Q \neq {}^3Q$ and $Q \neq {}^5Q$. We want to show that $Q = {}^2Q$ or $Q = {}^4Q$.

It follows from Lemma 2.1 that $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \neq {}^iQ(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ for $i = 1, 3, 5$. Of course, ${}^1Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset {}^5Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$.

Consider two cases:

(I) $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Then then it follows from Lemma 2.1 that $Q(\mathbf{R}^n, \mathcal{F}^p)_0 \subset {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Of course, $Q(\mathbf{R}^n, \mathcal{F}^p)_0 \neq {}^iQ(\mathbf{R}^n, \mathcal{F}^p)_0$ for $i = 1, 3$ because of the dimension argument. Then Lemma 4.1 implies $Q(\mathbf{R}^n, \mathcal{F}^p)_0 = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Hence $\dim Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = n - p$. Consider two subcases:

(a) $n - p \geq 2$. Then by Lemma 4.2(a) $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$, and then $Q = {}^2Q$ because of Lemma 2.1.

(b) $n - p = 1$. Then by Lemma 4.2(b), $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ (i.e. $Q = {}^2Q$ because of Lemma 2.1) or $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = \text{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$. So, we suppose that $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = \text{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$. Then from the naturality condition with respect to the homotheties $\tau \text{id}_{\mathbf{R}^n}$, $\tau \neq 0$, it follows that $Q(\mathbf{R}^n, \mathcal{F}^p)_{\tau(dx^1|0)} = \text{span}\{\frac{d}{dt}[\tau(dx^1|0) + t(dx^1|0)]_{t=0}\}$ for any $\tau \in \mathbf{R} - \{0\}$. On the other hand, $Q(\mathbf{R}^n, \mathcal{F}^p)_0 = \text{span}\{\frac{d}{dt}[t(dx^n|0)]_{t=0}\} \neq \text{span}\{\frac{d}{dt}[t(dx^1|0)]_{t=0}\}$. Contradiction.

(II) $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} - {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \neq \emptyset$. Then it follows from Lemma 4.3 that ${}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Then by Lemma 2.1, ${}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0 \subset Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Of course, $Q(\mathbf{R}^n, \mathcal{F}^p)_0 \neq {}^iQ(\mathbf{R}^n, \mathcal{F}^p)_0$ for $i = 3, 5$ because of the dimension argument. Then $Q(\mathbf{R}^n, \mathcal{F}^p)_0 = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$ because of Lemma 4.4. Hence $\dim Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = \dim Q(\mathbf{R}^n, \mathcal{F}^p)_0 = n + p$. Consider two subcases:

(a) $n + p < 2n - 1$. Then it follows from Lemma 4.5(a) that $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$, and then $Q = {}^4Q$ because of Lemma 2.1.

(b) $n + p = 2n - 1$. Then by Lemma 4.5(b), $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ (i.e. $Q = {}^4Q$) or $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} + \text{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \dots, n\}$. So, suppose that $Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} + \text{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \dots, n\}$. Then by the naturality condition with respect to the homotheties $\tau \text{id}_{\mathbf{R}^n}$, $\tau \neq 0$, we obtain that $Q(\mathbf{R}^n, \mathcal{F}^p)_{\tau(dx^1|0)} = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{\tau(dx^1|0)} + \text{span}\{T^*(\frac{\partial}{\partial x^i})_{\tau(dx^1|0)} : i =$

$2, \dots, n\}$ for any $\tau \in \mathbf{R} - \{0\}$. On the other hand, since $n \geq 2$, then $Q(\mathbf{R}^n, \mathcal{F}^p)_0 = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0 + \text{span}\{T^*(\frac{\partial}{\partial x^i})_0 : i = 1, \dots, n-1\} \neq {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0 + \text{span}\{T^*(\frac{\partial}{\partial x^i})_0 : i = 2, \dots, n\}$. Contradiction. $\square$

6. Similarly as in [3], we introduce the following definition. A *natural lifting of foliations to the cotangent bundle* is a system of foliations $Q(M, F)$ on T^*M projecting (by the cotangent bundle projection) onto F , for every n -manifold M and every p -dimensional foliation F on M , satisfying the following naturality condition: for any n -manifolds M, N , p -dimensional foliations F_1 on M and F_2 on N and every embedding $f: M \rightarrow N$ the assumption $Tf \circ F_1 = F_2 \circ f$ implies $TT^*f \circ Q(M, F_1) = Q(N, F_2) \circ T^*f$.

We have the following obvious corollary of Theorem 1.1.

Corollary 6.1. *Any natural lifting of foliations to the cotangent bundle is equal to 4Q , where 4Q is defined in Item 1.*

REFERENCES

1. GANCARZEWICZ J. "Liftings of functions and vector fields to natural bundles", Dissertationes Mathematicae CCXII. Warszawa. 1983.
2. KOLÁŘ I. and MICHOR P.W. and SLOVÁK J. "Natural Operations in Differential Geometry", to appear.
3. MIKULSKI W.M. "Natural liftings of foliations to the tangent bundle", Mathematica Bohemica, to appear.
4. NIJENHUIS A. "Natural bundles and their general properties", in Differential Geometry in Honor of K. Yano, Kinokuniya, Tokio. (1972), 317-343.
5. WOLAK R. "Geometric structures on foliated manifolds", Universidad de Santiago de Compostella 76 (1989).
6. YANO K. and ISHIHARA S. "Tangent and cotangent bundles". Marcel Dekker Inc., New York, 1973.

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