COMPACTNESS OF TRAJECTORIES OF DYNAMICAL SYSTEMS
IN COMPLETE UNIFORM SPACES

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In this paper we investigate the asymptotic behaviour of trajectory $\{\varphi_t(x)\}_{t\geq 0}$, for a semigroup of mappings $\{\varphi_t\}_{t\geq 0}$ of a Hausdorff space X into itself. More precisely : the main subject of our interest is to establish conditions equivalent to precompactness of the trajectory and of the set of limit points for a given point $x \in X$. This topic has already been studied in [7], [8] and [6]. In our case the space X is in addition equipped with a complete uniform structure $\mathcal{U}$ (see [4] for definition). We also assume the following four conditions for the family $\{\varphi_t\}$:

(i) $\varphi_0(x) = x$, for all $x \in X$

(ii) $\varphi_t \circ \varphi_s = \varphi_{t+s}$, for all $t,s \in R_+$

(iii) $\lim_{t \to s} \varphi_t(x) = \varphi_s(x)$, for all $s \in R_+$

(iv) for every $W \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that for all x, y with $(x,y) \in V$ and all $t \geq 0$ we have $(\varphi_t(x), \varphi_t(y)) \in W$.

The first three of the above conditions mean that the mappings φ_t form an one-parameter continuous semigroup acting on X. The last condition establishes its equicontinuity.
For a fixed element W of $\mathcal{U}$ by $\mathcal{U}_W$ we will denote the collection of all the elements V which fulfill (iv). We also write W_x instead of $\{ y : (x,y) \in W\}$. For contraction semigroups acting on subsets of Banach spaces the condition (iv) may be replaced by an adequate norm - condition. In this case many interesting results were obtained, dealing with limit properties of the trajectory $\gamma(x) = \{\varphi_t(x) : t \geq 0\}$ (see [1], [3], [5]). Subsequently in [2] were obtained some analogous results for nonextending semigroups acting

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on Polish spaces. The methods of proofs used there have let hope that further generalisations are possible.

We start by proving the following lemma, which is an adaptation of a well known result from [3] (Theorem 1) .

Lemma 1. For every $x \in X$, the set of limit points $w(x) = \bigcap_{s \geqslant 0} \overline{\{\varphi_t(x) : t \geqslant s\}}$ is either minimal or empty.

Proof. We have to show that $\overline{\gamma(y)} = w(x)$, for every $y \in w(x)$. The inclusion $\subseteq$ is immediate because for every $t \geqslant 0$, the point $\varphi_t(y)$ is a limit of $\varphi_t(x)$. Now let $z \in w(x)$ and U be an open neighbourhood of z. There exists an element W of the structure $\mathcal{U}$ such that for all α large enough $(\varphi_{t_\alpha}(x), v) \in W$ implies $v \in U$, where $t_\alpha \to \infty$ is some fixed net satysfying $\varphi_{t_\alpha}(x) \to z$. We can also easily find a net $s_\alpha \to \infty$ with $\varphi_{t_\alpha - s_\alpha}(x) \to y$. For $V \in \mathcal{U}_W$ we have $(\varphi_{t_\alpha - s_\alpha}(x), y) \in V$ for some α . Thus $(\varphi_{t_\alpha}(x), \varphi_{s_\alpha}(y)) \in W$, so $\varphi_{s_\alpha}(y) \in U$, hence $w(x) \subseteq \overline{\gamma(y)}$, and the minimality is proved.

By X_0 we shall denote the set of all $x \in X$ such that the trajectory $\gamma(x)$ is precompact.

Lemma 2. The set X_0 is closed and φ_t-invariant.

Proof. Let $x_\alpha \to x$, where $x_\alpha \in X_0$. For the precompactness of $\gamma(x)$ it is enough to show that $\gamma(x)$ is totally bounded with respect to $\mathcal{U}$ (see [4]). For $U \in \mathcal{U}$ let $W \in \mathcal{U}$ be such that $(b,a) \in W$, $(b,c) \in W$ and $(c,d) \in W$ imply $(a,d) \in U$. By equicontinuity, for some α we have $(\varphi_t(x_\alpha), \varphi_t(x)) \in W$ for every $t \geqslant 0$. Now $\gamma(x_\alpha)$ is precompact, thus there exists a finite set of points $y_n = \varphi_{t_n}(x_\alpha)$ such that W_{y_n} cover $\gamma(x_\alpha)$. Hence, for fixed $t \in R_+$, $(\varphi_t(x_\alpha), \varphi_{t_n}(x_\alpha)) \in W$ for some n. Also $(\varphi_{t_n}(x_\alpha), \varphi_{t_n}(x)) \in W$ and thus $(\varphi_t(x), \varphi_{t_n}(x)) \in U$. We have obtained a finite covering U_{z_n} of $\gamma(x)$, where $z_n = \varphi_{t_n}(x)$, so the precompactness of $\gamma(x)$ is proved. The invariantness of X_0 is obvious and so the proof is complete.

Theorem. Let $\{\varphi_t\}_{t \geqslant 0}$ be an equicontinuous semigroup acting on a complete uniform space X. Then the following conditions are equivalent :

a) $x \in X_0$

b) $w(x)$ is nonempty and compact

c) there exists a φ_t-invariant probability measure μ_x on $w(x)$

d) for every continuous function $F : X \to E$ (E is a Banach space) the Bochner integrals

$T^{-1}\int_0^T F(\varphi_t(x))\,dt$ are convergent for $T\to\infty$ to a limit

$\overline{F}(x)\in E$.

If the above holds then $\overline{F}$ is a continuous invariant function on X_0 and it equals $\int_{w(x)} F(y)\mu_x(dy)$, where μ_x is the unique invariant probability measure on $w(x)$.

Proof. a)$\Rightarrow$b) is obvious by the definition of X_0 and $w(x)$. b)$\Rightarrow$c) is the well known corollary of the Markov - Kakutani theorem. c)$\Rightarrow$b). Suppose that $w(x)$ is non-compact. Then it is not totally bounded and thus there exists an infinite collection of nonempty pairwise disjoint open sets of the form W_{z_n} , where $z_n\in w(x)$, and $W\in\mathcal{U}$. Since $w(x)$ is minimal we may (changing if necessary the set W) choose the points z_n of the form $\varphi_{t_n}(z_0)$ for some $z_0\in w(x)$. Now, for $V\in\mathcal{U}_W$ we have $\varphi_{t_n}(V_{z_0})\subseteq W_{z_n}$. By invariantness of μ_x the measures of the sets W_{z_n} are at least $\mu_x(V_{z_0})$. This is a contradiction since by minimality of $w(x)$ $\mu_x(V_{z_0})>0$ and, on the other hand, μ_x is finite. b)$\Rightarrow$d) see [1] Th. 3.2 and Corollary 3.1 . d)$\Rightarrow$a). Suppose $x\notin X_0$, i.e. $\gamma(x)$ is not totally bounded. An easy argument using the uniform structure allows as to find an infinite collection of open pairwise disjoint neighbourhoods U_n of certain points $x_n=\varphi_{t_n}(x)$ such that every convergent net is (starting from some index) contained in at most one of U_n's. We may also assume that for every n the set $U_n\cap\{\varphi_t(x),\ t\leqslant t_n\}$ is of the form $\{\varphi_t(x),\ t\in(t_n-\varepsilon,\ t_n]\}$. Let F_n be continuous functions on X with $F_n(x_n)=1$, $F_n=0$ out of U_n (see [4] for the existence of Uhryson functions on uniform spaces). The function $F=\sum_{n=1}^{\infty}\beta_n\cdot F_n$ is continuous which contradicts d) whenever β_n increases rapidly enough. To prove the last assertion of the Theorem consider $E=R$ and restrict all the functions F to the compact set $\overline{\gamma(x)}$. Now observe that the map $F\to\overline{F}(x)$ is a linear nonnegativ functional on $C(\overline{\gamma(x)})$. So, by the Riesz theorem it is represented by a Radon measure ν_x on $\overline{\gamma(x)}$, i.e. we can write $\overline{F}(x)=\langle F,\nu_x\rangle$. Taking $F\equiv 1$ we obtain that ν_x is a probability measure. To see the invariantness of ν_x denote $F_s=F\circ\varphi_s$ for $F\in C(\overline{\gamma(x)})$ and $s\geqslant 0$ and calculate :

$T^{-1}\int_0^T F_s(\varphi_t(x))\,dt \longrightarrow \langle F_s,\nu_x\rangle = \langle F,\nu_x\circ\varphi_s^{-1}\rangle$. On the other hand

$T^{-1}\int_0^T F_s(\varphi_t(x))\,dt = T^{-1}\int_s^{T+s} F(\varphi_t(x))\,dt =$

$$= T^{-1}(T+s)(T+s)^{-1}\int_0^{T+s} F(\varphi_t(x))\,dt - T^{-1}\int_0^{s} F(\varphi_t(x))dt \longrightarrow \langle F, \nu_x\rangle .$$

Our last step is to check that ν_x is supported by $w(x)$. Let $y \notin w(x)$. There exists $W \in \mathcal{U}$ such that y does not belong to the set $U = \bigcup_{z \in w(x)} W_z$ together with its open neighbourhood. But $\varphi_t(x) \in U$ for big t, hence for any continuous $F : X \to R$ with $F(y) = 1$ and $F = 0$ on U we have $F_s = 0$ on $\overline{\gamma(x)}$ and $\langle F, \nu_x\rangle = \langle F_s, \nu_x\rangle = 0$ for s big enough. The uniqueness of the measure ν_x follows from the well known Halmos - von Neumann theorem. We omit the easy standard approximation argument for proving the continuity of $\overline{F}$ on X_0 .

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