NORMS ON SUPER-REFLEXIVE BANACH SPACES

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1. Abstract. We study uniform convexity and smoothness properties satisfied by all the equivalent norms of a super-reflexive Banach space.

Introduction. G. Pisier proved that every super-reflexive Banach space has an uniformly convex equivalent norm with a modulus of convexity of power-type ([10]). A natural question is : what can be said of any equivalent norm on a super-reflexive Banach space ? We show that every equivalent norm has some uniform convexity and smoothness properties.

Notations. Let X be a Banach space and N be a norm on X, we note $B_N(X)$ the unit ball of X, $S_N(X)$ the unit sphere and X^* its dual. If F is a subset of X, conv(F) is the convex hull of F.

I. Strong extreme points.

Let us consider the notion of strong extreme point. This notion has been introduced par K. Kunen and H.P. Rosenthal ([7]).

Définition 1. Let C be a closed convex bounded set. A point x in C is a strong extreme point if for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that :

$$y,z \in C\ ,\ \left\| \frac{y+z}{2} - x \right\| < \eta(\varepsilon) \Rightarrow \| y - z \| < \varepsilon \quad .$$

If every point of the unit sphere is a strong extreme point of the unit ball, the norm is said midpoint locally uniformly rotund (MLUR).

Obviously, [a norm is locally uniformly rotund] $\Rightarrow$ [the norm is MLUR] $\Rightarrow$ [the norm is rotund]. The converse implications do not hold.

If x is a denting-point, then x is a strong extreme point and x is extreme. The converses are not true.

The modulus $\Delta(x,\varepsilon)$ which is defined below measures "how much" a point is a strong extreme point of the unit ball.

Definition 2. Let X be a Banach space with norm $\|.\|$. The modulus of strong extremality in x is the number :

$$\forall \varepsilon > 0, \ \Delta_{\|.\|}(x,\varepsilon) = \inf \{1-\lambda ; \exists \tau : \|\lambda x \pm \tau\| \leq 1, \|\tau\| \geq \varepsilon\}.$$

It is easy to show that x is a strong extreme point of the unit ball if and only if $\Delta_{\|.\|}(x,\varepsilon) > 0$, $\forall \varepsilon > 0$.

Let us give now the main result of this section. For any equivalent norm $\|.\|$ on a super-reflexive Banach space X, we let :

$$\Omega_{\|.\|}(K,q) = \{x \in S_{\|.\|}(X) : \Delta_{\|.\|}(x,\varepsilon) \geq K\varepsilon^q, \forall \varepsilon > 0\} \qquad (K > 0, \ q \geq 2).$$

With this notation, the following is true :

Theorem 3. [4], [5]. Let X be a super-reflexive Banach space and $\|.\|$ be an equivalent norm on X with modulus of convexity of power-type ($\delta_{\|.\|}(\varepsilon) \geq C\varepsilon^q$). N is any equivalent norm on X. Then, for every η, $0 < \eta < 1$, there exists $K(\eta) > 0$ such that :

$$B_N(X) \subseteq \operatorname{conv} [\, \Omega_N(K(\eta),q) \,] + \eta \, B_N(X).$$

Proof. The proof of this theorem is based on a technique of J. Lindenstrauss for obtaining strongly exposed points in weakly compact convex sets ([8]).

The theorem follows from a simple lemma.

Lemma 4. [4],[5]. Let $(Y,|\!|\!|.|\!|\!|)$ be an uniformly convex space with modulus of convexity $\delta_{|\!|\!|.|\!|\!|}$. Let $S : (X,N) \to (Y,|\!|\!|.|\!|\!|)$ be an isomorphism into Y. If S attains its norm in x, then x is a strong extreme point of $B_N(X)$ and moreover :

$$\Delta_N(x,\varepsilon) \geq \delta_{|\!|\!|.|\!|\!|}\left(\frac{2\varepsilon}{\|S\| \, \|S^{-1}\|}\right).$$

Remarks

1) In the case where dim X is finite, this result can be obtained more directly by using arguments of strong compactity.

2) The example of $X = \oplus_{\ell^2} \ell_n^\infty$ shows that the theorem is not true in general for a reflexive space X. It would be nice to know if the validity of theorem 3 characterizes the class of super-reflexive Banach spaces.

3) Let us introduce the notion of φ-strongly exposed point : in what follows we denote by φ an increasing function in $[0,1[$ such that $\varphi(0) = 0$.

Definition 5. [4] Let C be a subset of a Banach space X and $x \in C$. We say that x is φ-strongly exposed in C if there exists $f \in X^*$ such that

1. $f(x) = \sup\{f(y), y \in C\}$
2. if $y \in C$ satisfies $f(x) - f(y) \leq \varphi(\varepsilon)$ for some $\varepsilon \in]0,1[$ then $\|x - y\| \leq \varepsilon$.

Then f is called a φ-strongly exposing functional for x.

Let $\|.\|$ be a norm of a Banach space X, let us denote $E_{\|.\|}(\varphi)$ the set of the φ-strongly exposed points in the unit ball $B_{\|.\|}(X)$.

Proposition 6. [4] Let X be a super-reflexive Banach space and $\|.\|$ be an uniformly convex norm on X such that $\delta_{\|.\|}(\varepsilon) \geq C\varepsilon^q$, $\forall \varepsilon > 0$; N is an equivalent norm. Then, for every $\eta \in]0,1[$, there exist a function φ_η and a constant $\kappa(\eta)$ such that :

$$B_N(X) \subseteq \text{conv}\ [\ E_N(\varphi_\eta) \cap \Omega_N(\kappa(\eta), q)\ ] + \eta\ B_N(X).$$

Remark. By using an argument of J.M. Borwein ([1]) it is possible to show that the family of the φ_η-strongly exposing functionals for a point of the unit sphere is an η-net in $S(X^*)$ ([4], [5]).

II. Applications.

1. Quasi-transitive Banach spaces.

The theorem 3 implies

Corollary 7 [4]. A super-reflexive quasi-transitive Banach space is uniformly convex with modulus of convexity of power-type.

2. Uniform approximation property.

Definition 8 [6]. A Banach space X is said to have the λ-uniform approximation property (λ-u.a.p.) if $\forall \varepsilon > 0$, $\forall$ k integer, $\forall$ F subspace of X with dim F = k, there exists an operator $T : X \to X$ with

1) $rk(T) \leq n_X(k,\varepsilon)$
2) $\|T\| \leq \lambda$
3) $\|Tx - x\| \leq \varepsilon$ for $x \in B(F)$.

Where $n_X(k,\varepsilon)$ is an integer which depends on k and ε, but not on the space F.

J. Lindenstrauss and L. Tzáfriri have proved that a super-reflexive space X has 1-u.a.p. if and only if X* has 1-u.a.p. ([9]).

S. Heinrich extended this result to general spaces by using the ultrapowers ([6]). The theorem 3 permits to get their result and an explicit computation of $n_{X^*}(k,\varepsilon)$ for every equivalent norm on X.

Let X be a super-reflexive Banach space and $\varepsilon > 0$. By a result of R.E. Bruck ([2]) there exists an integer $p(\varepsilon)$ such that

$$\forall F \subset B(X^*),\ \text{conv}\, F \subseteq \text{conv}_{p(\varepsilon)} F + \varepsilon B(X^*) \quad .$$

Let k be an integer and F a subspace of dimension k, the cardinal of an ε-net of the unit sphere of F is maximized by $K.\varepsilon^{-k}$ where K is a constant which does not depend on F.

With these notations, we get

Theorem 9 [4] Let X be a super-reflexive Banach space. If X has 1-u.a.p. for an arbitrary equivalent norm then for every $\varepsilon > 0$, k integer, one has

$$n_{X^*}(k,9\varepsilon) \leq n_X(K\,\varepsilon^{-k}\, p(\varepsilon),\ \varphi_\varepsilon(\varepsilon)) .$$

3. Duality with smoothness properties.

Definition 10. A Banach space $(X,\|.\|)$ belongs to the class C if for every $\eta \in]0,1[$, there exists a function φ_η such that

$$B_{\|.\|}(X) \subseteq \text{conv}\, E_{\|.\|}(\varphi_\eta) + \eta\, B_{\|.\|}(X) \quad .$$

When this property of uniform exposition is transformed by duality, we obtain a condition of uniform smoothness, more precisely : let us recall a definition which has been introduced in ([3]). Let X be a Banach space. $\mathcal{D}(X)$ is the set of the x in the unit sphere where the norm is Fréchet-smooth and for every $x \in \mathcal{D}(X)$, we denote f_x the differential of this norm in x.

Definition 11. X is almost uniformly smooth (a.u.s.) if there exists a subset A of $\mathcal{D}(X)$ such that

a) $\forall \varepsilon \in]0,1[, \exists \delta(\varepsilon) > 0 : y \in B(X^*), x \in A$ and
$y(x) > 1 - \delta(\varepsilon) \Rightarrow \|y - f_x\| \leq \varepsilon$;

b) the set $\{f_x, x \in A\}$ is a $(1-\varepsilon)$-norming subset of X^*.

Let us point out that this terminology is different from the terminology we used in ([3]).

Proposition 12. [4]. X belongs to the class C if and only if X^* is almost uniformly smooth.

Propositions 6 and 12 give us the following result :

Proposition 13. Every super-reflexive space is almost uniformly smooth for every equivalent norm.

Remark.

The almost uniform smoothness property is far from implying reflexivity. Examples of a.u.s. spaces are given in [3] : $c_0(\Gamma)$, $\ell^\infty(\Gamma)$, $K(\ell^p,\ell^q)$, $L(\ell^p,\ell^q)$ $(1 < p, q < \infty)$.

If X and Y are a.u.s. and Y^* has the Radon-Nikodym property and the approximation property then the tensor-product $X \hat{\otimes}_\varepsilon Y$ is a.u.s. ([3]). The class of a.u.s. spaces is stable by c_0-direct sum ([3]).

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